APPROXIMATE MOMENTUM ANALYSES OF THE FLOW BEHIND AN UNSHROUDED ACTUATOR-DISK, NORMAL TO THE FREE-STREAM

By David H. Chester *

P.O.Box 381, Petach Tikva, 49103, ISRAEL.

ABSTRACT

These analyses extend the one-dimensional actuator-disk theory of propellers by Rankine. Radial velocity components of the axi-symmetric flow are expressed by using three approximate momentum theories, enabling the streamlines and flow properties to be directly obtained. Firstly, the velocity across each concentric stream-tube is taken at its mean value (although the overall velocity distribution across the disc may be non-uniform). Equations for the continuity of volume flow within the stream-tubes and for their velocity components are introduced with a vectorial representation that has both momentum and hodographic connotations. The resulting linear differential equation is integrated to give the shape of the streamlines. Relating the axial acceleration to the static pressure, yields the angle of flow at the edge of a stationary actuator-disk, which at 54·74 degrees closely matches measurements from a stationary rotor. An alternate assumption gives slightly different results, which agree with and explain the semi-empirical expression used by Landgrebe. A hybrid theory is also provided. Graphical presentation of the the theories is used to compare them with experiment. It is concluded that the variation of the streamline shapes with the velocity profile is small and that our knowledge of the flow properties has been usefully extended.

NOTATION

C Constant of integration (see section 2.1).

C1 Property of the slipstream volume flow rate, which = $V_{mb}\ r_{mb}^2$, see equation (2.5).

C2 Property of a streamline in equation (2.3), which = $V_m\ r_m^2\ (\tan \Theta_m)/(r_m - r_j)$, see equation (2.17).

D Constant of integration (see section 2.2).

E Constant of integration (see section 2.3).

F Property of a streamline in equation (2.4), which may = $-(r_j/r_m)\tan \Theta_m\ /(1 - r_j/r_m)$, see equation (2.14a).

f A core modification factor on r , which = $(1 + r_c^2/(r_m\ r_j))$, see equation (3.2a).

K Property of a streamline, which = $\tan \Theta_{ms}$

log Abbreviation for natural logarithm.

R Radius of curvature of a streamline.

r Radial distance from axi-symmetric axis.

t Time.

V Axial velocity component.

VR Radial velocity component.

x Axial distance from the actuator-disk.

δ Small incremental quantity (as prefix).

Θ Angle of the local flow relative to the axial direction.

Subscripts (which also may be taken in certain combinations).

b on the boundary of the slipstream.

c on the core of unchanged axial velocity.

j in the fully developed jet-stream.

m at the actuator-disk.

o free-stream value (of axial velocity).

s for stationary conditions (where $V_o = 0$).

sl along a streamline.

u on the axi-symmetric axis (where r = 0).

* Aeronautical Engineer,
Department 2444, TASHAN, I.A.I. Ltd.,
Ben-Gurion International Airport, 70100.

1. Introduction

The original one-dimensional Actuator-Disk Theory by Rankine (1865) and Froude (1895) References [1] and [2] , proved that the increment of induced velocity doubles between the disk and the jet-wake far downstream. The thrust and ideal propulsive efficiency were related to the jet and free-stream velocities. Initially this approach appeared to establish Momentum Theory as the means for determining fluid dynamics of propellers. But it was never extended to the specific details of the streamlines, nor was it used to explain the local flow properties by using more than Bernoulli's Equation, (which may be written in Newtonian terms as : pressure-change x area = mass-flow x velocity-increment).

The evasion of this path of enquiry carried the impetus for the development of propeller fluid-dynamics away from what initially appeared to be a most promising direction. The reasons for adopting the alternative approaches were :

a) The supremacy of the classical Potential-Flow Theory, involving velocity-potentials and stream-functions, as expressed by Laplace's Equation (with the later use of conformal transformations for two-dimensional flow around aerofoils and cascades). This theory suggested that the actual propeller situation of rotating blades in a free-stream is a very complicated problem and it implied that analysis by merely using momentum methods was inappropriate. Also since Laplace's Equation was derived from the momentum and continuity of a general element in the flow, it was probably felt that justice had already been done.

b) The Blade Element Theory of Propellers (having overcome its initial difficulties by using a modification of the Momentum Theory), proved to be a practical design tool. An essential assumption here was that the blade elements behaved independently, which was confirmed by wind-tunnel measurements on propellers by Lock [3] in 1924.

   For the theoretical distributions on lightly loaded propellers without contraction of their slipstreams, the Vortex Theory by Betz [4] in 1919 was developed to provide an exact (and complex) analysis by Goldstien [5] in 1929. This concept, which eclipsed the Momentum Theory approach, was also strongly influenced by the previous work on fixed-wing vortex aerodynamics.

c) The introduction of marine screw-propellers (after a dramatic contest with paddle-wheel propulsion) was so successful, that the design by the Blade Element Theory was accepted for many years. Ships' propellers were inevitably made to operate in the region of disturbed flow at the stern, where ideal efficiencies could not be expected in any case. Consequently no motivation was felt for improving the theory. Only when high-speed craft (using steam turbines) became feasible, did a need emerge to design the shape of the hull to provide the propeller with a smooth uniform entry flow. At this stage the Blade Element Theory was dominant and the streamline hull-lines were obtained by estimation.

d) The original Actuator-Disk Theory used the assumption of uniform pressure and velocity distributions over the disk. Extending the general theory to two dimensions by making the distribution non-uniform seemed to be a step that was just too hard to take. Even when a potential-flow model was introduced by Koning [6] in 1929, he took a distribution of doublets having uniform strength on the disk. The conditions along the axis of asymmetry were then applied across all of the slip-stream. The solution expressed the variation of flow parameters with axial displacement, but was found to be unsatisfactory.

   Similarly in a recent technical note, Gibson [7] obtained the flow field of cylindrically arranged ring vortices without including contraction effects. The difficulties in finding the exact solution were avoided by this simplification and the model provided for a uniform velocity over the disk. His expression for the variation of axial velocity with axial position agreed with that of Koning.

e) Eventually contraction effects were taken more seriously and the subsequent helicopter rotor theories developed using Vortex Theory and finite elements for the blades and downstream flow, see Reference [8] . Shed vortices that assume "free-wake" flow patterns were found to provide the most satisfactory results, although the demand on the digital computer was heavy, making good solutions difficult to obtain until recent times. They are still hard to understand in analytic terms.

   Flax [9] showed that for a hovering rotor the flow models by the Vortex and Momentum methods were identical. However no effort was subsequently made to substitute the latter into the analysis of helicopter rotors, due perhaps to the Shed Vortex Theory holding precedence.

Only in more general situations does Momentum Theory find use in aerodynamics. For example, the down-wash behind a lifting body or wing is related to the rate of change of momentum experienced by the air that passes over it. This is equated to the lift. But when particular details of the flow are required at the element itself, it is left to the more complicated analytic methods to provide the solutions.

In this paper an attempt will be made to apply the Momentum Theory to the flow at and beyond an unducted actuator-disc facing the free-stream, without the previous restrictive assumptions being applied. In particular, taking a uniform velocity over the disk and the neglect of the component of radial velocity no longer will be accepted.

2. Theoretical Analysis.

The flow past the actuator-disk (shown in Fig.1), occupies the following three regions:

a) Up-stream, where the free-stream accelerates as it encounters the progressively reducing pressures ahead of the disk.

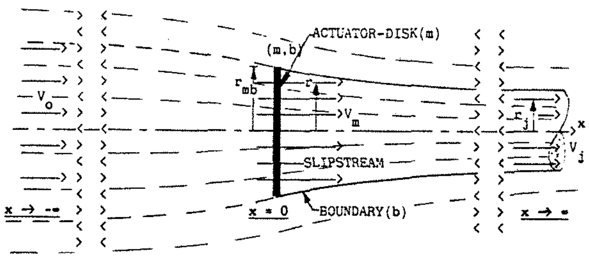

FIGURE 1. <u>THE FLOW THAT IS INDUCED BY THE ACTUATOR-DISK PRESSURES</u>

b) In the isolated slipstream after the disk (where $x > 0$), the free-stream axial velocity V_o having already grown to V_m , increases further, as the flow leaves the raised pressure field behind the disk. Eventually it reaches the value of V_j far downstream. With the development of this axial velocity, the the slipstream reduces in diameter. This contraction affects

c) the flow that remains outside the slipstream and aft of the disk. With progress in the axial direction the velocity reduces, ultimately returning to that of the free-stream (whilst that within the slipstream achieves its asymptotic jet-stream value).

These conditions are summarized in Table 1 , which introduces some additional properties of the flows.

TABLE 1. <u>BOUNDARY-CONDITIONS FOR THE FLOW AROUND AN ACTUATOR-DISK</u>

QUANTITY	LOCATION IN FLOW		
Description	In the free-stream	At the disk	In fully developed jet-stream
axial distance x	$-\infty$	0	$+\infty$
axial velocity V	V_o	V_m	V_j
radial distance r	-	r_m	r_j
flow direction Θ (negative quantity)	0	Θ_m (min.)	0
$\tan\Theta = dr/dx$	0	min.	0
acceleration dV/dt	0	max.	0

The one-dimensional momentum theory of References [1] and [2] may be applied to any streamline in the slipstream and it provides :

$$V_m = (V_j + V_o)/2 \qquad (1.1).$$

Although this result was originally derived for a slipstream having a uniform velocity distribution, the expression also applies to stream-tubes in a slipstream having an axial velocity that varies with radial position. Even when the axial velocity is combined with radial and tangential components, equation (1.1) is still true. Only the most general properties have been described above.

2.1 <u>Mean Velocity Theory for Flow Within Concentric Stream-Tubes</u>

The axi-symmetric flow within the slipstream consists of numerous streamlines that lay within concentric stream-tubes. In applying equation (1.1), the <u>mean value of the axial velocity</u> is taken over the cross-section of a particular stream-tube. The variable radial velocity distribution implies that different mean axial velocity-ratios may occur along various stream-tubes, with a corresponding axial displacement. Then the continuity of the ensuing flow produces contractions in the stream-tubes' diameters that are not necessarily in proportion.

2.1.1 <u>Due to the continuity of volume flow</u>, the stream-tubes contract with passage down-stream according to :

$$r_m^2\, V_m = r^2\, V = r_j^2\, V_j \qquad (1.2)$$

for the axi-symmetric flows. This expression is applicable to any of the concentric stream-tubes including the one containing the outer bounding streamline (where $r_m = r_{mb}$ and $r_j = r_{jb}$).

Comparing the two extreme conditions of equation (1.2) :

$$r_j^2 = r_m^2\, V_m/V_j \qquad (1.3)$$

and using equation (1.1) :

$$r_j = r_m\, [(1 + V_o/V_j)/2]^{\frac{1}{2}} \qquad (1.4).$$

But when the disk is stationary ($V_o = 0$) , the use of this equation results in :

$$r_j/r_m = 1/\sqrt{2} = 0{\cdot}7071 \qquad (1.4a).$$

In practice, the slipstream behind a stationary disk does not contract by this amount and some of the assumptions that lead to this result are invalid. It is suggested here that the uniform velocity distribution for a stream-tube taken in equation (1.3) is inapplicable to the whole of the slipstream. In this case, since each "active" stream-tube must contract by the same ratio, it is deduced that there also exists a central core of "passive" streamlines (that cause no thrust, acceleration nor contraction of the jet). This model of the flow now approximately corresponds to the overall situation. To allow for the radius of the core, equation (1.2) is modified giving:

$$(r_m^2 - r_c^2)\, V_m = (r^2 - r_c^2)\, V = (r_j^2 - r_c^2)\, V_j \qquad (1.5).$$

where $r_m \geq r \geq r_j > r_c > 0$.

The velocities used here are the mean values over the stream-tubes, <u>not including the core area</u>. By using the radii that occur on the boundary of the slipstream, from equation (1.5) :

$$r_{mb}^2\ V_m - r_{jb}^2\ V_j = r_c^2\ (V_m - V_j)$$

$$\text{or } r_c/r_{mb} = \left[\frac{(r_{jb}/r_{mb})^2\ V_j/V_m - 1}{V_j/V_m - 1}\right]^{\frac{1}{2}} \qquad (1.5a).$$

This relationship is illustrated in Fig.2.

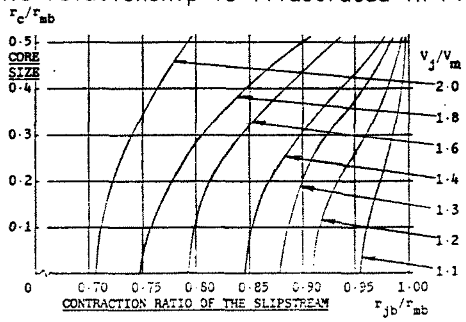

FIGURE 2. VARIATION OF THE RELATIVE CORE SIZE WITH THE CONTRACTION RATIO OF THE SLIPSTREAM FOR VARIOUS JET-STREAM VELOCITY RATIOS - equation (1.5a)

Equations (1.3), (1.4) and (1.4a) are also affected, but they are of little further interest.

Now differentiate equation (1.5) with respect to x :

$$(r^2 - r_c^2)\ \frac{dV}{dx} + 2\ r\ V\ \frac{dr}{dx} = 0 \qquad (1.6)$$

$$\text{or} \quad \frac{dV}{dx} = -\ \frac{2\ r\ V}{r^2 - r_c^2}\ \tan\Theta \qquad (1.6a)$$

$$\text{where :} \quad \frac{dr}{dx} = \tan\Theta \qquad (a.1)$$

the angle Θ being that of the local flow direction, as explained in the appendix. (This includes the streamlines that occur on the boundary of the stream-tube.) The radial velocity component may now be determined from equation (1.6a) :

$$VR = V\ \frac{dr}{dx} = V\ \tan\Theta = -\ \frac{r^2 - r_c^2}{2\ r}\ \frac{dV}{dx} \qquad (1.7).$$

At the disk, the maximum value of radial velocity is found from this equation as :

$$VR_m = V_m\ \tan\Theta_m = -\ (1 - (r_c/r_m)^2)\ \frac{r_m}{2}\left[\frac{dV}{dx}\right]_m \qquad (1.8).$$

However, this expression is not directly useful and a different approach for determining the radial velocity component is adopted below.

2.1.2 The velocity vector or hodographic presentation shown in Fig.3. is now used to express the variation of the conditions along each streamline. *This important concept for the momentum of* the overall flow of inclined actuator-disks was proposed by Hafner [15] for lifting rotors. The flow directions were later found by Chester [11].

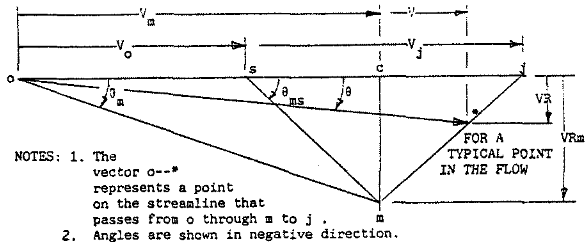

FIGURE 3 VELOCITY COMPONENTS ON A TYPICAL STREAMLINE

A similar approach is taken here using the mean velocities described above. The added momentum components are taken as laying along straight lines. This simplifying assumption enables an analytic solution to be obtained although it violates the exact hodographic equations of Reference [4].

Both from equation (1.7) and the vector diagram, it is seen that there is a linear relationship between the radial velocity component VR and the change in the local axial velocity. Between its asymptotes at either the fully developed jet-stream or the free-stream, this radial velocity can now be written as:

$$VR = 2\ VR_m\ \frac{V_j - V}{V_j - V_o} \quad \text{behind the disk, where} \quad x > 0$$

$$\text{or } V\ \tan\Theta = \frac{V_j + V_o}{V_j - V_o}\ (V_j - V)\ \tan\Theta_m \quad \text{with } V > V_m \qquad (1.9a)$$

and similarly ahead of the disk, where $x < 0$:

$$V\ \tan\Theta = \frac{V_j + V_o}{V_j - V_o}\ (V - V_o)\ \tan\Theta_m \quad \text{with } V < V_m \qquad (1.9b)$$

(at the disk, where $x = 0$, these expressions become the same).

$$\text{If} \quad K = \frac{V_j + V_o}{V_j - V_o}\ \tan\Theta_m \qquad (1.9c),$$

then from the vector diagram : $K = \tan\Theta_{ms}$, which is the slope of the flow at the same position on a stationary actuator-disk. The size of this term depends upon the particular streamline involved.

Equation (1.9a) may be rewritten as :

$$\tan\Theta = K\ (V_j/V - 1)$$

and with equations (1.5) and (a.1), this becomes:

$$\tan\Theta = \frac{dr}{dx} = K\ [(r^2 - r_c^2)/(r_j^2 - r_c^2) - 1] \qquad (1.10).$$

Although mean velocities of the stream-tubes were used so far in the analysis, the radii appearing in equation (1.10) are on the surfaces of the stream-tubes. They are therefore true properties *of streamlines. For conditions at the disk :*

$$(r_j^2 - r_c^2)\ \tan\Theta_m = K\ (r_m^2 - r_j^2)\ ,$$

hence :

$$K/(r_j^2 - r_c^2) = \tan\Theta_m / (r_m^2 - r_j^2) \quad (1.10a).$$

On separating the variables in equation (1.10) with equation (1.10a) :

$$\frac{dr}{r^2 - r_j^2} = \frac{K}{r_j^2 - r_c^2}\,dx = \frac{\tan\Theta_m}{r_m^2 - r_j^2}\,dx \quad (1.11).$$

After integration this becomes :

$$-(1/r_j)\coth^{-1}(r/r_j) = x\,(\tan\Theta_m)/(r_m^2 - r_j^2) + C \quad (1.12)$$

where C is the constant of integration. (It is noted that the alternative solution of the integral on the left-hand side, has logarithms and is less useful here.) Equation (1.12) may be written as :

$$r_j/r = \tanh[-\,x\,r_j\,(\tan\Theta_m)/(r_m^2 - r_j^2) - C\,r_j] \quad (1.13).$$

At the disk, $x = 0$, $r = r_m$ and :

$$r_j/r_m = \tanh(-\,C\,r_j)$$

$$\text{or } -\,C\,r_j = \tanh^{-1}(r_j/r_m) \quad .$$

Substituting back into equation (1.13) and re-arranging gives :

$$\frac{r}{r_m} = (r_j/r_m)\coth\left[\frac{-(x/r_m)(r_j/r_m)\tan\Theta_m}{1 - (r_j/r_m)^2} + \tanh^{-1}(r_j/r_m)\right] \quad (1.14).$$

Hence the family of streamlines is determined by values of the two parameters : $\tan\Theta_m$ and (r_j/r_m) applied directly in equation (1.14). The effect of the core is indirectly retained only by the distribution of $\tan\Theta_m$ over the disc.

Due to the assumption of straight hodographic lines, this result will probably not be very accurate for actual propellers and rotors. Equation (1.14) is also affected by the approximations for the velocity distribution across the slipstream that has yet to be determined properly.

2.1.3 <u>The axial acceleration at the disk</u> is now examined. From equations (a.1) and (1.6a) , after re-arrangement and differentiation with respect to x:

$$\frac{d^2V}{dx^2} = \frac{-\,2\,r}{r^2 - r_c^2}\,\frac{dV}{dx}\,\frac{dr}{dx} - 2\,V\left[\frac{1}{r^2 - r_c^2} - \frac{2\,r^2}{(r^2 - r_c^2)^2}\right]\left[\frac{dr}{dx}\right]^2 - \frac{2\,r\,V}{r^2 - r_c^2}\,\frac{d^2r}{dx^2}$$

and substituting again from equation (1.6a) :

$$\frac{d^2V}{dx^2} = \frac{4\,r^2\,V}{(r^2 - r_c^2)^2}\left[\frac{dr}{dx}\right]^2 - \frac{2\,V}{r^2 - r_c^2}\left[1 - \frac{2\,r^2}{r^2 - r_c^2}\right]\left[\frac{dr}{dx}\right]^2 - \frac{2\,r\,V}{r^2 - r_c^2}\,\frac{d^2r}{dx^2}$$

and

$$\frac{d^2V}{dx^2} = \frac{-\,2\,V}{r^2 - r_c^2}\left[\left[1 - \frac{4\,r^2}{r^2 - r_c^2}\right]\left[\frac{dr}{dx}\right]^2 + r\,\frac{d^2r}{dx^2}\right] \quad (1.15).$$

At the disk $x = 0$ and the axial acceleration is a maximum value. This implies that $[d^2V/dx^2]_m = 0$ and using equations (a.1) and (a.2) :

$$\left[1 - \frac{4\,r_m^2}{r_m^2 - r_c^2}\right]\tan^2\Theta_m + \frac{r_m}{\cos^2\Theta_m}\left[\frac{d\Theta}{dx}\right]_m = 0$$

hence :

$$\left[\frac{d\Theta}{dx}\right]_m = \frac{\sin^2\Theta_m}{r_m}\left[\frac{4\,r_m^2}{r_m^2 - r_c^2} - 1\right] \quad (1.16).$$

Because Θ has its minimum value at the disk one might suppose that the first differential, namely equation (1.16) , becomes zero and the angle too. However this clearly does not occur in practice and the use of this expression to find the value of the angle is invalid, due to the singularity when $x = 0$. If we now consider the variation of r_m and Θ_m , the distribution over the actuator-disk can be found by :

$$\left[\frac{d\Theta}{dr}\right]_m = \left[\frac{d\Theta}{dx}\right]_m\left[\frac{dx}{dr}\right]_m = \left[\frac{d\Theta}{dx}\right]_m\frac{1}{\tan\Theta_m}$$

then from equation (1.16) :

$$\left[\frac{d\Theta}{dr}\right]_m = \left[\frac{4\,r^2_m}{r^2_m - r_c^2} - 1\right]\frac{\sin\Theta_m\cos\Theta_m}{r_m}$$

and with the separation of the variables, prior to integration over the disk :

$$\frac{3\,r^2_m + r_c^2}{r_m\,(r_m^2 - r_c^2)}\,dr_m = \frac{d\Theta_m}{\sin\Theta_m\cos\Theta_m}$$

After integration this yields :

$$\log\left[\frac{(r_m^2 - r_c^2)^2}{r_m}\right] = \log[\tan\Theta_m] + C \quad ,$$

C being the constant of integration.

Using conditions at the boundary, where $\Theta_m = \Theta_{mb}$ and $r_m = r_{mb}$, this equation reduces to:

$$\frac{(r_m^2 - r_c^2)^2\,r_{mb}}{(r_{mb}^2 - r_c^2)^2\,r_m} = \frac{\tan\Theta_m}{\tan\Theta_{mb}} \quad (1.17).$$

When $r_m = r_c$, the value of $\tan\Theta_m$ on the core radius is zero, as anticipated from the comments made at the end of the previous section. A non-dimensional form of this expression is shown in Fig.4a for the parameters (r_c/r_{mb}) , (r_m/r_{mb}) and $(\tan\Theta_m)/(\tan\Theta_{mb})$. When the core is of negligible size, at the disc it appears that the local slope of the streamlines increases as the cube of the radial position.

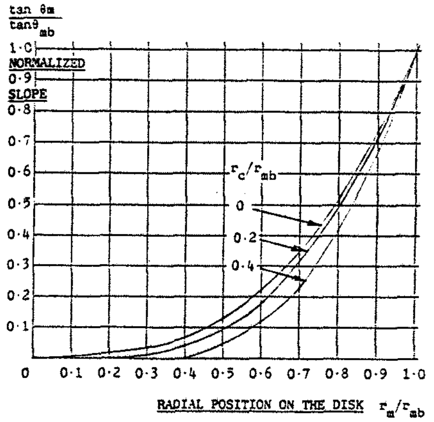

FIGURE 4a. DISTRIBUTION ACROSS THE DISK OF THE SLOPE OF THE STREAMLINES, FOR DIFFERENT CORE SIZES - equation (1.17)

Using equation (1.6a) for conditions at the disk

$$\left[\frac{dV}{dx}\right]_m = \frac{-2\, r_m V_m}{r_m^2 - r_c^2} \tan \Theta_m$$

and substituting from equation (1.17) for $\tan \Theta_m$

$$\left[\frac{dV}{dx}\right]_m = \frac{-2\, r_m V_m}{r_m^2 - r_c^2} \frac{(r_m^2 - r_c^2)^2\, r_{mb}}{(r_{mb}^2 - r_c^2)^2\, r_m} \tan \Theta_{mb}$$

$$\left[\frac{dV}{dx}\right]_m = \frac{1}{V_m}\left[\frac{dV}{dt}\right]_m = -2\, r_{mb} V_m \tan \Theta_{mb} \frac{r_m^2 - r_c^2}{(r_{mb}^2 - r_c^2)^2} \qquad (1.18).$$

This expression is shown on Fig.4b.

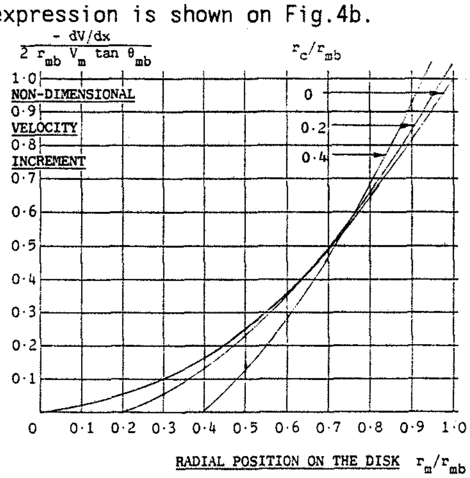

FIGURE 4b. DISTRIBUTION ACROSS THE DISK OF THE VELOCITY INCREMENT FOR DIFFERENT SIZES OF THE CORE - equation (1.18).

When the core is of negligible size it appears that the local acceleration increases with the square of the radial position. Similar results to these are applicable further downstream, since the flow is continuous. The last two expressions do not seem to be very typical of flows through actual propellers or rotors and consequently in part 2.2 of this work we will attempt to introduce less constricting assumptions.

2.1.4 For the acceleration along the boundary of the free-jet (of a stationary disk, where $V_o = 0$) it has been shown in the appendix that :

$$\frac{dV_b}{dx} = -V_{jb} \sin \Theta_{bs} \frac{d\Theta_{bs}}{dx} \qquad (a.7).$$

Now from equation (1.6a), for the conditions being taken :

$$\frac{dV_b}{dx} = \frac{-2\, r_b V_b}{r_b^2 - r_c^2} \tan \Theta_{bs}$$

By combining these two expressions to eliminate dV_b/dx and after re-arrangement one obtains :

$$\frac{d\Theta_{bs}}{dx} = \frac{2\, r_b}{r_b^2 - r_c^2}\left[\frac{V_b}{V_{jb}}\right]\frac{1}{\cos \Theta_{bs}}$$

But according to equation (a.6) the last two terms cancel and :

$$\frac{d\Theta_{bs}}{dx} = \frac{2\, r_b}{r_b^2 - r_c^2} \qquad (1.19).$$

Reference to this form of equation has been made in the appendix. After deriving equation (a.5) there, this type of equation was found to result in zero acceleration along the central axis. At this singular line and within the core, the expression is not workable, but on the boundary equation (1.19) can be solved by the substitution

$$\frac{d\Theta}{dx} = \frac{d\Theta}{dr}\frac{dr}{dx} = \frac{d\Theta}{dr} \tan \Theta$$

and by separating the variables it gives :

$$\tan \Theta_{bs}\, d\Theta_{bs} = 2\, r_b\, dr_b/(r_b^2 - r_c^2)$$

After integration this yields :

$$-\log(\cos \Theta_{bs}) = \log(r_b^2 - r_c^2) + C ,$$

where C is the constant of integration. Using the conditions at the disk (where $x = 0$) this reduces to :

$$(r_b^2 - r_c^2) \cos \Theta_{bs} = (r_{mb}^2 - r_c^2) \cos \Theta_{mbs}$$

$$\text{or} \quad \left[\frac{r_b^2 - r_c^2}{r_{mb}^2 - r_c^2}\right] = \frac{\cos \Theta_{mbs}}{\cos \Theta_{bs}} \qquad (1.20).$$

The parallel streamlines at the fully contracted jet define the lower limit in the relationship between r_{jb}/r_{mb} , r_c/r_{mb} and $\cos \Theta_{mbs}$ for any of these flows:

$$\left[\frac{r_{jb}^2 - r_c^2}{r_{mb}^2 - r_c^2}\right] = \cos \Theta_{mbs} \qquad (1.21),$$

because the angle $\Theta_{jbs} = 0$.

2.1.5 Comparing the two equations for $d\Theta/dx$, at the outer corner of the stationary disk, point (mb) of Fig.1, equations (1.16) and (1.19) are set equal, then :

$$\frac{\sin^2\Theta_{mbs}}{r_{mb}}\left[\frac{4\ r_{mb}^2}{r_{mb}^2 - r_c^2} - 1\right] = \frac{2\ r_{mb}}{r_{mb}^2 - r_c^2}$$

or $\sin^2\Theta_{mbs} = 2\ r_{mb}^2/(3\ r_{mb}^2 + r_c^2)$ (1.22)

Combining equations (1.21) and (1.22) to eliminate the angle according to the rule $\sin^2\Theta + \cos^2\Theta = 1$, provides :

$$\left[\frac{r_{jb}^2 - r_c^2}{r_{mb}^2 - r_c^2}\right]^2 + \frac{2\ r_{mb}^2}{3\ r_{mb}^2 + r_c^2} = 1$$

and :

$$r_{jb} = \left[\left[\frac{r_{mb}^2 + r_c^2}{3\ r_{mb}^2 + r_c^2}\right]^{\frac{1}{2}}(r_{mb}^2 - r_c^2) + r_c^2\right]^{\frac{1}{2}} \quad (1.23)$$

The values of Θ_{mbs} and r_{jb}/r_{mb} that are obtained from these last two expressions are calculated and presented in Table 2. It is seen that there is only a very small effect on these quantities by the core parameter r_c/r_m.

TABLE 2. VARIATIONS OF SLIPSTREAM CONTRACTION r_{jb}/r_{mb} AND FLOW ANGLE Θ_{mbs} WITH CORE SIZE r_c/r_{mb} FOR A STATIONARY ACTUATOR-DISK

r_c/r_{mb}	0·0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40
r_{jb}/r_{mb}	0·7598	0·7606	0·7639	0·7688	0·7756	0·7840	0·7940	0·8053	0·8179
$-\Theta_{mbs}$	54·74°	54·70°	54·60°	54·43°	54·20°	53·91°	53·56°	53·16°	52·71°

When the core radius is zero, the flow angle is given, using equation (1.22) , by :

$$\sin^2\Theta_{mbs} = 2/3$$

or : $\Theta_{mbs} = \sin^{-1}[-(2/3)^{\frac{1}{2}}]$

$= -\ 54{\cdot}74$ degrees (1.24),

the positive root being omitted, since it is physically impossible (unless the device is acting as a braked windmill). Substitution back into equation (1.21) yields :

$r_{jb}/r_{mb} = (1 - 2/3)^{\frac{1}{4}} = 0{\cdot}7598$ (1.25).

Even with the assumptions used in this approximate theory, these values of Θ_{mbs} and r_{jb}/r_{mb} agree closely with the experimentally determined results (see part 3 of this work) and at least the flow near the edge of the slipstream has been successfully represented.

2.2 Approximate Axial-Flow Theory

Unlike the previous theory, the assumption of the mean velocity distribution across the concentric stream-tubes is now dropped. Instead we consider the flow within an element of the slipstream. Equations (1.1) and (1.5) still correspond to the overall or general mean flow in the axial direction x , but the expression for the local continuity of the volume-flow of the stream-tubes is derived instead for an element shown in Fig.5. The streamline properties C1 and C2 are used below, instead of the slipstream flow constants that appeared in the Mean Velocity Theory. This

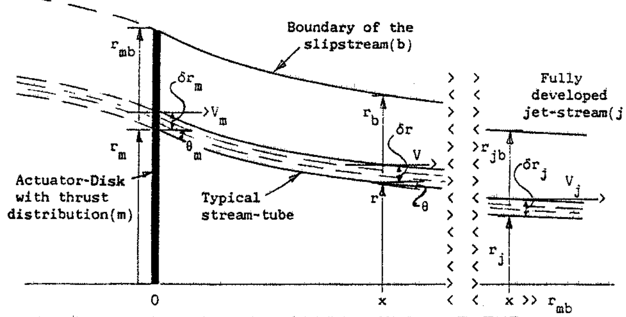

FIGURE 5. ELEMENTARY STREAM-TUBE OF THE SLIPSTREAM

approach is applicable to individual streamlines and it takes a differential equation form, which is introduced to complement equation (1.5) for continuity :

$$V_m\ r_m\ \delta r_m = V\ r\ \delta r = V_j\ r_j\ \delta r_j \quad (2.1)$$

Re-arranging gives $\delta r_j = \delta r_m (V_m/V_j)(r_m/r_j)$ (2.1a).

Using equation (1.5) to include the core, provides:

$$(r_m^2 - r_c^2)\ V_m = (r_j^2 - r_c^2)\ V_j$$

which yields : $r_m^2 = (r_j^2 - r_c^2)\ V_j/V_m + r_c^2$.

With equation (2.1a) this results in :

$$\delta r_j = \delta r_m\ (V_m/V_j)[(1 - (r_c/r_j)^2)V_j/V_m + (r_c/r_j)^2]^{\frac{1}{2}}$$

or $\delta r_j = \delta r_m\ (V_m/V_j)^{\frac{1}{2}}[1 - (r_c/r_j)^2(1 - V_m/V_j)]^{\frac{1}{2}}$

and $\delta r_j = \delta r_m\ [(1 + V_o/V_j)/2]^{\frac{1}{2}}[1 - (r_c/r_j)^2(1 - V_o/V_j)/2]^{\frac{1}{2}}$ (2.2).

When the core radius is zero this reduces to:

$$\delta r_j = \delta r_m[(1 + V_o/V_j)/2]^{\frac{1}{2}} \quad (2.2a).$$

This is similar to equation (1.4) which it replaces - it is given for comparison purposes.

Divide the variable part of equation (2.1) by δx and take the limiting values as δx , $\delta r \to 0$. Then along a streamline, using the partial differentiation notation that is now obligatory:

$$V\ r\left[\frac{\partial r}{\partial x}\right]_{sl} = C2 \quad \text{(a streamline property)} \quad (2.3).$$

Unlike the total differential coefficients, the partial differential expression (2.3) does not equal $\tan\Theta$. Should it equal this quantity, then far downstream the product of V_j and r_j would have to be infinite. For conditions in the jet-stream, this product has a finite value which may be anticipated by writing the total differential equation :

$$V\, r \left[\frac{dr}{dx} - F\right] = C2 \qquad (2.4)$$

instead of equation (2.3) . The quantity F varies from streamline to streamline, but it is not a function of x nor of r during passage along a streamline (because for this to be true the variable part would be included within the differential itself). Comparing equations (2.3) and (2.4) :

$$\frac{dr}{dx} = \left[\frac{\partial r}{\partial x}\right]_{sl} + F$$

The total differential comprises of two terms where the second term F adopts a particular value for each streamline. Taking the average flow properties, from equation (1.5) :

$V\,(r^2 - r_c^2) = C1$ (a streamline property)

and $V\, r = C1\, r/(r^2 - r_c^2)$ (2.5).

Equating the equations (2.4) and (2.5) :

$$\frac{dr}{dx} - F = \frac{C2}{V\, r} = \frac{C2}{C1\, r}\,(r^2 - r_c^2) \qquad (2.6).$$

Where the jet-stream has become fully developed, the slope $(dr/dx)_j = 0$ then :

$$- F = \frac{C2}{C1\, r_j}\,(r_j^2 - r_c^2) \qquad (2.7)$$

and applying this to equation (2.6) produces :

$$\frac{dr}{dx} + \frac{C2}{C1\, r_j}\,(r_j^2 - r_c^2) = \frac{C2}{C1\, r}\,(r^2 - r_c^2)$$

or

$$\frac{dr}{dx} = \frac{C2}{C1}\left[\frac{(r^2 - r_c^2)}{r} - \frac{(r_j^2 - r_c^2)}{r_j}\right] \qquad (2.8).$$

The bracketed expression on the R.H.S. can be written as :

$$= (r^2\, r_j - r_c^2\, r_j - r_j^2\, r + r_c^2\, r)/(r\, r_j)$$

$$= (r\, r_j\,(r - r_j) + r_c^2(r - r_j))/(r\, r_j)$$

$$= (r - r_j)(r\, r_j + r_c^2)/(r\, r_j)$$

and the variables of equation (2.8) can be separated giving :

$$\frac{r\, r_j\, dr}{(r - r_j)(r\, r_j + r_c^2)} = \frac{C2}{C1}\, dx \qquad (2.8a).$$

When this is integrated it yields:

$$\frac{- r_j}{r_j^2 + r_c^2}\left[- r_j \log(r - r_j) - \frac{r_c^2}{r_j}\log(r_j\, r + r_c^2)\right] = \frac{C2}{C1}\, x + D$$

where D is the constant of integration.

At $x = 0$, $r = r_m$, and D may be found by this substitution and consequently :

$$\frac{r_j^2}{r_j^2 + r_c^2}\left[\log\frac{r - r_j}{r_m - r_j} + \left[\frac{r_c}{r_j}\right]^2 \log\frac{r\, r_j + r_c^2}{r_m\, r_j + r_c^2}\right] = \frac{C2}{C1}\, x \qquad (2.9).$$

(Far down-stream, where the slipstream has fully developed: $x = \infty$, $r = r_j$ and $\log 0 = -\infty$, which agrees when it is substituted into the above equation, C2 actually being a negative quantity, see below.)

Re-arranging equation (2.8a) and using equation (a.1) from the appendix, at the disk (where $x = 0$ $r = r_m$ and $\Theta = \Theta_m$) gives :

$$\frac{r_m\, r_j \tan\Theta_m}{(r_m - r_j)(r_m\, r_j + r_c^2)} = \frac{C2}{C1} \qquad (2.10).$$

Hence after substituting back, equation (2.9) becomes :

$$\frac{r_j^2}{r_j^2 + r_c^2}\left[\log\frac{r - r_j}{r_m - r_j} + \left[\frac{r_c}{r_j}\right]^2 \log\frac{r\, r_j + r_c^2}{r_m\, r_j + r_c^2}\right] = \frac{r_m\, r_j \tan\Theta_m}{(r_m - r_j)(r_m\, r_j + r_c^2)}\, x$$

and :

$$\left[\log\frac{r - r_j}{r_m - r_j} + \left[\frac{r_c}{r_j}\right]^2 \log\frac{r\, r_j + r_c^2}{r_m\, r_j + r_c^2}\right] = \frac{(x/r_m)(1 + (r_c/r_j)^2)\tan\Theta_m}{(1 - r_j/r_m)(1 + r_c^2/(r_m\, r_j))} \qquad (2.11).$$

When the core does not exist $r_c = 0$ and this may be simplified to give :

$$\frac{r}{r_m} = \frac{r_j}{r_m} + \left[1 - \frac{r_j}{r_m}\right]\exp\left[\frac{\tan\Theta_m}{1 - r_j/r_m}\,\frac{x}{r_m}\right] \qquad (2.12).$$

There is a striking similarity between this formula and the proposal from the measurements in Reference [13] . Unlike the Mean Velocity Theory, the core radius directly affects the shape of the streamlines in equation (2.11).

The angle Θ may be found from equations (2.8a) (2.10) and (a.1) :

$$\tan\Theta = \frac{r - r_j}{r_m - r_j}\,\frac{r_m}{r}\,\frac{r\, r_j + r_c^2}{r_m\, r_j + r_c^2}\tan\Theta_m \qquad (2.13)$$

which simplifies when $r_c = 0$ to produce :

$$\tan\Theta = \frac{r - r_j}{r_m - r_j}\tan\Theta_m \qquad (2.13a).$$

Using equations (2.7) and (2.10) :

$$- F = \frac{C2}{C1\, r_j}\,(r_j^2 - r_c^2)$$

or

$$- F = \frac{r_m \tan\Theta_m\,(r_j^2 - r_c^2)}{(r_m - r_j)(r_m\, r_j + r_c^2)} \qquad (2.14)$$

which simplifies when $r_c = 0$ to produce :

$$- F = \frac{r_j/r_m}{1 - r_j/r_m}\tan\Theta_m \qquad (2.14a).$$

If we introduce into equation (2.14a) the theoretical values on the boundary of the slip-stream - numbers that were derived from the Mean Velocity Theory, a value of F_{bs} will be found for the stationary disk. Then :

$$F_{bs} = \frac{-r_{jb}/r_{mb}}{1 - r_{jb}/r_{mb}} \tan \Theta_{mbs} = \frac{0{\cdot}7598}{1 - 0{\cdot}7598} \sqrt{2} = 4{\cdot}4743 \qquad (2.15).$$

The values of F that are significant for flow inside the slipstream and/or for when the free-stream velocity is not zero (non-stationary disk), will vary between this number and zero.

Now $C1 = V_m (r_m^2 - r_c^2) = V_j (r_j^2 - r_c^2)$ (2.16)

equation (2.10) therefore results in :

$$C2 = \frac{r_m r_j V_m (r_m^2 - r_c^2)}{(r_m - r_j)(r_m r_j + r_c^2)} \tan \Theta_m$$

$$\text{or } C2 = \frac{r_m r_j V_j (r_j^2 - r_c^2)}{(r_m - r_j)(r_m r_j + r_c^2)} \tan \Theta_m \qquad (2.17).$$

This expression can also be written as :

$$C2 = \frac{V_j r_j^2 \{1 - (r_c/r_j)^2\}}{(r_m - r_j)\{1 + r_c^2/(r_m r_j)\}} \tan \Theta_m \qquad (2.17a).$$

where the curly brackets may be regarded as containing modifications of the basic expression due to the effects of the core.

2.3 Hybrid Theory

In this analysis, a solution is is obtained by combining parts of the two previous theories. From the Mean Velocity Theory, equation (1.9c) when substituted into equation (1.9a) gives :

$$V \tan \Theta = K (V_j - V)$$

$$\text{or } V (\tan \Theta + K) = K V_j \qquad (3.1).$$

From the Approximate Axial Flow Theory, equation (2.4) with the substitution of equations (2.17a) and (a.1) from the appendix, results in :

$$V r (\tan \Theta - F) = C2 = \frac{V_j (r_j^2 - r_c^2) \tan \Theta_m}{(r_m - r_j)(1 + r_c^2/(r_m r_j))} \qquad (3.2).$$

From equation (1.10a) :

$$\tan \Theta_m (r_j^2 - r_c^2)/(r_m - r_j) = K (r_m + r_j)$$

and with : $f = (1 + r_c^2/(r_m r_j))$ (3.2a),

equation (3.2) then becomes :

$$V r (\tan \Theta - F) = K V_j (r_m + r_j)/f \qquad (3.3).$$

For the Hybrid Theory, the expression $K V_j/V$ is eliminated from equations (3.1) and (3.3) which are then combined to produce :

$$(\tan \Theta + K)(r_m + r_j) = (\tan \Theta - F) r f \qquad (3.4).$$

This can also be written as :

$$\frac{1}{\tan \Theta} = \frac{r f - (r_m + r_j)}{F r f + K (r_m + r_j)} \qquad (3.5).$$

The value of K can be found from equation (3.4) and taking the conditions in the fully developed slipstream, $x = \infty$, $r = r_j$ and $\Theta = 0$ provides :

$$K (r_m + r_j) = - F r_j f \qquad (3.4a).$$

Substitute for $K (r_m + r_j)$ in equation (3.5) to give :

$$\frac{1}{\tan \Theta} = \frac{dx}{dr} = \frac{r f - r_m - r_j}{F f (r - r_j)} = \left[1 - \frac{r_m + r_j (1 - f)}{(r - r_j)}\right] \frac{1}{F} \qquad (3.6).$$

Now multiply each side by dr and integrate to obtain :

$$x + E = (r - (r_m + r_j(1 - f))\log[r - r_j])/F \qquad (3.7)$$

where E is the constant of integration.

At the disk $x = 0$ and $r = r_m$, hence :

$$E = (r_m - (r_m + r_j (1 - f))\log[r_m - r_j])/F$$

Substituting this back into equation (3.7) gives :

$$x = \left[r - r_m - r_m\left[1 + (r_j/r_m)(1 - f)\right]\log\frac{r - r_j}{r_m - r_j}\right]\frac{1}{F} \qquad (3.8).$$

From equation (3.4) :

$$F = - K (r_m + r_j)/(r_j f) \quad ,$$

and substituting from equation (1.10a):

$$F = - \tan \Theta_m (r_j^2 - r_c^2)/((r_m - r_j) f r_j)$$

$$\text{or } F = \frac{-(r_j/r_m) \tan \Theta_m}{(1 - r_j/r_m)} \frac{(1 - (r_c/r_j)^2)}{f} \qquad (3.9).$$

This expression for F is now substituted into equation (3.8) to finally produce :

$$x/r_m = \left[\frac{r}{r_m} - 1 - \left[1 + \frac{r_j}{r_m} (1 - f)\right]\log\frac{r - r_j}{r_m - r_j}\right] \frac{(1 - r_m/r_j) f}{(1 - (r_c/r_j)^2) \tan \Theta_m} \qquad (3.10).$$

When the effect of the core is negligible this reduces to :

$$\frac{x}{r_m} = \left[\frac{r}{r_m} - 1 - \log\frac{r - r_j}{r_m - r_j}\right]\frac{(1 - r_m/r_j)}{\tan \Theta_m} \qquad (3.10a).$$

3. Discussion - Comparison Between the Theories.

It is of interest to compare the results of the three theories for the streamlines, expressed by previously derived algebraic formulae. The graphs that derive from numerical analyses, are provided in this section to illustrate the shapes of the various functions. A comparison is also given with data from experiment - using the real flows behind a rotor having a finite number of blades. Although the blades actually impart unsteady motion to the slipstream, flow data provided by the helical wake of one blade, will be used for purposes of comparison to the momentum theories.

Three aspects of the streamlines are examined here. Firstly, we consider the slopes of the bounding flow of the slipstream for a stationary actuator-disk. This is of particular concern at the disk where there is a singularity in the first theory, as noted in the discussion that followed equation (1.16).

The second subject of interest is the shape or form of the streamlines themselves. Where families of such curves exist it is sufficient to look at the results for one of them. Since the greatest changes occur at the boundary of the slipstream, it is here that our attention to these theories will be directed.

Thirdly the effect of the core, which is an expression of the non-uniformity of the velocity distribution across the disk, is to be examined.

3.1 Relationship Between the Flow Angle and the Contraction Ratio

Before considering variations with distance downstream, it is useful to examine the connection between the flow angle at the edge of the stationary disc and the contraction ratio of the slipstream. Only the first theory provides an expression for this relationship. The other two theories are without any additional constraint of this kind.

For the Mean Velocity Theory, the angle of flow at the boundary of the disk is related to the contraction ratio of the fully developed slipstream. Equation (1.21) applies at its boundary, as previously found. This approximation is not very far from the actual situation.

Then :
$$\left[\frac{(r_{jb}/r_{mb})^2 - (r_c/r_{mb})^2}{1 - (r_c/r_{mb})^2}\right] = \cos \Theta_{mb}$$

which is plotted in Fig.6.

Two values of the core radius ratio r_c/r_{mb} of zero and 0·25 are shown on this graph.

If we use the slipstream conditions occurring at the stationary disk, that were obtained from equations (1.24) and (1.25) , then Fig.6 also shows the ratio of the radii and the flow angle at the disk to be :

$$r_{jb}/r_{mb} = 0{\cdot}7598 \quad ; \quad \Theta_{mbs} = -54{\cdot}74 \text{ degrees.}$$

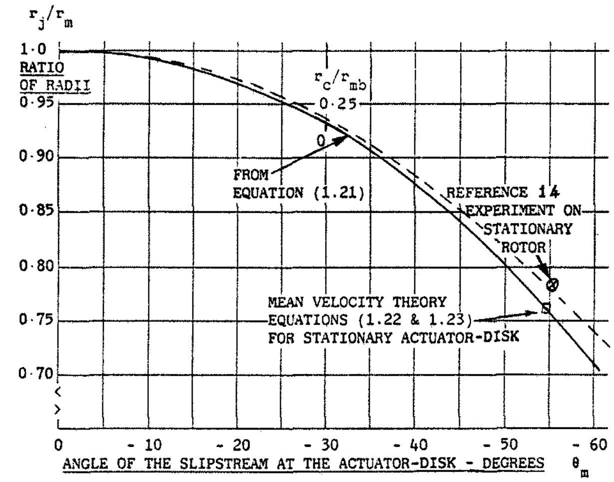

FIGURE 6. RELATIONSHIP BETWEEN THE RADIUS OF THE FULLY-DEVELOPED SLIPSTREAM AND FLOW ANGLE AT THE DISK

This point lays very close to the values :

$$r_{jb}/r_{mb} = 0{\cdot}785 \; ; \quad \Theta_{mbs} = -55{\cdot}4 \pm 0{\cdot}8 \text{ degrees,}$$

which were from measurements of the helical tip vortex leaving one of the three blades of an XV-15 rotor for the V-22 multi-mission tilt-rotor aircraft, see Reference [14]. These data are included in the figure, for comparison.

The value of the core radius ratio of $r_c/r_{mb} = 0{\cdot}25$ is suggested after a comparison of the measurements with the results in Table 2. (This is why the quarter size core was selected for use in Fig.6). This value will also be adopted in the subsequent figures.

3.2 The Slopes of the Streamlines Behind a Stationary Actuator-Disc Θ_s

For the Mean Velocity Theory, the slope of the boundary streamline may now be determined. Equation (1.10) is used for the relationship between the flow angle Θ_{bs} and the slipstream contraction ratio r_b/r_{mb} . The the theoretical value of the flow angle at the edge of the disk Θ_{mbs} is first introduced. The resulting curves are plotted in Figs.7a and 7b. These results for zero core and for one having a quarter of the diameter of the disk, will be compared with those from the other two theories.

For the Approximate Axial Flow Theory, a non-dimensional form of equations (2.13a) or (2.13) are used (after dividing the radii by r_m).

When the core is zero, the ratio $r_{jb}/r_{mb} = 0{\cdot}7598$ and $\Theta_{mbs} = -54{\cdot}74$ degrees are used. For the quarter sized core the ratio of radii here is 0·7840 and the angle of flow at the disk equals - 53·91 degrees, as given in Table 2. This enables the relationships between the radius ratio and the streamline slope to be determined and the resulting curves are plotted on Figs.7a and 7b.

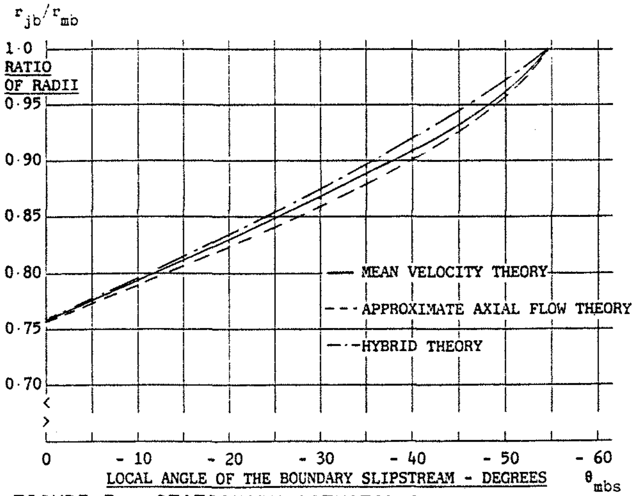

FIGURE 7a. STATIONARY ACTUATOR-DISK - RELATIONSHIP BETWEEN THE RADIUS OF THE SLIPSTREAM BOUNDARY AND LOCAL FLOW ANGLE, WITHOUT A CORE ($r_C = 0$)

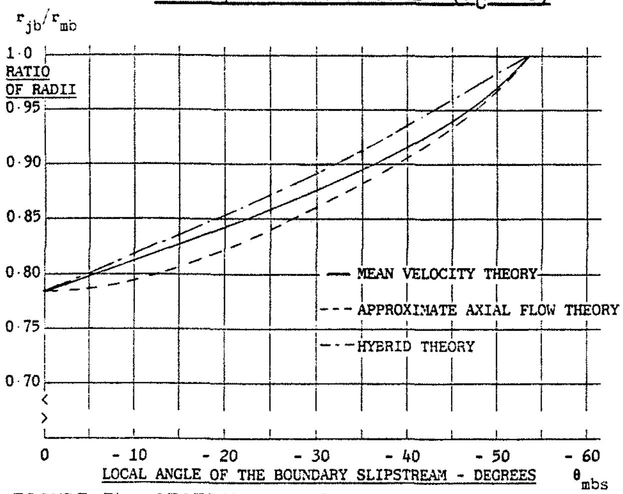

FIGURE 7b. STATIONARY ACTUATOR-DISK - RELATIONSHIP BETWEEN THE RADIUS OF THE SLIPSTREAM BOUNDARY AND LOCAL FLOW ANGLE, WITH A CORE ($r_C = 0{\cdot}25$)

For the Hybrid Theory, the angle of the flow is given by the reciprocal of equation (3.5) using $K = \tan \Theta_{mbs}$. A non-dimensional form of this equation is used and the relationship between the radius ratio and the streamline slope is directly obtained. For zero core the value of F is found from equation (3.9), where $f = 1{\cdot}0$ and $r_c = 0$. Θ_{mbs} and r_{jb}/r_{mb} are taken as above.

For the quarter sized core, equation (3.5) is used as before but with the revised values of radius ratio, flow angle and with $f = 1{\cdot}0797$. The curves are plotted on Figs.7a and 7b.

3.3 The Shape of the Slipstream Boundary

For the Mean Velocity Theory the shape is given by equation (1.14). When the value of r_{jb}/r_{mb} is introduced, this equation becomes :

$$r_b/r_{mb} = 0{\cdot}7598 \coth[1{\cdot}4142\ (x/r_{mb})\ 0{\cdot}7598/(1 - 0{\cdot}7598^2) + \tanh^{-1}(0{\cdot}7598)]$$

$$r_b/r_{mb} = 0{\cdot}7598 \coth[2{\cdot}5425\ (x/r_{mb}) + 0{\cdot}9958] \quad (4.1).$$

This expression is plotted on Fig.8a. Here the core effect is expressed by modifications to the angle of flow at the disc Θ_{mbs} and to the contraction ratio r_{jb}/r_{mb}. Taking the values of these quantities for the quarter sized core (from Table 2), the above expression is slightly changed, the results being plotted in Fig.8b.

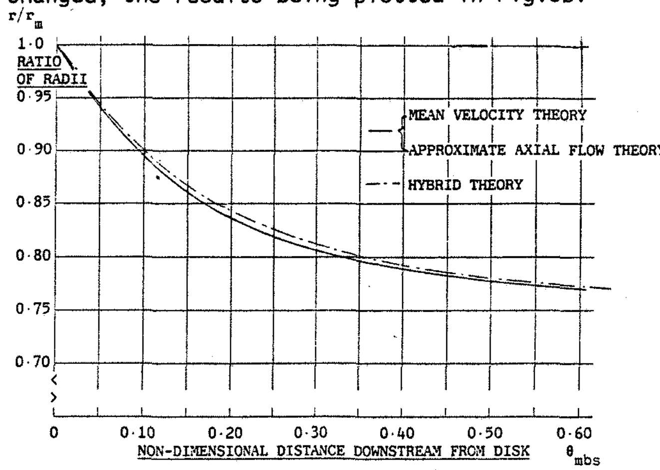

FIGURE 8a. STATIONARY ACTUATOR-DISK - RELATIONSHIP BETWEEN THE RADIUS OF THE SLIPSTREAM AND DISTANCE DOWNSTREAM OF THE DISK, WITHOUT A CORE ($r_C = 0$)

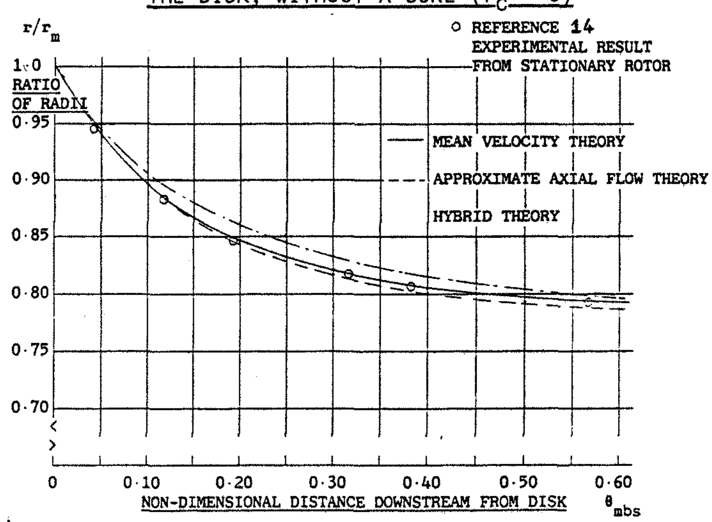

FIGURE 8b. STATIONARY ACTUATOR-DISK - RELATIONSHIP BETWEEN THE RADIUS OF THE SLIPSTREAM AND DISTANCE DOWNSTREAM OF THE DISK, WITH A CORE ($r_C = 0{\cdot}25\ r_{mb}$)

For the Approximate Axial Flow Theory the shape is given by equations (2.11) and (2.12), the latter applying when the core is neglected. Using this expression, when the values of Θ_{mbs} and r_{jb}/r_{mb} are introduced the equation becomes :

$$r_b/r_{mb} = (r_{jb}/r_{mb}) + (1 - r_{jb}/r_{mb}) \exp\left[\frac{-1{\cdot}414}{1 - r_{jb}/r_{mb}}\ (x/r_{mb})\right]$$

$$= 0{\cdot}7598 + (1 - 0{\cdot}7598) \exp\left[\frac{-1{\cdot}414}{1 - 0{\cdot}7598}\ (x/r_{mb})\right]$$

or $r_b/r_{mb} = 0{\cdot}7598 + 0{\cdot}2402 \exp[-5{\cdot}8874\,(x/r_{mb})]$ (4.2).

This expression is also plotted on Fig.8a.

For the case when the core $r_c/r_{mb} = 0{\cdot}25$, equation (2.11) is applicable with suitable values of Θ_{mbs} and r_{jb}/r_{mb} introduced from Table 2, as before. The resulting curve is also plotted in Fig.8b.

For the Hybrid Theory, the shape is given by equations (3.10) and (3.10a). After taking the non-dimensional forms for the radii and when the values of Θ_{mbs} and r_{jb}/r_{mb} are introduced for no core effect, equation (3.10a) becomes :

$$\frac{x}{r_m} = \left[r/r_m - 1 - \log\frac{r/r_m - 0{\cdot}7598}{1 - 0{\cdot}7598}\right]\frac{1 - 1/0{\cdot}7598}{-\sqrt{2}}$$

$$\text{or}\quad x/r_m = 0{\cdot}2235\left[r/r_m - 1 - \log\frac{r/r_m - 0{\cdot}7598}{0{\cdot}2402}\right] \qquad (4.3).$$

In this approach we assume successive values of the radius ratio r/r_m and calculate the non-dimensional displacement x/r_m. These results are plotted on Fig.8a.

For the case when the core is a quarter of the disk size, equation (3.10) applies and the various factors due to the core must be included and this produces :

$$x/r_m = 0{\cdot}2414\left[r/r_m - 1 - 0{\cdot}9375\log\frac{r/r_m - 0{\cdot}7840}{0{\cdot}216}\right] \qquad (4.3a).$$

Using the same technique as for equation (4.3) this result is plotted in Fig.8b.

3.4 Discussion

It is suggested that the relationship between the angle of flow at the disk Θ_{mb} and the contraction ratio of the slipstream r_{jb}/r_{mb}, that was derived in the Mean Velocity Theory (illustrated in Fig.6), is applicable across the whole slipstream and not only on its boundary. This proposal is based not only on equation (1.19) for the flow at the boundary of the slipstream, but also on equation (a.5) with $(dV/dx)_u$ set equal to zero. For the case of no core both these equations have the same form and they apply at the central and outside radial positions of the flow respectively.

From this theory, it is seen that the streamlines have a similar shape. However their radial positions are not uniformly proportional to the properties on the disk. The contraction ratio of the concentric stream-tubes varies non-linearly with radius. When the core is neglected, the streamlines leave the disk at flow angles which vary according to the cube of the relative radial position and at velocities that vary with the square of this ratio, see equations (1.17) and (1.18). With the core included these relationships are modified, as may be seen in Figs. 4a and 4b.

For a non-stationary disk (V_o is not zero), the flow angle at the disk should be corrected for the velocity ratio V_j/V_o, see equation (1.9c) - the same procedure as described above being applicable. Although for both the Approximate Axial Flow and the Hybrid Theories no such contraction-ratio rule has been found, it is reasonable to suppose that their families of streamlines behave similarly. In the Approximate Axial Theory the nature of the exponential function actually permits linear proportionality (because mathematically all the exponential streamlines here have the same form), but as mentioned above this is not necessarily true.

Figs. 7a & 7b and 8a & 8b show the close similarity between the resulting theoretical curves for the slope of the boundary streamline and its shape as a function of non-dimensional distance downstream x/r_m. The insensitivity of the physics to the range of assumptions used in the three theories, suggests that an exact streamline theory is of little additional value. Even when the slopes are compared in Figs.7a and 7b (where greater differences would be expected, due to the magnification of errors arising from differentiation), there is good agreement.

4. CONCLUSIONS

4.1 Sufficient information is contained by the momentum and continuity relationships of steady perfect-fluid flow, to enable the shapes of the streamlines to be determined by analysis. This has been achieved to good accuracy in the case of an axi-symmetric slipstream behind an unshrouded actuator-disk, by using algebra. The results were obtained directly without the need for iterative computation.

4.2 Three parameters are required to define the shape of a streamline. These non-dimensional quantities are :

a) the flow angle at a specific axial position, which in this case is the actuator-disk,

b) the contraction ratio of the flow contained in the associated volume of revolution about the the asymmetric axis, called the stream-tube, that lays between the disk and a point far downstream and

c) the radius of the core, that consists of a central unaccelerated region of fluid. The associated stream-tube (acting as a boundary) is parallel to the axial or free-stream direction. The core may be of zero size.

4.3 At the outer boundary of the stream-tube, the flow angle (item a) above) has been found from theoretical and experimental methods. The values for a stationary actuator-disk and for an actual rotor agreed very closely. The second parameter also was found to correlate well for a core having a 1/4 of the diameter of the disk.

4.4 When the disk has velocity relative to the free-stream, the angles of the flow may be related to those for the stationary disk by taking the velocity components. This vectorial approach also has hodographic connotations.

4.5 For a stationary actuator-disk, items a) and b) above are related by a condition for no acceleration along the outer boundary of the slipstream. This condition also appears to be applicable on the boundary of the central region or the axis, where however a singularity occurs.

4.6 It was found that the radius of the stream-tubes varies with axial position according to exponential functions that use the base of natural logarithms e . One result obtained from theory was actually the exponential decay function and the other theories provide expressions that resemble this but have more complex forms.

4.7 Three momentum theories that were based on different assumptions, yielded almost the same streamline shapes. The shapes were insensitive to the distribution of velocity across the slip-stream. These properties imply that it would be possible to find with fair accuracy the stream-lines of other situations too, without having to solve precisely the classical equations of potential-flow.

Appendix A - General "Two-Dimensional" Streamline Properties

Using cylindrical-polar coordinates, a typical streamline lays on a single ray-plane of the coordinate system. For the axi-symmetric flow that is of interest here, the rays are parallel to the general flow direction (along the axis). At a point (x,r) in this plane, the arc of a streamline has a local slope $\tan\Theta$, an axial velocity V and a radius of curvature R . By avoiding expressions that are associated with volume flow changes, a "two-dimensional" approach may be used on the plane described above. The following general formulae are applicable to the fluid particle dynamics along any streamline in this axi-symmetric flow.

(i) Streamline Slope, dr/dx

$$dr/dx = \tan\Theta \qquad (a.1).$$

Differentiating with respect to x gives :

$$d^2r/dx^2 = (1/\cos^2\Theta)\,(d\Theta/dx) \qquad (a.2).$$

(ii) Streamline Radius, R

$$R = \left[1 + \left[\frac{dr}{dx}\right]^2\right]^{3/2} \Bigg/ \frac{d^2r}{dx^2}$$

The radius R is a positive quantity because the acceleration towards the center of the arc is in the directions of increasing x and r .

From equations (a.1) and (a.2) :

$$R = (1 + \tan^2\Theta)^{3/2} \Bigg/ \left[\frac{d\Theta}{dx}\,\frac{1}{\cos^2\Theta}\right] = \frac{1}{\cos\Theta}\,\frac{1}{d\Theta/dx}$$

and $1/R = \cos\Theta\, d\Theta/dx$ (a.3).

(iii) Acceleration Component in x Direction, dV/dt

Due to the curvature of the flow, a point on a streamline experiences centripetal acceleration. The component of this acceleration in the axial direction is determined below. For flows of the kind being considered here, this acceleration component is positive. It reduces in size as the flow angle - Θ approaches zero. The difference in the axial acceleration due to curvature is therefore given (after introducing a negative sign) by :

$$\frac{dV}{dt} - \left[\frac{dV}{dt}\right]_u = -\frac{V^2}{R}\,(1 + \tan^2\Theta)\sin\Theta$$

where the suffix u refers to the uniform velocity distribution condition that applies close to and along the axis of the flow.

Then using equation (a.3) :

$$\frac{dV}{dt} - \left[\frac{dV}{dt}\right]_u = -V^2\,(1 + \tan^2\Theta)\sin\Theta\cos\Theta\,\frac{d\Theta}{dx}\,.$$

Now : $\frac{dV}{dx} = \frac{dV}{dt}\,\frac{dt}{dx} = \frac{dV}{dt}\,\frac{1}{V}$

Hence: $$\frac{dV}{dx} = \left[\frac{dV}{dx}\right]_u - V\tan\Theta\,\frac{d\Theta}{dx} \qquad (a.4).$$

Taking equation (1.6a) :

$$\frac{dV}{dx} = \frac{2\,r\,V}{r^2 - r_c^2}\tan\Theta$$

we may substitute in equation (a.4) to obtain :

$$\left[\frac{dV}{dx}\right]_u = -V\tan\Theta\left[\frac{2r}{r^2 - r_c^2} - \frac{d\Theta}{dx}\right] \qquad (a.5).$$

Provided that the core radius r_c is not 0 , this limit as r and $\Theta \to 0$, will be zero. When $r_c = 0$ it might be claimed that $(dV/dx)_u$ does not necessarily = 0. This is examined in section 2.1.

(iv) Free-Jet Streamline, Axial Acceleration, dV_b/dt

The "free-jet" exists when there are stationary external flow conditions outside the slipstream, (i.e. $V_o = 0$). The static pressure on the boundary being constant, means that the speed of the particles that move along this streamline

(V_{jb}) is also constant - zero acceleration being produced in the local flow direction. However, due to the curvature of the streamline, the axial velocity component varies according to :

$$V_b = V_{jb} \cos \Theta_{bs} \qquad (a.6)$$

where the double suffix "bs" applies at the boundary when the disc is stationary. This equation is differentiated with respect to x to find the axial "acceleration" on the boundary :

$$\frac{dV_b}{dx} = -V_{jb} \sin \Theta_{bs} \frac{d\Theta_{bs}}{dx} \qquad (a.7).$$

This is another way of expressing part of equation (a.4) using equation (a.6) with $V/\cos \Theta = V_{jb}$ taken as a constant.

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