A CONSTANT PRESSURE LIFTING LINE THEORY FOR THE CALCULATION OF SUBSONIC LINEARIZED FLOWS

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Abstract

The theory presented herein is related to the supersonic lifting line one (SLLT). It is based on the small perturbations assumptions for potential linearized subsonic flows.

A constant distribution of bound vortices is assumed along the chord. By imposing the kinematic condition, which requires the flow to be attached to the wing skeleton, an integral equation involving the spanwise loading is obtained. This equation is solved by means of a numerical functional collocation method.

Results are presented concerning the chordwise position of the collocation point which yields agreement with reference theoretical and experimental data.

Nomenclature

$A=b^2/S$ aspect ratio

A_i i-th coefficient in the trigonometrical series expansion of the circulation

b wing span

c wing chord

C_L lift coefficient $= \dfrac{L}{\rho/2\, U_\infty^2\, S}$

C_{L_α} lift slope coefficient $= \dfrac{\partial C_L}{\partial \alpha}$

M Mach number

q taper coefficient $= 1 - \dfrac{c_r}{c_t}$

S wing surface

U_∞ free stream velocity

x, y wing plane coordinates (see Fig.1)

α wing incidence

Λ sweep angle (see Fig.1)

Γ circulation

$\bar{\Gamma}$ non-dimensional circulation

$$\bar{\Gamma} = \frac{\Gamma}{2\, b\, U_\infty \alpha}$$

subscripts

LE leading edge

M control point

P current point

r root

t tip

T tail

TE trailing edge

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1. Introduction

The present paper deals with a Constant Pressure Lifting Line Theory (CPLLT) which extends the supersonic one (SLLT)(1) to the case of subsonic incompressible flows. Besides the obvious interest of having an integrated subsonic-supersonic lifting line method, the investigation aims also to gather analytical data regarding the control point position. This is of interest also for the study of related constant pressure (CPM) panel methods (e.g. Woodward(2)) based on the Boundary Element Method.

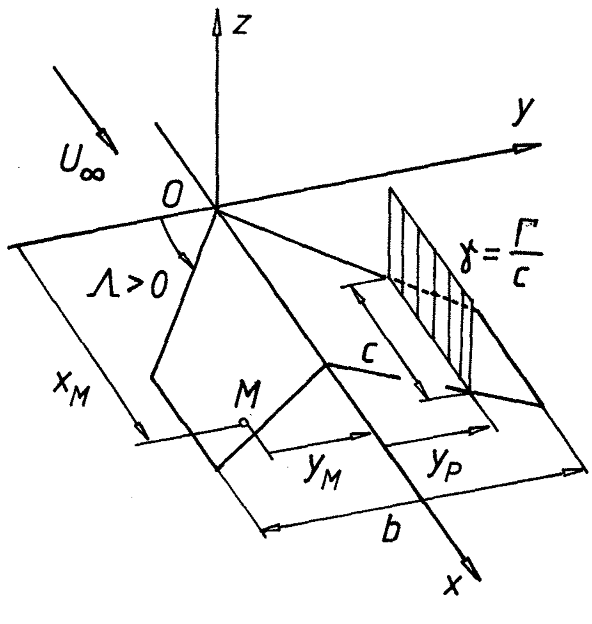

Fig.1 CPLLT Wing Geometry

For incompressible flow conditions the integral equation relating spanwise loading and angle of attack takes the form[3]

$$\lim_{\varepsilon\to 0}\left[2\,\frac{\Gamma(y_M)\,H(x_M,\,y_M,\,y_M)}{\varepsilon}+\right.$$

$$+\int_{-b/2}^{y_M-\varepsilon}\frac{\Gamma(y_P)\,H(x_M,\,y_M,\,y_P)}{(y_M-y_P)^2}\,dy_P\;+$$

$$+\int_{y_M+\varepsilon}^{b/2}\frac{\Gamma(y_P)\,H(x_M,\,y_M,\,y_P)}{(y_M-y_P)^2}\,dy_P\;=$$

$$=4\pi U_\infty\alpha(y_M) \qquad (1)$$

where,

$$H(x_M,\,y_M,\,y_P)=\frac{1}{c(y_P)}\,\{\,c(y_P)\,+ \qquad (2)$$

$$+\sqrt{[x_M-x_{LE}(y_P)]^2+(y_M-y_P)^2}\;-$$

$$-\sqrt{[x_M-x_{TE}(y_P)]^2+(y_M-y_P)^2}\,\}$$

For the solution of integral equation (1) the usual trigonometrical series development is used

$$\Gamma(\theta)=2\;b\;U_\infty\alpha\sum_{i=1}^{N}A_i\;\sin\,i\theta \qquad (3)$$

where the angular spanwise coordinate θ is related to y by the relation

$$y=-\,(b/2)\,\cos\theta \qquad (4)$$

Even if the flow is assumed to be incompressible ($M_\infty=0$), some compressibility effects can be accounted for by means of the Prandtl-Glauert correction.

2. The choice of the control point

2.1. General

The basic assumption that led to the present lifting line theory is the constant pressure (vorticity) distribution along the chord. This is by no means a natural choice for subsonic flow, but enables a unified approach for both subsonic and supersonic regimes.

The main error is due to the infringement of the Kutta-Joukowski condition requiring the trailing edge vorticity to be zero. The constant distribution of vorticity implies an infinite trailing edge induced velocity. Therefore, the trailing edge behaves like an error emitter and this effect is of special importance for the near field properties.

Special care should be taken in finding a suitable position of the collocation point in order to ensure correct results for the present theory.

Both global and local corrections of the chordwise control point position will be examined.

These corrections will be useful not only for the constant pressure SLLT lifting line model, but also for the related panel methods of Woodward type.

2.2. Global control point position

2.2.1. Small aspect ratio wings

In the analysis concerning the limiting case A →0 it will be assumed that all the spanwise control points have the same abscissa x_M and are located at the same relative chordwise position p, where

$$p(y_M) = \frac{1}{c(y_M)} [x_M - x_{LE}(y_M)] \qquad (5)$$

From (2) it follows that

$$H(x_M, y_M, y_p) = 2\ p \qquad (6)$$

The integral equation (1) becomes

$$p \oint_{-b/2}^{b/2} \frac{d\Gamma}{dy_p} \frac{dy_p}{y_M - y_p} = 2\pi U_\infty \alpha \qquad (7)$$

The integral in (7) is calculated according to Cauchy's principal part. The solution of (7) is the elliptic span loading, known to generate a constant downwash in the lifting line approximation[5]. It results

$$\Gamma(y) = \frac{b\ U_\infty \alpha}{p} \sqrt{1 - \left(\frac{y}{b/2}\right)^2} \qquad (8)$$

The corresponding lift slope coefficient becomes

$$C_{L_\alpha}^{CPLLT} = \frac{\pi A}{2\ p} \qquad (9)$$

The target value is[4]

$$C_{L_\alpha}^{Theory} = \frac{\pi A}{2} \qquad (10)$$

and thus we obtain

$$p = 1 \qquad (11)$$

2.2.2. Large aspect ratio wings

This limiting case refers to infinite (two-dimensional) aspect ratio wings. The spanwise variation of parameters will be neglected and, accordingly, the induced velocity may be calculated by a simple chordwise integral

$$\frac{\Gamma}{2\pi c} \oint_0^c \frac{dx_p}{x_M - x_p} = U_\infty \alpha \cos\Lambda \qquad (12)$$

Formula (12) yields the circulation as

$$\Gamma = \frac{\pi c\ U_\infty \alpha \cos\Lambda}{\frac{1}{2} \ln \frac{p}{1-p}} \qquad (13)$$

The lift slope coefficient can be written as

$$C_{L_\alpha}^{CPLLT} = \frac{2\pi \cos\Lambda}{\frac{1}{2} \ln \frac{p}{1-p}} \qquad (14)$$

Since the exact theoretical value for this case is[6]

$$C_{L_\alpha}^{Theory} = 2\pi \cos\Lambda \qquad (15)$$

the chordwise position of the control point results

$$p = \frac{e^2}{e^2+1} \cong 0.88 \qquad (16)$$

2.2.3. General case

Finding a control point position to ensure good correlation between the present lifting line theory and proven theoretical and experimental data for the case of usual aspect ratio wings is a rather complicated problem. Moreover, no "exact" theoretical data are available for wings of general planform, implying that the applicability of any analytical solution would be restricted.

Therefore, an empirical formula was derived for the dimensionless parameter p (5) giving the chordwise position of the control point

$$\bar{p} = 0.88 + 0.12 / \exp\ (A/\cos\Lambda_M) \qquad (17)$$

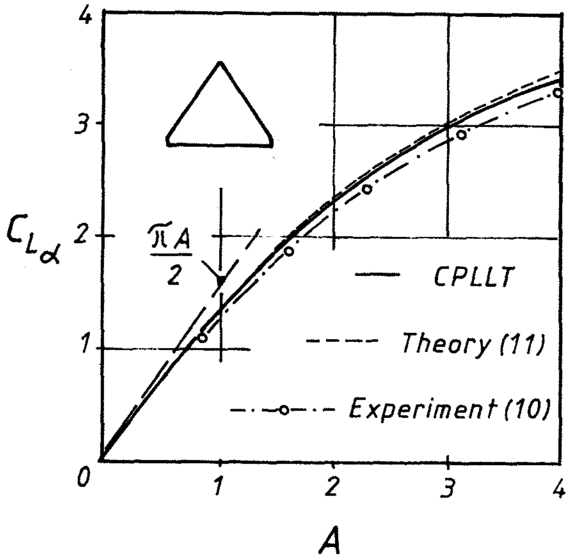

Fig.2 Lift slope coefficient of triangular wings vs. aspect ratio

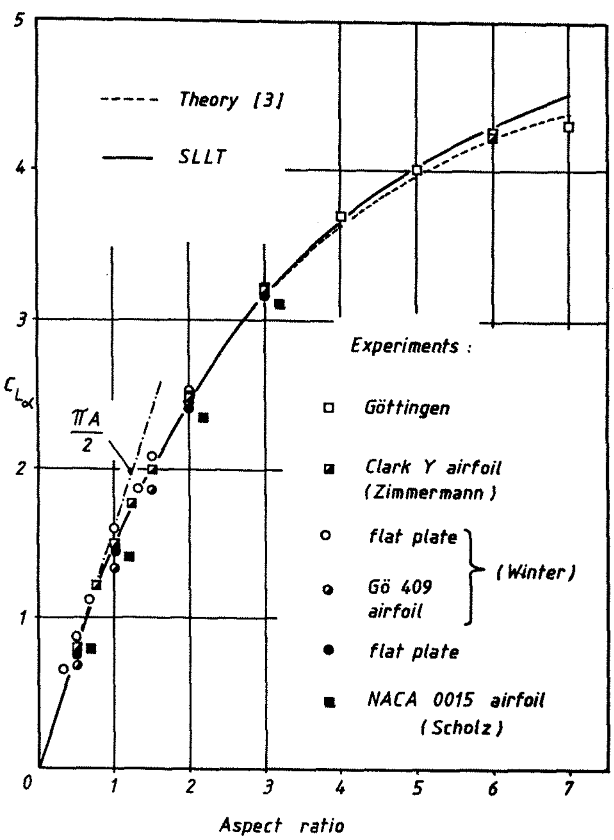

Fig.3 Lift slope coefficient of rectangular wings as a function of aspect ratio

Obviously, formula (17) leads to the analytical results (11) and (16) for small and large aspect ratios A, respectively.

Based on formula (17) a comparison is presented in Fig.2 concerning the lift slope coefficient of triangular wings as a function of aspect ratio.

Similar data are plotted in Fig.3 for rectangular wings.

Finally, the results of the present lifting line theory are compared in Fig.4 for the case of untapered swept wings. The reference data for Figs.2-4 have been taken from Schlichting and Truckenbrodt [12].

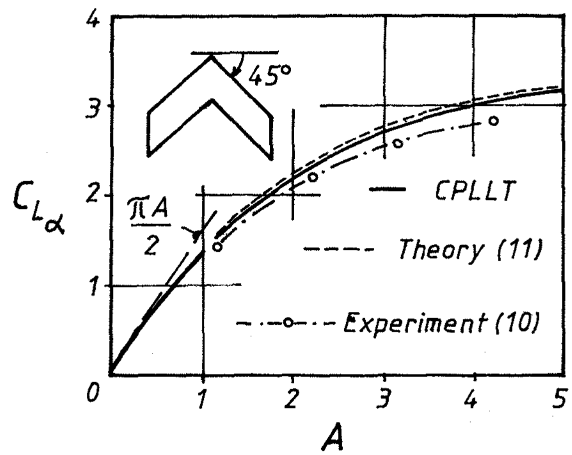

Fig.4 Lift coefficient slope of untapered swept wings vs. aspect ratio

2.3. Local corrections of the control point position

2.3.1. Infinite cranked wing

In this section it will be discussed the influence of a sweep discontinuity on the induced velocity for an infinite wing, see Fig.5.

It will be assumed that both circulation and chord are constant which implies that the effect of the trailing (downstream) vortices will be neglected.

$$-dw(M)=\frac{\gamma\, dx_P}{4\pi d_1}\left[1+\frac{(x_M-x_P)\sin\Lambda+y_M\cos\Lambda}{MP}\right]+ \qquad (18)$$

$$+\frac{\gamma\, dx_P}{4\pi d_2}\left[1+\frac{(x_M-x_P)\sin\Lambda-y_M\cos\Lambda}{MP}\right]$$

where γ is the distributed vorticity,

$$\gamma = \Gamma / c$$

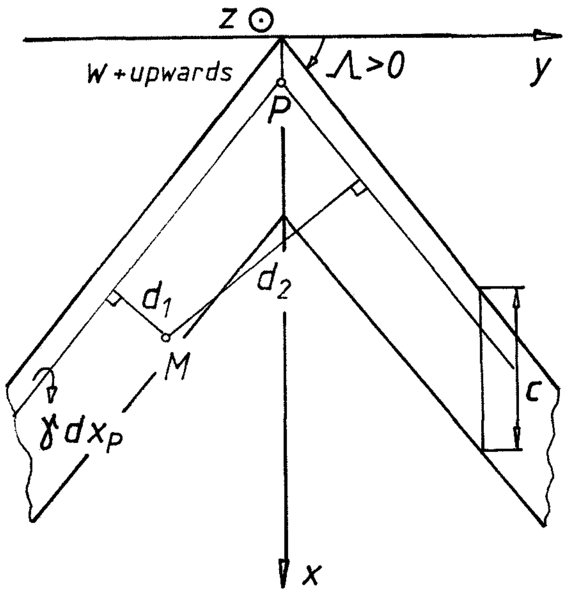

Fig.5 Infinite cranked wing

The integration of (18) yields

$$-\frac{4\pi w(M)\cos\Lambda}{\gamma} = \ln\left|\frac{x_M^2 - b_M^2}{a_M^2 - a_M^2}\right| + \qquad (19a)$$

$$+ 2\sin\Lambda \ln|E_1| + \ln|E_2|$$

where,

$$a_M = c - x_M \qquad (19b)$$

$$b_M = y_M \operatorname{tg}\Lambda \qquad (19c)$$

$$E_1 = \frac{x_M + r_{LE}}{a_M + r_{TE}} \qquad (19d)$$

$$r_{LE}^2 = x_M^2 + y_M^2 \qquad (19e)$$

$$r_{TE}^2 = a_M^2 + y_M^2 \qquad (19f)$$

$$E_2 = \frac{x_M - r_{LE}\sin\Lambda}{x_M + r_{LE}\sin\Lambda} \cdot \frac{a_M - r_{TE}\sin\Lambda}{a_M + r_{TE}\sin\Lambda} \qquad (19g)$$

In the limit $y_M \to 0$, for points situated on the wing surface (i.e. $0<x_M<c$), the induced velocity takes the form

$$\lim_{y_M\to 0}\left[-\frac{2\pi w(M)\cos\Lambda}{\gamma} = \ln\frac{x_M}{c - x_M} + 2\sin\Lambda \ln\frac{4x_M(c - x_M)}{y_M^2} + \ln\frac{1-\sin\Lambda}{1+\sin\Lambda}\right. \qquad (20)$$

We shall notice first that for straight wings formula (20) leads an equivalent of the correlation already established (16) between the chordwise position of the control point p (5) and the local distribution of vorticity

$$\gamma = \pi U_\infty \alpha \qquad (21)$$

which is valid only for the no sweep case considered.

Next, it will be observed that according to formula (20) the induced velocity becomes infinite along the center line. The same result has been observed by Kuchemann and Weber[6] in a somewhat different analysis.

This behaviour of the induced velocity will clearly affect the loading along each line where the sweep is discontinuous (center line, tip, cranks). Either a local modification of the chordwise control point position "p" or a wing singularity may be used in order to tackle this problem.

2.3.2. Control point position correction

Based on the analysis presented before, an attempt will be made at defining a suitable correction formula for the chordwise position of the collocation point in the regions where discontinuities of the sweep angle occur.

For positive (see Fig.5) angles of sweep formula (20) yields

$$\lim_{\frac{y_M}{c}\to 0}\frac{x_M}{c} = \left(\frac{y_M}{2c}\right)^{e_1} \cdot K^{e_2} \qquad (22)$$

where,

$$e_2 = \frac{1}{1 + |\sin\Lambda|} \qquad (23)$$

$$e_1 = 2\,|\sin\Lambda|\,e_2 \qquad (24)$$

$$k = e^{2}\,\frac{1+\sin\Lambda}{1-\sin\Lambda} \qquad (25)$$

For negative sweep angles, it results

$$\lim_{\frac{y_M}{c} \to 0} \frac{x_M}{c} = 1 - \left(\frac{y_M}{2\,c}\right)^{e_1} \cdot k^{-e_2} \qquad (26)$$

To keep the induced velocity constant when approaching a crank line, formulas (22) and (26) indicate that the control point must move towards the leading edge or trailing edge according to the "sign" (see Fig.5) of the kink. Starting from these analytical results the following empirical formula has been derived

$$p = \bar{p} - \Delta p_r - \Delta p_t \qquad (27)$$

where $\bar{p}$ is given by (17) and

$$\Delta p_r = f_1 \,/\, \exp\left[f_2 \cdot k^{e_2} \cdot \left(\frac{y_M}{2\,c}\right)^{e_1}\right] \qquad (28)$$

$$\Delta p_t = f_3 \,/\, \exp\left[f_4 \cdot k^{e_2} \cdot \left(\frac{y_M}{2\,c}\right)^{e_1}\right] \qquad (29)$$

$$f_1 = \begin{cases} \bar{p} & \text{for } \Lambda > 0 \\ \bar{p}-1 & \text{for } \Lambda < 0 \end{cases} \qquad (30)$$

$$f_3 = \begin{cases} \bar{p}-1 & \text{for } \Lambda > 0 \\ \bar{p} & \text{for } \Lambda < 0 \end{cases} \qquad (31)$$

$$f_2 = \begin{cases} 1.8\, f_A \,/\, \bar{p} & \text{for } \Lambda > 0 \\ 1.3\, f_A \,/(1-\bar{p}) & \text{for } \Lambda < 0 \end{cases} \qquad (32)$$

$$f_4 = \begin{cases} f_A \,/(1-\bar{p}) & \text{for } \Lambda > 0 \\ f_A \,/\, \bar{p} & \text{for } \Lambda < 0 \end{cases} \qquad (33)$$

$$f_A = 1 + 2 \,/\, A^{2} \qquad (34)$$

2.3.3. Control point position validation

Since formula (27) is not entirely supported by a theoretical analysis, it must be tested against a number of reference cases.

In order to make this comparison possible, the "direct" approach which allows the calculation of spanwise loading for a wing of given planform and for an imposed distribution of control points $x_M(y_M)$ by means of the integral equation (1) has been inversed. In the "inversed" procedure, the planform and span loading are known and the spanwise distribution of collocation points $x_M(y_M)$ results.

For this purpose, equation (1) is solved iteratively using a Newton-Raphson type method.

In Figs.6 and 7 are presented the results for two wings of aspect ratio A=2.75 and taper coefficient q=0.5, one straight, the other swept (Λ_{25}=50°).

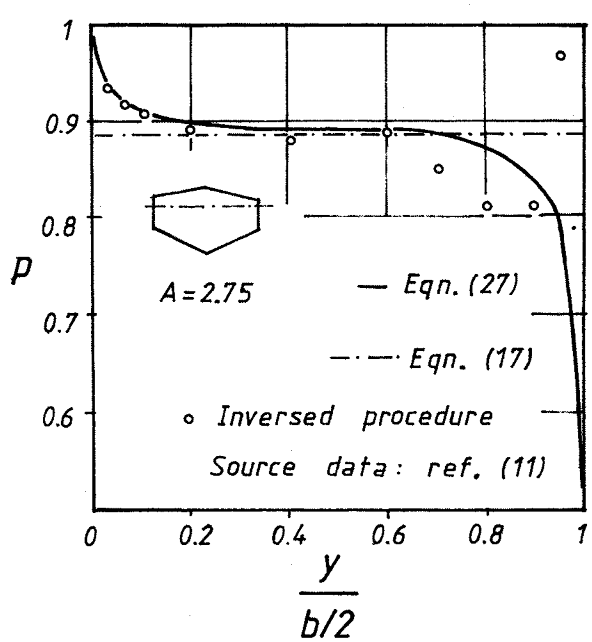

Fig.6 Control point distribution for the straight tapered wing of ref.(11)

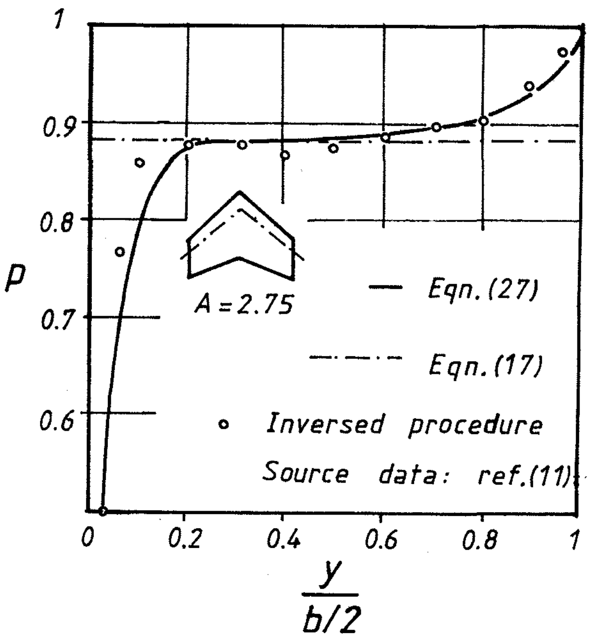

Fig.7 Control point distribution for the swept tapered wing of ref.(11)

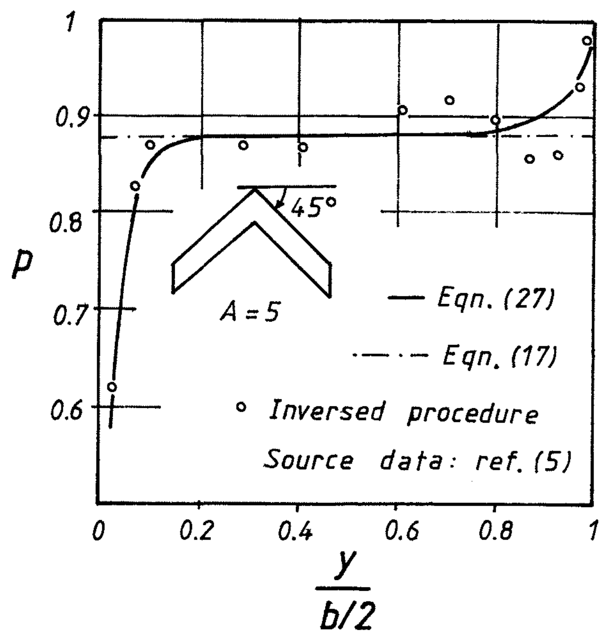

Fig.9 Control point distribution for the swept untapered wing of aspect ratio A=5 from ref.(5)

Similar comparisons are presented in Figs.8 and 9 for a triangular clipped wing of aspect ratio A=3 and for an untapered swept wing of aspect ratio A=5. Both have the leading edge sweep angle Λ_0=45° and have been extracted from ref.[2].

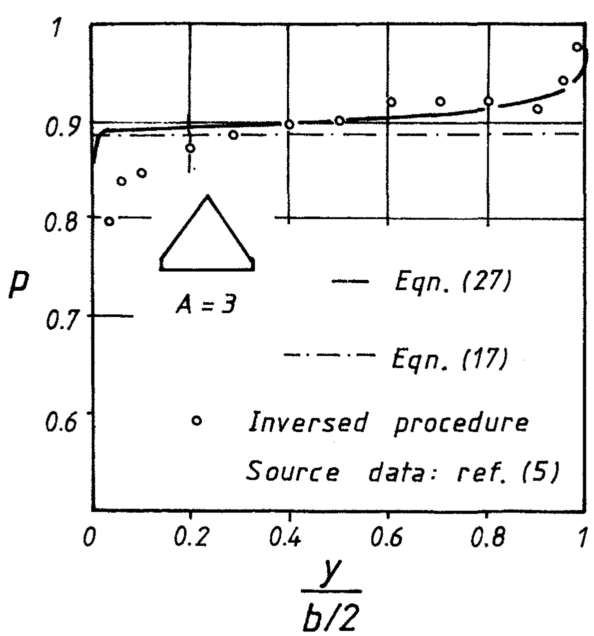

Fig.8 Control point distribution for the triangular clipped wing of aspect ratio A=3 from ref.(5)

The comparisons in the preceding diagrams show a reasonable correlation between the proposed formula (27) for the collocation point spanwise distribution and reference theoretical results. It seems that a better formula could be derived, especially concerning the tip region, which has been treated in formula (27) like the center of a wing with reversed sweep (see ref.(5)).

However, it should be realised that a total agreement is almost impossible to be reached because of the intricate local aspects involved, which preclude an explicit solution.

3. Results

A convergence test is presented in Fig.10 for the case of the untapered swept wing of aspect ratio A=5 in Fig.9. The first coefficient in the trigonometrical series development (3) of the circulation A_1 and the spanwise position of the load application point on a semi-wing are showed to converge very fast. The required computer time refers to BASIC program run on Apple IIe (8-bit microprocessor).

The comparisons in Figs.11-14 are intended to complement those of Figs.6-9.

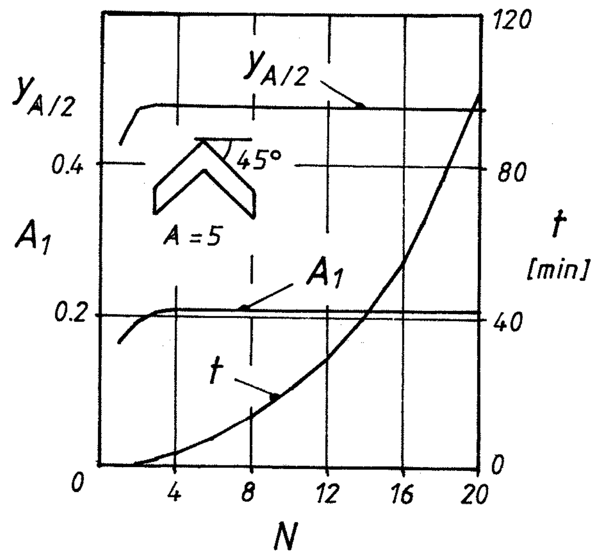

Fig.10 Convergence test for the swept wing of Fig.9

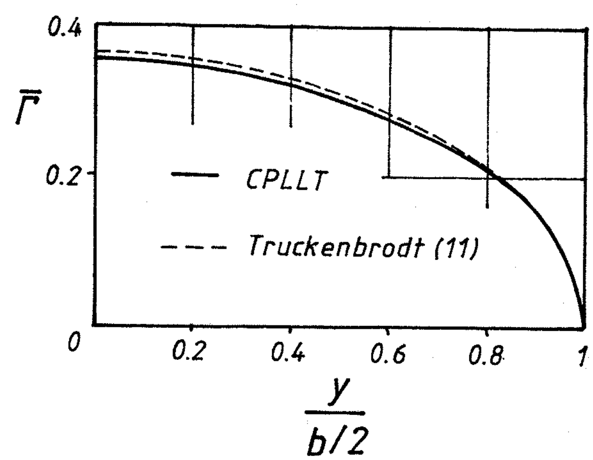

Fig.11 Loading on straight tapered wing of Fig.6

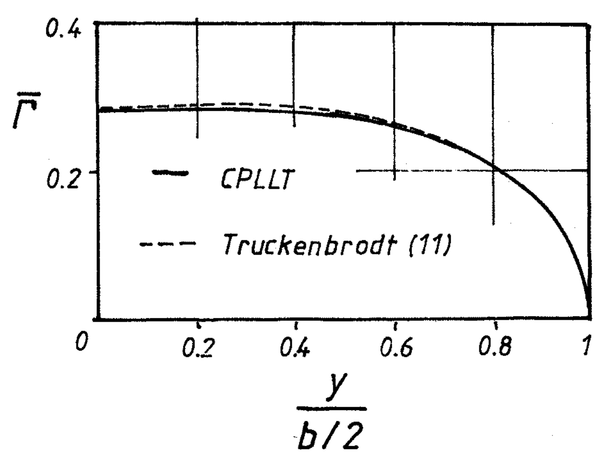

Fig.12 Loading on swept tapered wing of Fig.7

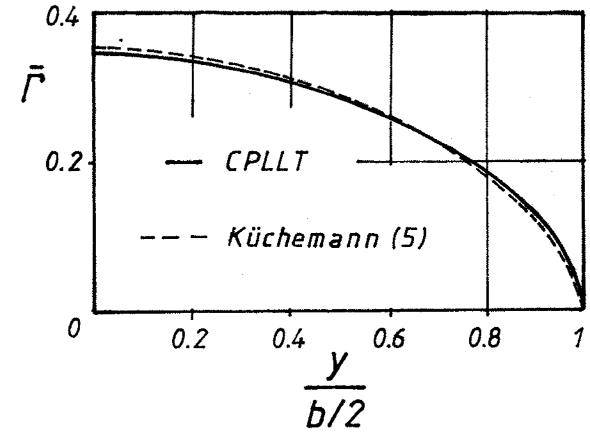

Fig.13 Loading on clipped triangular wing of Fig.8

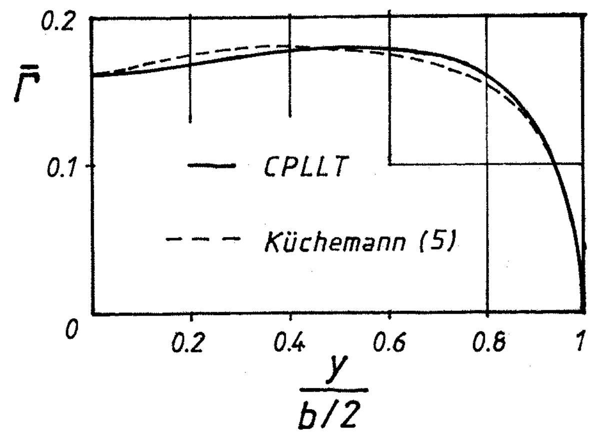

Fig.14 Loading on swept untapered wing of Fig.9

They refer to the same wings, and enable us to examine the sensitivity of Constant Pressure Lifting Line Theory (CPLLT) relative to changes in the chordwise position of the control points.

The agreement of CPLLT with reference data is generally satisfactory.

A test case for wing-fuselage interference is presented in Fig.15. It concerns a straight tapered (q=0.58) wing of aspect ratio A=4.83 at Mach number M=0.3 mid-mounted without incidence on a circular fuselage of relative radius R/b=0.1

The reference theoretical data have been calculated according to the theory of Multhopp[14] by Ferrari[15]. The present approach is broadly similar to these.

Finally, a downwash comparison is presented in Fig.16.

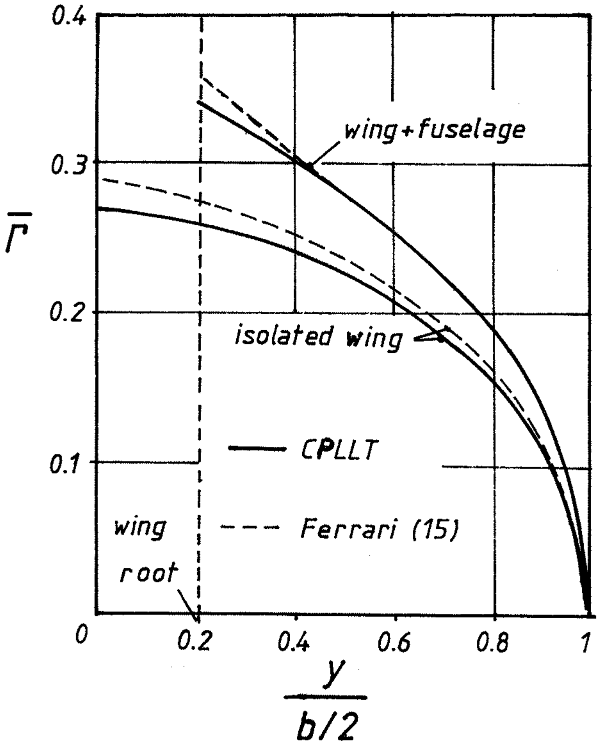

Fig.15 Wing-fuselage interference

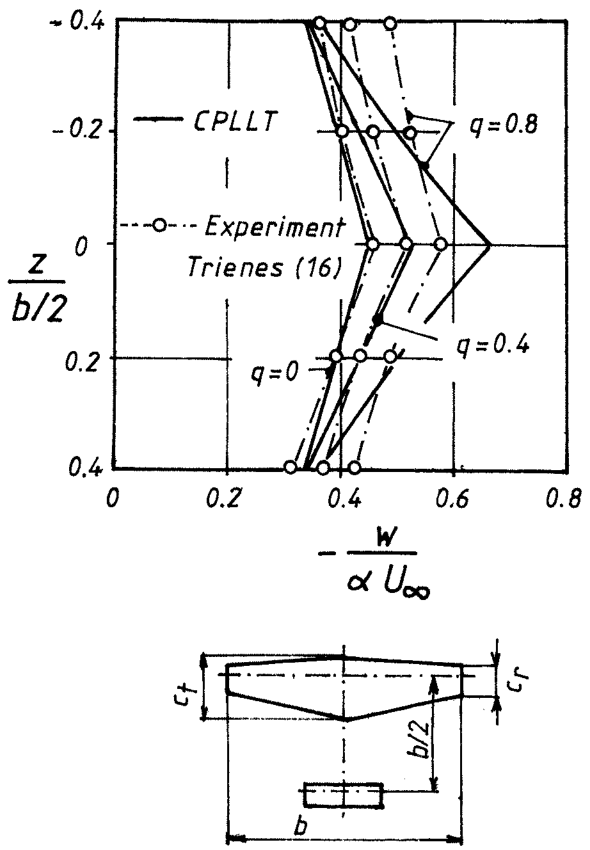

Fig.16 Downwash field induced by straight wings of different taper coefficients

4. Conclusions

The main result of the present paper refers to the chordwise position of the control point for the constant pressure approximation. A formula (27) is presented which has been derived starting from analytical considerations and extended empirically to accomodate finite aspect ratio effects.

It is shown that the control point is largely influenced by sweep discontinuities. For instance, it reaches the leading edge for the center line of a swept wing, although normally it is situated in the vicinity of the trailing edge.

From the examination of Figs.6-9 and Figs.11-14 it can be inferred that the relative differences in terms of loading are roughly of the same order as those in terms of control point position. Since for swept wings the chordwise control point position p (5) exhibit large variations, it appears that the precision of CPLLT based on formula (27) decreases.

The Constant Pressure Lifting Line Theory (CPLLT) presented seem to enable satisfactory results concerning span loading. However, the local correction devised for the chordwise control point position does not fully compensate the singular behaviour due to sweep discontinuities and this is reflected in the rather unsatisfactory estimation of downwash, see Fig.16.

Further investigations are necessary to assess if a completely satisfactory lifting line theory based on the constant pressure assumption is feasible.

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