NONLINEAR INTERACTION IN A THREE-DIMENSIONAL COMPRESSIBLE BOUNDARY LAYER

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Abstract

A perturbation method for analysis of nonlinear wave interaction in three-dimensional compressible boundary layer is developed. The method is based on a biorthogonal eigenfunction system for three - dimensional compressible boundary layers. It is assumed that characteristic space and time scales of the disturbances are much less than space and time scales of nonlinear development and an averaging technique in intermediate scales may be applied. The method is a generalization of Zelman's results for two-dimensional incompressible boundary layer. As an example a three-wave interaction is considered. A numerical example for so called Tollmien-Schlichting wave interaction shows possibility of amplification for rather broad packets without an exact resonance synchronizm for three-wave interaction.

1. Introduction

Laminar-turbulent transition in boundary layer flow occurs as a result of unstable disturbances amplification. At sufficiently low level of external forcing (free stream turbulence, acoustic, vibrating of the streamlined surface etc) a considerable part of the transition zone may be described by the linear hydrodynamic stability theory. A final stage of the development is a nonlinear one, and consideration of disturbance nonlinear interactions may help in understanding of possible scenarios in the transition to turbulence. The fundamental experiments by Klebanoff et al[1] discovered appearance and growth of three-dimensional structures in transition zone, and the transition was accompanied by occurrence of high frequency spikes in oscilloscope observations. Another picture of transition to turbulence was found by Kachanov et al[2]. In their experiment a primary disturbance was induced in boundary layer by a vibrating ribbon, and an amplification of rather broad packet of disturbances at half frequency of the primary disturbance was found. Experimental investigations[3]–[6] have shown an opportunity of various scenarios on the final stage of laminar-turbulent transition. A discussion of physical features in the nonlinear mechanisms is given in the review by Kachanov[7]. Herbert and Morkovin[8] called the first type of transition as K-breakdown. The second type does not have generally accepted name, and we call it as N-type of transition in accordance with terminology of the review[7]. It was established that for relatively large value of primary disturbance amplitude the K-breakdown mechanism reveals itself. When the primary disturbance amplitude is relatively small, the N-type does play the leading role. In application to a natural boundary layer flow transition it means that there are primary waves that were selected due to their amplification in linear stage; and their amplitude values relative to other background disturbance amplitudes determine possibility of N-type or K-type scenario.

Nonlinear interactions in boundary layer flow have been investigated in many theoretical works. The most part of the models was based on the so called weakly nonlinear stability the-

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ory. The corresponding bibliography may be found in Herbert's review[9]. Among them we would like to point out the papers[10]–[12] that have the crucial role in the wave-resonant scheme of transition[7]. A theoretical opportunity of subharmonic resonance was pointed out by Craik[10]. It was suggested to consider a resonant triade of a two-dimensional Tollmien-Schlichting wave with two subharmonic oblique waves in a two-dimensional incompressible boundary layer. If the frequency of a primary wave is equal to ω_1 and a wave number in the downstream direction is equal to α_1, the oblique resonant waves have frequency $\omega_2 = \omega_1/2$ and a wave number component in downstream direction $\alpha_2 = \alpha_1/2$. The oblique waves have equal values of transversal wave number components $\pm\beta$. In the work by Nayfeh and Bozatly[11] four waves interaction was considered to explain three-dimensionality in K-breakdown. Because in this model the Craik's type of resonance also plays an important role, the theory was called as Craik-Nayfeh-Bozatly model. Zelman and Maslennikova[12] analyzed resonance interactions in a boundary layer flow with averaging method that is adjustable to any kind of wave packet interactions in scope of weakly nonlinear theory. In terms of the method they successfully compared theoretical results with experimental data. Explanation of the method and its application are in the paper[13].

Although theoretical results by Zelman and Maslennikova are in good agreement with experimental data for N-type of breakdown, the first detailed comparison of theoretical and experimental data was obtained by Herbert[14],[15] in terms of the secondary instability model. In this model a composition of a mean flow and a primary wave is considered as a basic flow for secondary instability analysis. There are the following assumptions:
1. Mean flow in the boundary layer is considered as a parallel one; 2. The primary Tollmien-Schlichting wave has locally constant amplitude; 3. Nonlinear self-interaction for secondary disturbance is neglected. The secondary instability model provides a convenient numerical method to analyze a subharmonic resonance. Later the secondary instability method was applied to compressible two-dimensional boundary layers[16]. The secondary instability method as Zelman's method demonstrates a possibility of broad band packet amplification. We consider the secondary instability method as a useful tool to analyze subharmonic resonance interactions, while the Zelman's averaging method is a regular perturbation method, which contains a procedure to take into account next terms of the asymptotical solution; and it allows to analyze other kinds of nonlinear interactions. Moreover, after assumption about scaling in space and time, the secondary instability equations may be reduced to the equation for the subharmonic disturbance in the Zelman's method.

The problem of nonlinear interaction in three-dimensional boundary layer becomes very complicated. Lekoudis[17] found many resonant triades in boundary layer over a swept-back wing, but an interaction of the disturbances was not considered. The first application of a perturbation method to a nonlinear problem in the incompressible tree-dimensional boundary layer was done by El-Hady[18]. It was shown that there are various opportunities for the resonance interaction because in three-dimensional boundary layer different types of disturbances may exist (Tollmien-Schlichting waves, cross-flow instability, Görtler instability and so called vertical vorticity disturbances). The calculations[18] showed a strong amplification of disturbances due to nonlinear triad interactions. Nonlinear interactions in three-dimensional incompressible boundary layers were also considered in scope of secondary instability theory[19],[20].

In summary of the introduction, we would like to point out that the problem of nonlinear interactions in three-dimensional compressible boundary layers has not been considered yet. The goal of the present paper is to formulate a procedure that allows to generalize Zelman's results for 3D compressible boundary layer flow.

2. Biorthogonal eigenfunction system

Success in the perturbation method depends on the appropriate choosing of corresponding linear solution presentation, because in the next order of magnitude, which is associated with nonlinear interactions, functions are calculated with the help of this basic solution. The main tool of amplitude equation evaluating is solvability condition for equations obtained in the next orders of magnitude. Usually there is a non-uniform system of ordinary differential equations and the non-uniform part of the system must be orthogonal in some sense to the solution obtained in the main order of magnitude. Thus, the first step in the generalization of the perturbation method is to formulate the basic system of functions for linear problem and to point out an orthogonality relationship.

A biorthogonal eigenfunction system for spatially growing disturbances in two- dimensional compressible boundary layer was proposed in the paper[21]. A three-dimensional version of the system was published in[22]. The formulation of the eigenfunction system suggests quasi parallel boundary layer flow, because the characteristic scale of disturbances is comparable with boundary layer thickness and much less than scale of non-uniformity of the basic flow.

Let's consider a plane parallel three-dimensional boundary lay flow of a compressible gas. We select as coordinate system: y is the distance along the normal to the streamlined surface; Ox axis is along the velocity vector of the mean flow at the edge of the boundary layer; and Oz axis is along the streamlined surface and normal to Ox axis. Navier-Stokes equations are being written in dimensionless form by using the displacement thickness δ^* calculated with mean velocity profile in the x - direction; and the velocity scale U_p is the velocity at the edge of the boundary layer. We measure the time in units δ^*/U_p, the pressure is referred to $\varrho_e U_p^2$. The temperature, viscosity and density are also measured in units of corresponding quantities outside the boundary layer: T_e, μ_e and ϱ_e.

We define a complex vector-function $\mathbf{A}$ with 16 component: $A_1 = u - x$ velocity component; $A_2 = \partial u/\partial y$; $A_3 = v$ - y - velocity component; $A_4 = P$ – pressure disturbance; $A_5 = T$ – temperature disturbance; $A_6 = \partial T/\partial y$; $A_7 = w$ – z-velocity component; $A_8 = \partial w/\partial y$; $A_9 = \partial A_1/\partial x$; $A_{10} = \partial A_3/\partial x$; $A_{11} = \partial A_5/\partial x$; $A_{12} = \partial A_7/\partial x$; $A_{13} = \partial A_1/\partial z$; $A_{14} = \partial A_3/\partial z$; $A_{15} = \partial A_5/\partial z$; $A_{16} = \partial A_7/\partial z$. Linearized Navier-Stokes equations are being written in the following form

$$H_0\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial}{\partial y}\left(L_0\frac{\partial \mathbf{A}}{\partial x}\right) + L_1\frac{\partial \mathbf{A}}{\partial y} = \quad (2.1)$$
$$H_{10}\mathbf{A} + H_2\frac{\partial \mathbf{A}}{\partial x} + H_3\frac{\partial \mathbf{A}}{\partial z};$$

where H_j and L_j are matrices 16×16. Their non-zero elements are presented in the Appendix A1. The following biorthogonal system $\{\mathbf{A}_{\alpha\beta}, \mathbf{B}_{\alpha\beta}\}$ is introduced.

$$-i\omega H_0\mathbf{A}_{\alpha\beta} + \frac{d}{dy}\left(L_0\frac{d\mathbf{A}_{\alpha\beta}}{dy}\right) + \quad (2.2)$$
$$+L_1\frac{d\mathbf{A}_{\alpha\beta}}{dy} = H_{10}\mathbf{A}_{\alpha\beta} + i\alpha H_2\mathbf{A}_{\alpha\beta} + i\beta H_3\mathbf{A}_{\alpha\beta};$$
$$y = 0: \quad A_{\alpha\beta 1} = A_{\alpha\beta 3} = A_{\alpha\beta 5} = A_{\alpha\beta 7} = 0;$$
$$y \to \infty: \quad |A_{\alpha\beta j}| < \infty \quad (j = 1, ..., 16).$$

$$i\overline{\omega}H_0^*\mathbf{B}_{\alpha\beta} + \frac{d}{dy}\left(L_0^*\frac{d\mathbf{B}_{\alpha\beta}}{dy}\right) - \quad (2.3)$$
$$-L_1^*\frac{d\mathbf{B}_{\alpha\beta}}{dy} = H_{10}^*\mathbf{B}_{\alpha\beta} - i\overline{\alpha}H_2^*\mathbf{B}_{\alpha\beta} - i\overline{\beta}H_3^*\mathbf{B}_{\alpha\beta};$$
$$y = 0: \quad B_{\alpha\beta 2} = B_{\alpha\beta 4} = B_{\alpha\beta 6} = B_{\alpha\beta 8} = 0;$$
$$y \to \infty: \quad |B_{\alpha\beta j}| < \infty \quad (j = 1, ..., 16).$$

In Eqs (2.2), (2.3) and what is following α, β, ω are in general case complex numbers; $*$ denotes adjoint matrix; the bar above indicates complex conjugation. The biorthogonal system has both discreet and continuous spectra. Instability waves exist among the discreet spectrum solutions that tend to zero at $y \to \infty$. The continuous spectrum solutions are limited outside the boundary layer. If α and β are real parameters we deal with the temporal stability theory when disturbances may grow in time. If ω is a real parameter, the disturbances may grow in space. We are considering ω and β real; α may be a complex parameter. For each fixed ω and β we can repeat the procedure of the work[21] to obtain the conclusion about completeness of the introduced system. The following orthogonality relationship is valid

$$< H_2\mathbf{A}_{\alpha\beta}; \mathbf{B}_{\gamma\beta} >= \Delta_{\alpha\gamma}; \quad (2.4)$$
$$< \mathbf{A}, \mathbf{B} >\equiv \lim_{\epsilon \to 0}\sum_{j=1}^{16}\int_0^\infty e^{-\epsilon y}A_j\overline{B}_j\,dy;$$

where $\Delta_{\alpha\gamma} = \delta_{\alpha\gamma}$ is the Kronecker symbol, if one of the numbers belongs to the discreet spectrum; $\Delta_{\alpha\gamma} = \delta(\alpha - \gamma)$ is the delta-function, if both numbers belong to the continuous spectrum. In numerical analysis it is more convenient to calculate $\mathbf{A}_{\alpha\beta}$ and $\mathbf{B}_{\alpha\beta}$ with an arbitrary normalization. Thus, in a general case we have in the Eq. (2.4) a factor that depends on the normalization. Certainly, the final results are independent of the normalization.

Although the proposed form of biorthogonal eigenfunction system is a convenient one for orthogonality analysis, computational procedure may be more embarrassing than in regular stability analysis, where eight ordinary differential equations are considered. Therefore, the next important step in our method is to carry out a correspondence between the solutions of the Eqs. (2.2), (2.3) and the solutions in regular formulation of stability problem in compressible gas.

Linear stability equations in three- dimensional compressible boundary layer are written in the regular form of the system of eight equations[23],[24]

$$\frac{dZ_{\alpha\beta}}{dy} = H_s Z_{\alpha\beta}, \qquad (2.5)$$

where the vector $\mathbf{Z}$ contains only eight components that are the same as the first eight components of the vector $\mathbf{A}_{\alpha\beta}$; H_s is a matrix 8×8, its non-zero elements are presented in the Appendix A2. The boundary conditions for the solution are the following

$$\begin{aligned} y=0: &\quad Z_{\alpha\beta 1} = Z_{\alpha\beta 3} = \qquad (2.5a)\\ &\quad Z_{\alpha\beta 5} = Z_{\alpha\beta 7} = 0;\\ y\to\infty: &\quad |Z_{\alpha\beta j}| < \infty \quad (j=1,...,8). \end{aligned}$$

The adjoint problem may be written in the form

$$-\frac{d\mathbf{Y}_{\alpha\beta}}{dy} = H_s^* \mathbf{Y}_{\alpha\beta}; \qquad (2.6)$$

with the following boundary conditions

$$\begin{aligned} y=0: &\quad Y_{\alpha\beta 2} = Y_{\alpha\beta 4} = \qquad (2.6a)\\ &\quad Y_{\alpha\beta 6} = Y_{\alpha\beta 8} = 0;\\ y\to\infty: &\quad |Y_{\alpha\beta j}| < \infty \quad (j=1,...,8). \end{aligned}$$

The correspondence between vector $\mathbf{Y}_{\alpha\beta}$ and the vector $\mathbf{B}_{\alpha\beta}$ is presented in the Appendix A2. Also there are the following important relationships, which may be established by direct substitution:

$$\begin{aligned} &< H_0\mathbf{A}_{\alpha\beta}, \mathbf{B}_{\alpha\beta} >= \quad i < \frac{\partial H_s}{\partial\omega}\mathbf{Z}_{\alpha\beta}, \mathbf{Y}_{\alpha\beta} >;\\ &< H_2\mathbf{A}_{\alpha\beta}, \mathbf{B}_{\alpha\beta} >= -i < \frac{\partial H_s}{\partial\alpha}\mathbf{Z}_{\alpha\beta}, \mathbf{Y}_{\alpha\beta} >;\\ &< H_3\mathbf{A}_{\alpha\beta}, \mathbf{B}_{\alpha\beta} >= -i < \frac{\partial H_s}{\partial\beta}\mathbf{Z}_{\alpha\beta}, \mathbf{Y}_{\alpha\beta} > . \end{aligned}$$

3. The perturbation method

We write the nonlinear equations for disturbances in the following form

$$\begin{aligned} &H_0\frac{\partial\mathbf{A}}{\partial t} + \frac{\partial}{\partial y}\left(L_0\frac{\partial\mathbf{A}}{\partial x}\right) + L_1\frac{\partial\mathbf{A}}{\partial y} = \qquad (3.1)\\ &H_{10}\mathbf{A} + H_2\frac{\partial\mathbf{A}}{\partial x} + H_3\frac{\partial\mathbf{A}}{\partial z} + \epsilon H_4(\mathbf{A})\mathbf{A}; \end{aligned}$$

where the operator H_4 is associated with nonlinear interactions; ϵ – is the amplitude parameter. We consider the boundary layer flow as a parallel one. Non-parallel effects may be considered also in the scope of the method[13]. We assume that the parameters ω, β are real and the initial data allow to consider disturbances as a sum of narrow packets with wave numbers $\mathbf{k}^{(s)} = (\alpha_r^{(s)}, \beta^{(s)})$ and frequencies $\omega^{(s)}$. We suppose existence $\mathbf{k}_0 = (\alpha_{r0}, \beta_0)$ that $k > k_0 \gg \epsilon$; and the main idea is based on assumption of different scales in space and time: 1. The 'fast' scale in space corresponds to a wavelength $\lambda \le 1/k_0$ and the 'fast' time scale is ω_0^{-1}, where ω_0 is a character value of the disturbance frequency; 2. The scales of nonlinear distortion in space and time have order of magnitude $O(\epsilon^{-1})$ and we can introduce the 'slow' variables $x_1 = \epsilon x; z_1 = \epsilon z; t_1 = \epsilon t$. To apply the Zelman's technique we assume existence of intermediate scales L_* and T_*: $\lambda \ll L_* \ll \lambda\epsilon^{-1}$; $\omega_0^{-1} \ll T_* \ll (\omega_0\epsilon)^{-1}$. More precisely we have to talk about scales X_* and Z_* in $x-$ and $z-$ directions correspondingly, and the following inequalities must be valid:

$$\epsilon^{-1} \gg (\alpha_0 X_*, \beta_0 Z_*, \omega_0 T_*) \gg 1.$$

Due to the introduced scaling the disturbance behavior in a local area is similar to linear development. On the large scales we consider a nonlinear distortion.

Let's present a solution of the Eqs. (3.1) as a sum of narrow wave packets

$$\begin{aligned} &\mathbf{A} = \sum_s \int_{\omega^{(s)}-\Delta\omega}^{\omega^{(s)}+\Delta\omega} d\omega \int_{\beta^{(s)}-\Delta\beta}^{\beta^{(s)}+\Delta\beta} d\beta\, \widetilde{\mathbf{A}^{(s)}}\, e^{i\theta_s} \approx \qquad (3.2)\\ &\sum_s \mathbf{A}_s e^{i\theta_s} = \sum_s\sum_{n=0} \epsilon^n \mathbf{A}_s^{(n)}(y, x_1, z_1, t_1) e^{i\theta_s}; \end{aligned}$$

where

$$\theta_s(x,z,t) = \alpha_r^{(s)}x + \beta^{(s)}z - \omega^{(s)}t.$$

If $\beta^{(s)}, \omega^{(s)}$ are prescribed, the complex eigenvalue $\alpha^{(s)} = \alpha_r^{(s)} + i\alpha_i^{(s)}$ may be found from the linear problem (2.2). The zero order vector-function is written as

$$\mathbf{A}_s^{(0)} = \mathbf{A}_{s0}(y)a_s(x_1, z_1, t_1); \qquad (3.3)$$

where $\mathbf{A}_{s0}(y)$ is a solution of the problem (2.2) with corresponding parameters $\omega^{(s)}, \beta^{(s)}$. Therefore, the nonlinear effects in the zero order are taken into account due to the slow amplitude function a_s and its derivative is expressed as the series

$$\frac{da_s}{dx_1} = -\epsilon^{-1}\alpha_i a_s + \sum_{n=1}^{\infty} \epsilon^{n-1} F_{sn}, \qquad (3.4)$$

where F_{sn} are unknown functions and we assume that $O(\alpha_i) = O(\epsilon)$. After substituting (3.2), (3.3), (3.4) into (3.1) we obtain a non-uniform equation for $\mathbf{A}_s^{(1)}$:

$$\sum_s \left\{ H_0 \mathbf{A}_s^{(1)} i\frac{\partial\theta_s}{\partial t} + L_1 \frac{\partial \mathbf{A}_s^{(1)}}{\partial y} + \right. \qquad (3.5)$$

$$\frac{\partial}{\partial y}\left(L_0 \frac{\partial \mathbf{A}_s^{(1)}}{\partial y}\right) - H_{10}\mathbf{A}_s^{(1)} - H_2\mathbf{A}_s^{(1)} i\frac{\partial\theta_s}{\partial x} -$$

$$\left. - H_3\mathbf{A}_s^{(1)} i\frac{\partial\theta_s}{\partial z} \right\} e^{i\theta_s} =$$

$$\sum_s \left\{ H_2 \mathbf{A}_{s0} F_{s1} + H_3 \mathbf{A}_{s0} \frac{\partial a_s}{\partial z_1} - H_0 \mathbf{A}_{s0} \frac{\partial a_s}{\partial t_1} \right\} +$$

$$H_4 \left[\sum_p \mathbf{A}_{p0} a_p e^{i\theta_p} \right] \left(\sum_q \mathbf{A}_{q0} a_q e^{i\theta_q} \right) + O(\epsilon).$$

To illustrate the method we restrict our consideration with subharmonic type of resonance, when nonlinear effect occurs in the first order of magnitude. Without loss of accuracy we add in the left and in the right parts of the Eq. (3.5) term $H_2\mathbf{A}_s^{(1)}\alpha_i e^{i\theta_s}$ to obtain in the left side the operator of the linear problem (2.2). We multiply (3.5) by $exp(-i\theta_m)$ and integrate with respect x, z, t on intermediate scales X_*, Z_*, T_*:

$$\frac{1}{X_* Z_* T_*} \int_x^{X_*+x} dx \int_z^{Z_*+z} dz \int_t^{T_*+t} dt \left\{ ... \right\} e^{i\theta_m}.$$

Therefore, we obtain the following equation:

$$\left\{ H_0 \mathbf{A}_m^{(1)} i\frac{\partial\theta_m}{\partial t} + L_1 \frac{\partial \mathbf{A}_m^{(1)}}{\partial y} + \right. \qquad (3.6)$$

$$\frac{\partial}{\partial y}\left(L_0 \frac{\partial \mathbf{A}_m^{(1)}}{\partial y}\right) - H_{10}\mathbf{A}_m^{(1)} -$$

$$\left. H_2\mathbf{A}_m^{(1)} \left(i\frac{\partial\theta_s}{\partial x} - \alpha_i \right) - H_3\mathbf{A}_m^{(1)} i\frac{\partial\theta_m}{\partial z} \right\} =$$

$$\left\{ H_2 \mathbf{A}_{m0} F_{m1} + H_3 \mathbf{A}_{m0} \frac{\partial a_m}{\partial z_1} - H_0 \mathbf{A}_{m0} \frac{\partial a_m}{\partial t_1} \right\} +$$

$$\sum_{p,q} \mathbf{G}^{(pq)} h_{m,p+q} a_p a_q + O(\epsilon);$$

$$h_{m,p+q} = \frac{1}{X_* Z_* T_*} \int_x^{X_*+x} dx \times$$

$$\int_z^{Z_*+z} dz \int_t^{T_*+t} dt e^{i\theta_p + i\theta_q - i\theta_m}.$$

The vector $\mathbf{G}^{(pq)}$ originates from the nonlinear terms in the Eq. (3.5). If $\Delta\omega = |\omega_m - \omega_p - \omega_q| \gg \epsilon$, $\Delta k = |k_m - k_p - k_q| \gg \epsilon$, the factor $h_{m,p+q}$ in the Eq. (3.6) equal to zero. The Eq. (3.6) is a non-uniform equation for $\mathbf{A}_m^{(1)}$ and the left part is the same as the equation for the linear problem (2.2). To avoid a singular character of the solution in (3.6) the right side must be orthogonal to the solution $\mathbf{B}_{m0}$ obtained from the Eq. (2.3) with corresponding parameters α, β, ω. Thus, we obtain solvability condition that gives a definition of the unknown function F_{m1}:

$$F_{m1} = -\frac{\omega_\beta}{\omega_\alpha}\frac{\partial a_m}{\partial z_1} - \frac{1}{\omega_\alpha}\frac{\partial a_m}{\partial t_1} - \qquad (3.7)$$

$$\sum_{p,q} \frac{< \mathbf{G}^{(pq)}, \mathbf{B}_{m0} >}{< H_2 \mathbf{A}_{m0}, \mathbf{B}_{m0} >};$$

$$\omega_\alpha = -\frac{< H_2 \mathbf{A}_{m0}, \mathbf{B}_{m0} >}{< H_0 \mathbf{A}_{m0}, \mathbf{B}_{m0} >};$$

$$\omega_\beta = -\frac{< H_3 \mathbf{A}_{m0}, \mathbf{B}_{m0} >}{< H_0 \mathbf{A}_{m0}, \mathbf{B}_{m0} >}.$$

4. Boundary-layer model

In numerical example we consider a boundary layer flow over an infinit span swept wing in the so called local self-similar approach. In this model of the boundary layer the governing equations are reduced to a system of ordinary differential equations with Lees-Dorodnitsyn variables depending on the local pressure gradient and on the Mach number outside the boundary layer. The equations for the two dimensional boundary layer in a compressible gas were considered by *Li and Nagmatsu*[25] (see also the book[26]), and the equations could be generalized for the three dimensional case[27].

The coordinate system used for the boundary layer model is depicted in Fig. 1, where s is the distance from the leading eadge along the surface contour, and z^* is the distance along the leading eadge.

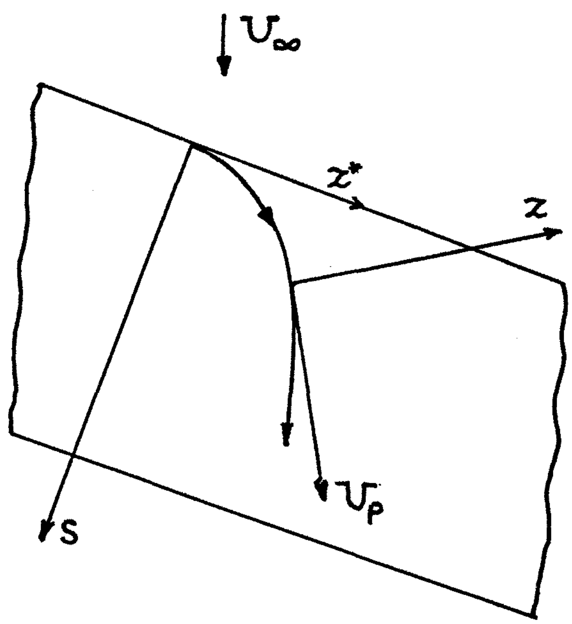

Fig.1 The coordinate system in the analysis over a swept-back wing.

We use the variables

$$\xi = \int_0^s \varrho_e \mu_e U_e \, ds, \qquad (4.1)$$

$$\eta = \frac{\varrho_e U_e}{\sqrt{2\xi}} \int_0^y \frac{\varrho}{\varrho_e} \, dy. \qquad (4.2)$$

where U_e is $s-$ velocity component outside the boundary layer (we remind that U_p is a total velocity outside the boundary layer). The pressure is constant across the boundary layer and the mean flow parameters are assumed to be independent of z^*. Thus, we postulate for stream function, $s-$ and z^*- components velocity, total enthalpy the following presentations:

$$\psi(s,\eta) = \Phi(s) f(\eta); \qquad (4.3)$$

$$U(s,\eta) = U_e(s) f'(\eta); \qquad (4.4)$$

$$W(s,\eta) = W_e(s) q(\eta); \qquad (4.5)$$

$$I(s,\eta) = I_e(s) g(\eta); \qquad (4.6)$$

where a prime denotes a derivative. Due to the assumption on the independence relative to z^*, $W_e = W_\infty = const$. Also we have $I_e(s) = I_\infty = const$, where subscript ∞ denotes free stream parameters. As a result, the following system of ordinary differential equations may be obtained in the local self-similar model:

$$(Cf'')' + ff'' + \left[\frac{\varrho_e}{\varrho} - (f')^2\right]\beta_H = 0; \qquad (4.7)$$

$$(Cq')' + fq' = 0; \qquad (4.8)$$

$$\left(\frac{C}{Pr} g'\right)' + fg' = \qquad (4.9)$$

$$\frac{1 - Pr}{Pr} \frac{(\gamma - 1)M^2}{1 + \frac{(\gamma-1)}{2} M^2} \times [C(f'f'' \cos^2 \Psi_p + qq' \sin^2 \Psi_p)]';$$

$$C = \frac{\varrho \mu}{\varrho_e \mu_e}; \qquad (4.10)$$

$$\beta_H = \frac{2\xi}{U_e} \frac{dU_e}{d\xi}. \qquad (4.11)$$

Ψ_p is the angle between the vector U_p of the flow velocity outside the boundary layer and the $s-$ component velocity U_e; M is the Mach number; Pr is the Prandtl number; γ is the specific heat ratio. In the absence of suction from the boundary layer, the boundary conditions for the Eqs. (4.7) - (4.9) are:

$$\eta = 0: \quad f = f' = q = 0; \qquad (4.12)$$

$$\eta \to \infty: \quad f', q, g \to 1. \qquad (4.13)$$

The boundary condition for the enthalpy depends on heat transfer (cooling or heat insulated wall):

$$g'(0) = g'_w \quad \text{or} \quad g(0) = g_w. \qquad (4.14)$$

The system of Eqs. (4.7) - (4.9) was solved numerically with the fourth-order Runge-Kutta method with a constant step across the boundary layer.

5. Numerical example of nonlinear interaction

We consider three Tollmien-Schlichting waves with amplitudes $\epsilon_1, \epsilon_2, \epsilon_3$ in a boundary layer with the local parameters $\Psi_p = 10°$; $\beta_H = 0.1$; $M = 0.6$; $R = 2600$ that correspond to modelling of boundary layer flow in middle part of an airfoil.

The dimensionless frequency parameter $F = \omega\mu_e/U_p^2$ was chosen for the waves as $F_1 = 15.6 \cdot 10^{-6}$; $F_2 = F_3 = F_1/2$. In this example we use the length scale δ^* based on the $s-$component profile in a boundary layer.

The stability equations (2.5) and (2.6) (direct and adjoint problems) were solved with forth-order Runge-Kutta method; and a procedure of ortonormalization for linear independent solutions was used. The eigenfunctions in (2.5) were normalized with the velocity disturbance amplitude equal to 1.

The first wave was taken with the $z-$component of the wavenumber $\beta_1 = 0$ and the corresponding eigenvalue was found equal to $\alpha_1 = 0.1364 - i0.2577 \cdot 10^{-3}$. The second wave was chosen with $\beta_2 = 0.1$ and the corresponding eigenvalue is equal to $\alpha_2 = 0.6703 \cdot 10^{-1} + i0.4740 \cdot 10^{-2}$. The scale parameters were chosen as $X_* = Z_* = 200$. The scales satisfy to assumptions of the averaging method for disturbances with amplitude $< 0.5\%$. To analyze an opportunity of a wide band amplification we varied the $z-$component of the wave number β_3. At $\beta_3 = -0.1$ we have a synchronizm with accuracy $O(\epsilon)$ ($\alpha_3 = 0.6683 \cdot 10^{-1} + i0.7561 \cdot 10^{-2}$).

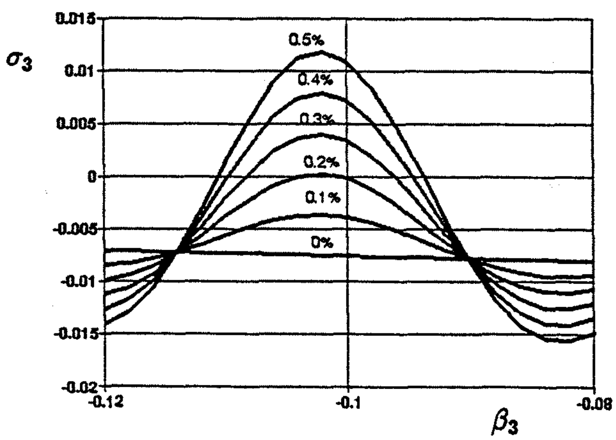

Fig. 2 The growth rate σ_3 vs β_3 for various amplitudes ϵ_1.

The spatial growth rate $\sigma_3 = -Im(\alpha_3) + \epsilon_1 \Delta_3$ was calculated for $\epsilon_2 = \epsilon_3$ with phase shift equal to zero. The nonlinear effect in the growth rate is described by the term with the factor Δ_3. In the Fig. 2 we can see that the nonlinear interaction may cause an amplification of the waves from a wide band of the wave number $z-$component.

The analogous result for σ_2 is shown in Fig. 3.

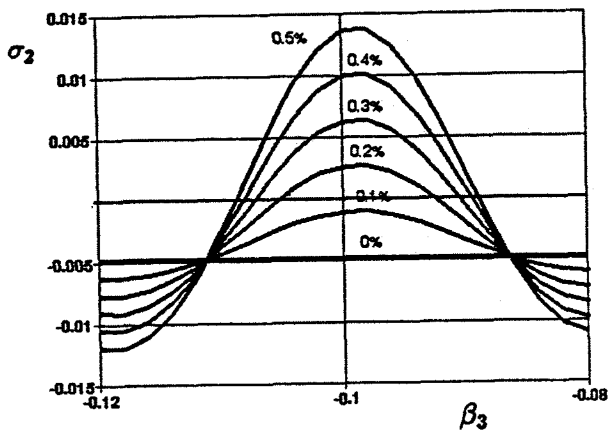

Fig. 3 The growth rate σ_2 vs β_3 for various amplitudes ϵ_1.

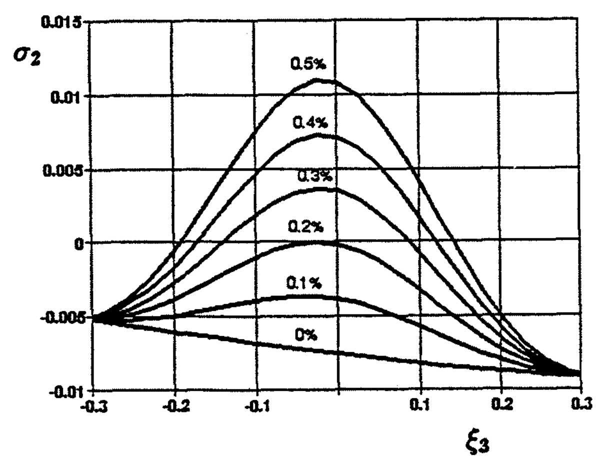

Fig. 4 The growth rate σ_3 vs detunning parameter ξ_3 for various amplitudes ϵ_1.

To illustrate a role of frequency detunning we considered $\beta_3 = -\beta_2$ and varied F_3. The results are shown in Fig. 4. The detunning parameter is defined as $\xi_3 = 1 - 2\omega_3/\omega_1$.
Thus, we can see that a wide band of wave packets may be amplified due to the nonlinear interaction without exact synchronizm.

6. Concluding remark

In the summary we would like to emphasize that the presented method allows to consider two- or three-dimensional flows, compressible or incompressible ones. In case of a supersonic boundary layer there is an opportunity nonlinear interaction of normal modes with external disturbances (for example, with acoustic disturbances); and the proposed procedure also can be applied to this problem. Any difference in problems is reflected in dimension of the system of equations and in explicit form of the matrices.

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Appendix A1

We denote the ratio of the second viscosity to the first one as e and introduce $r = 2(e+2)/3$; $m = 2(e-1)/3$; $D = d/dy$; M – Mach number; R – Reynolds number; Pr – Prandtl number; γ - specific heat ratio. The subscript s denotes a profile of the basic mean flow. $\mu_s' = d\mu_s/dT_s$. H^{ij} denotes an i,j-th element of the matrix H.

$H_0^{21} = -R/(\mu_s T_s)$;
$H_0^{34} = \gamma M^2$; $H_0^{35} = -1/T_s$;
$H_0^{43} = 1/T_s$;
$H_0^{64} = RPr(\gamma-1)M^2/\mu_s$;
$H_0^{65} = -RPr/(\mu_s T_s)$;
$H_0^{87} = -R/(\mu_s T_s)$;
$L_0^{43} = -\mu_s r/R$;
$L_1^{jj} = 1 \quad (j = 1, ..., 8)$;
$L_1^{2,10} = m+1$; $L_1^{8,14} = m+1$.
$H_{10}^{12} = H_{10}^{56} = H_{10}^{78} = 1$;
$H_{10}^{jj} = 1 \quad (j = 9, ..., 16)$;
$H_{10}^{22} = -D\mu_s/\mu_s$; $H_{10}^{23} = RDU_s/(\mu_s T_s)$;
$H_{10}^{25} = -D(\mu_s' DU_s)/\mu_s$; $H_{10}^{26} = -\mu_s' DU_s/\mu_s$;
$H_{10}^{33} = DT_s/T_s$;
$H_{10}^{62} = -2Pr(\gamma-1)M^2 DU_s$;
$H_{10}^{63} = RPrDT_s/(\mu_s T_s)$;
$H_{10}^{65} = -\mu_s' Pr(\gamma-1)M^2[(DU_s)^2 + (DW_s)^2]/\mu_s - D(\mu_s' DT_s)/\mu_s$;
$H_{10}^{66} = -2D\mu_s/\mu_s$; $H_{10}^{68} = -2Pr(\gamma-1)M^2 DW_s$;
$H_{10}^{83} = RDW_s/(\mu_s T_s)$; $H_{10}^{85} = -D(\mu' DW_s)/\mu_s$;
$H_{10}^{86} = -\mu' DW_s/\mu_s$; $H_{10}^{88} = -D\mu_s/\mu_s$;
$H_2^{91} = H_2^{10,3} = H_2^{11,5} = H_2^{12,7} = -1$;
$H_2^{21} = RU_s/(\mu_s T_s)$; $H_2^{23} = -D\mu_s/\mu_s$;
$H_2^{24} = R/\mu_s$; $H_2^{29} = -r$;
$H_2^{31} = -1$; $H_2^{34} = -\gamma M^2 U_s$; $H_2^{35} = U_s/T_s$;
$H_2^{41} = mD\mu_s/R$; $H_2^{42} = (m+1)\mu_s/R$;
$H_2^{43} = -U_s/T_s$; $H_2^{45} = \mu_s' DU_s/R$;
$H_2^{4,10} = \mu_s/R$;
$H_2^{63} = -2Pr(\gamma-1)M^2 DU_s$;
$H_2^{64} = -RPr(\gamma-1)M^2 U_s/\mu_s$;
$H_2^{65} = RPrU_s/(\mu_s T_s)$; $H_2^{6,11} = -1$;
$H_2^{87} = RU_s/(\mu_s T_s)$; $H_2^{8,12} = -1$;
$H_3^{13,1} = H_3^{14,3} = H_3^{15,5} = H_3^{16,7} = -1$;
$H_3^{21} = RW_s/(\mu_s T_s)$; $H_3^{2,12} = -(m+1)$;
$H_3^{2,13} = -1$;
$H_3^{34} = -\gamma M^2 W_s$; $H_3^{35} = W_s/T_s$; $H_3^{37} = -1$;
$H_3^{43} == -W_s/T_s$; $H_3^{45} = \mu_s' DW_s/R$;
$H_3^{47} = mD\mu_s/R$;
$H_3^{48} = (m+1)\mu_s/R$; $H_3^{4,14} = \mu_s/R$;
$H_3^{63} = -2Pr(\gamma-1)M^2 DW_s$;
$H_3^{64} = -RPr(\gamma-1)M^2 W_s/\mu_s$;
$H_3^{65} = RPrW_s/(\mu_s T_s)$; $H_3^{6,15} = -1$;
$H_3^{83} = -D\mu_s/\mu_s$; $H_3^{84} = R/\mu_s$;
$H_3^{87} = RW_s/(\mu_s T_s)$;
$H_3^{89} = -(m+1)$; $H_3^{8,16} = -r$.

Appendix A2

We denote $\hat{\omega} = \omega - \alpha U_s - \beta W_s$; $\chi = (R/\mu_s - ir\gamma M^2\hat{\omega})^{-1}$; $Q = R\chi/\mu_s$. The non-zero elements of the matrix H_s are the following[23]:

$$H_s^{12} = H_s^{56} = H_s^{78} = 1;$$
$$H_s^{21} = \alpha^2 + \beta^2 - i\hat{\omega}R/(\mu_s T_s);$$
$$H_s^{22} = -D\mu_s/\mu_s;$$
$$H_s^{23} = -i\alpha(m+1)DT_s/T_s - i\alpha D\mu_s/\mu_s + RDU_s/(\mu_s T_s);$$
$$H_s^{24} = i\alpha R/\mu_s + (m+1)\gamma M^2\alpha\hat{\omega};$$
$$H_s^{25} = -\alpha(m+1)\hat{\omega}/T_s - D(\mu_s' DU_s)/\mu_s;$$
$$H_s^{26} = -\mu_s' DU_s/\mu_s;$$
$$H_s^{31} = -i\alpha;\ H_s^{33} = DT_s/T_s;\ H_s^{34} = i\gamma M^2\hat{\omega};$$
$$H_s^{35} = -i\hat{\omega}/T_s;\ H_s^{37} = -i\beta;$$
$$H_s^{41} = -i\chi\alpha(rDT_s/T_s + 2D\mu_s/\mu_s);$$
$$H_s^{42} = -i\chi\alpha;$$
$$H_s^{43} = \chi[-\alpha^2 - \beta^2 + i\hat{\omega}R/(\mu_s T_s) + rD^2T_s/T_s + rD\mu_s DT_s/(\mu_s T_s)];$$
$$H_s^{44} = -ir\chi\gamma M^2[\alpha DU_s + \beta DW_s - \hat{\omega}DT_s/T_s - \hat{\omega}D\mu_s/\mu_s];$$
$$H_s^{45} = i\chi[r(\alpha DU_s + \beta DW_s)/T_s + \mu_s'(\alpha DU_s + \beta DW_s)/\mu_s - r\hat{\omega}D\mu_s/(\mu_s T_s)];$$
$$H_s^{46} = -i\chi r\hat{\omega}/T_s;$$
$$H_s^{47} = -ir\chi\beta DT_s/T_s - 2i\chi\beta D\mu_s/\mu_s;$$
$$H_s^{48} = -i\chi\beta;$$
$$H_s^{62} = -2(\gamma-1)M^2 PrDU_s;$$
$$H_s^{63} = -2i(\gamma-1)M^2 Pr(\alpha DU_s + \beta DW_s) + RPrDT_s/(\mu_s T_s);$$
$$H_s^{64} = i(\gamma-1)M^2 PrR\hat{\omega}/\mu_s;$$
$$H_s^{65} = \alpha^2 + \beta^2 - iRPr\hat{\omega}/(\mu_s T_s) - (\gamma-1)M^2 Pr\mu_s'[(DU_s)^2 + (DW_s)^2]/\mu_s - D^2\mu_s/\mu_s;$$
$$H_s^{66} = -2D\mu_s/\mu_s;$$
$$H_s^{68} = -2(\gamma-1)M^2 PrDW_s;$$
$$H_s^{83} = -i(m+1)\beta DT_s/T_s - i\beta D\mu_s/\mu_s + RDW_s/(\mu_s T_s);$$
$$H_s^{84} = (m+1)\gamma M^2\beta\hat{\omega} + i\beta R/\mu_s;$$
$$H_s^{85} = -(m+1)\beta\hat{\omega}/T_s - D(\mu_s' DW_s)/\mu_s;$$
$$H_s^{86} = -\mu_s' DW_s/\mu_s;$$
$$H_s^{87} = \alpha^2 + \beta^2 - i\hat{\omega}R/(\mu_s T_s);$$
$$H_s^{88} = -D\mu_s/\mu_s;$$

The relationships for the vectors **B** and **Y** are the following

$$B_1 = Y_1 - i\overline{\alpha}L_0^{43}\overline{Q}Y_4;$$
$$B_2 = Y_2;$$
$$B_3 = Y_3 + i\overline{\alpha}L_1^{2,10}Y_2 + i\overline{\beta}L_1^{8,14}Y_8 - L_0^{43}\overline{Q}Y_4H_s^{33} + L_0^{43}\frac{d}{dy}(\overline{Q}Y_4);$$
$$B_4 = \overline{Q}Y_4;$$
$$B_5 = Y_5 + \overline{H}_s^{46}Y_4;$$
$$B_6 = Y_6;$$
$$B_7 = Y_7 - i\overline{\beta}L_0^{43}\overline{Q}Y_4;$$
$$B_8 = Y_8;$$
$$B_9 = -i\overline{\alpha}rB_2 - i\overline{\beta}(m+1)B_8;$$
$$B_{10} = -(m+1)\frac{dB_2}{dy} + \frac{i\overline{\alpha}\mu_s}{R}B_4;$$
$$B_{11} = -i\overline{\alpha}B_6;$$
$$B_{12} = -i\overline{\alpha}B_8 - i\overline{\beta}(m+1)B_2;$$
$$B_{13} = -i\overline{\beta}B_2;$$
$$B_{14} = -(m+1)\frac{dB_8}{dy} + \frac{i\overline{\beta}\mu_s}{R}B_4;$$
$$B_{15} = -i\overline{\beta}B_6;$$
$$B_{16} = -i\overline{\beta}rB_8.$$

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