MODELLING OF INTERMITTENT FLOWS WITH THE K-E LOW REYNOLDS NUMBER TURBULENCE MODEL AND CONDITIONED NAVIER-STOKES EQUATIONS

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Abstract

In flows with a high intensity turbulent mean part, laminar boundary layers undergo transition through direct excitation of turbulence. This is called by-pass transition. Regions form that are intermittently laminar and turbulent. By-pass transition is typical for turbomachinery flows. Classical turbulence modelling based on global time averaging is not valid in intermittent flows. To take correctly account of the intermittency, conditioned averages are necessary. These are averages taken during the fraction of time the flow is turbulent or laminar respectively. Starting from the Navier-Stokes equations, conditioned continuity, momentum and energy equations are derived for the laminar and turbulent parts of an intermittent flow. The turbulence is described by the $k-\varepsilon$ model. The supplementary parameter introduced by the conditioned averaging is the intermittency factor. In the calculations, this factor is prescribed in an algebraic way. Results for flat plate test cases are given.

Conditioned averages

We define an intermittency function I(x,y,z,t) with value 1 in a turbulent region and value 0 in a non-turbulent, say laminar, region. The time-averaged value of this function during some time interval T is defined as the intermittency factor

$$\gamma = \frac{1}{T}\int_0^T I(x,y,z,t)dt = \gamma(x,y,z,t).$$

The time interval T is chosen to be large with respect to the time scales of the turbulence, but still small with respect to the time scales of the mean flow.

Nomenclature

$C_f = 2\tau_w/\rho U_\infty^2$	skin friction coefficient.
f_μ, f_2	damping functions.
I	intermittency function.
k	turbulence kinetic energy.
$Re_x = U_\infty x/\nu$	distance Reynolds number.
$Re_\theta = U_\infty \theta/\nu$	momentum thickness Reynolds number.
Tu	turbulence intensity (%).
$\overline{u}$	global time average.
$\overline{u}_t$	average during turbulent state.
u_l	value during laminar state.
u', v'	fluctuating velocity components.
$u_\tau = \sqrt{\tau_w/\rho}$	friction velocity.
$u^+ = \overline{u}/u_\tau$	velocity in wall units.
$y^+ = yu_\tau/\nu$	distance in wall units.
γ	intermittency factor.
δ_1	displacement thickness.
ε	turbulence dissipation.
μ	dynamic viscosity.
ρ	density.
θ	momentum boundary layer thickness.

Sub- and Superscripts

l	laminar state.
t	turbulent state.
tot	sum of mean molecular and Reynolds quantities during turbulent state.
tr	transition.
~	conditioned Favre-averaging.
–	conditioned Reynolds-averaging.

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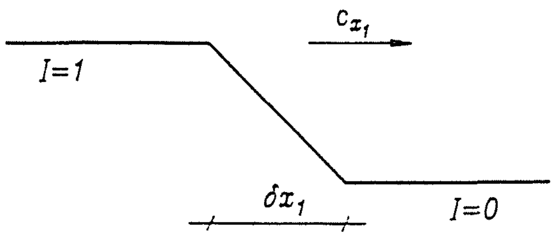

Figure 1: Upgoing front

We calculate first the turbulent conditioned mean value of a quantity for Reynolds-averaging. Afterwards, we verify the results for Favre-averaging. As an example we take the velocity component in x-direction: u. This quantity can be decomposed in mean and fluctuating components by

$$\begin{aligned} u &= \overline{u} + u' & \\ u &= \overline{u}_t + u'_t & \text{for I=1} \\ u &= u_l & \text{for I=0.} \end{aligned}$$

Laminar fluctuations are neglected here. The turbulent mean value and fluctuation satisfy

$$\overline{Iu'_t} = 0 \qquad \overline{Iu} = \gamma \overline{u}_t = \frac{1}{T}\int_0^T I u dt.$$

The laminar mean value and the global mean value satisfy

$$\overline{(1-I)u} = (1-\gamma)u_l \quad \text{and} \quad \overline{u} = \gamma \overline{u}_t + (1-\gamma)u_l.$$

Further, we derive the equations for the conditioned averages of a space or time derivative quantity. The turbulent conditioned average of a space derivative term $\frac{\partial u}{\partial x}$ is defined by $\overline{I\frac{\partial u}{\partial x}}$. During the turbulent phase we decompose by

$$u = \overline{u}_t + u'_t \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial \overline{u}_t}{\partial x} + \frac{\partial u'_t}{\partial x}.$$

We accept that the quantity $\frac{\partial u'_t}{\partial x}$ is uncorrelated, like u'_t, such that the time average during the turbulent phase is zero. As a consequence the contributions of the turbulent phase in the integral defining the mean value is $\gamma\frac{\partial \overline{u}_t}{\partial x}$. Further, there are contributions coming from the fronts between turbulent and laminar zones. Fig. 1 shows schematically the passage of an upgoing front, i.e. a front where the state changes from laminar to turbulent.

During the passage of the front, the space derivative is seen as

$$\frac{\partial u}{\partial x} = \frac{u_l - \overline{u}_t}{\delta x_1} - \frac{u'_t}{\delta x_1},$$

where we accept that the front is spaced over a distance δx_1. We take now as convention that we consider the front passage as part of the turbulent phase. The contribution of an upgoing front to the integral defining the turbulent mean value is given by

$$\frac{1}{T}(u_l - \overline{u}_t)\frac{\delta t_1}{\delta x_1},$$

where δt_1 is the time interval during which the front passes. We consider the mean value of u'_t even during this small time interval as being zero. The quantity $\frac{\delta x_1}{\delta t_1}$ represents the passage velocity c_{x_1} of the upgoing front in x-direction.

Similarly the contribution from a downgoing front is given by

$$\frac{1}{T}(\overline{u}_t - u_l)\frac{\delta t_2}{\delta x_2}.$$

The contribution from one passage of a turbulent zone is

$$\frac{1}{T}(\overline{u}_t - u_l)(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}). \tag{1}$$

By integrating over many passages, a sum of terms of form (1) appears . The interpretation of this sum is straithforward. We consider the definition of the intermittency factor on the position P(x,y,z) and on a position P', an infinitesimal distance δx further in x-direction. When the upgoing front passes at time t_1 at the position P, it passes at time $t_1 + \frac{\delta x}{c_{x_1}}$ at the position P'. Similarly, the downgoing front passes at times t_2 and $t_2 + \frac{\delta x}{c_{x_2}}$. Neglecting higher order variations of c_{x_1} and c_{x_2} during the passage, the fraction of time turbulent flow is seen at point P is given by

$$\gamma = \frac{\Sigma(t_2 - t_1)}{T},$$

while at point P' it is

$$\gamma + \delta\gamma = \frac{\Sigma(t_2 - t_1)}{T} + \frac{\delta x}{T}\Sigma(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}).$$

Hence

$$\frac{\partial \gamma}{\partial x} = \frac{1}{T}\Sigma(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}).$$

This results in the rule for a space derivative

$$\overline{I\frac{\partial u}{\partial x}} = \gamma\frac{\partial \overline{u}_t}{\partial x} + (\overline{u}_t - u_l)\frac{\partial \gamma}{\partial x}. \tag{2}$$

This rule is valid for every other space direction.

The laminar conditioned mean value is simply

$$\overline{(1-I)\frac{\partial u}{\partial x}} = (1-\gamma)\frac{\partial u_l}{\partial x}. \tag{3}$$

The sum of the expressions (2) and (3) gives

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial(\gamma \bar{u}_t + (1-\gamma)u_l)}{\partial x} = \frac{\partial \bar{u}}{\partial x},$$

which, of course, should be the result.

Following a similar reasoning, conditioned mean values for a time derivative quantity can be constructed. We consider $\overline{I\frac{\partial u}{\partial t}}$. The contribution of the turbulent phase to the integral defining the mean value is $\gamma\frac{\partial \bar{u}_t}{\partial t}$. The front contributions are respectively

$$\frac{(\bar{u}_t - u_l)}{\delta t_1} \quad \text{and} \quad \frac{(u_l - \bar{u}_t)}{\delta t_2}.$$

For γ constant in time, there is complete compensation of these two terms. For varying γ, there is a resultant contribution. Over a time interval T, the passage of the upgoing fronts is advanced in the mean by the amount $\frac{1}{2}\frac{\partial \gamma}{\partial t}T$, while the passage of the downgoing fronts is retarded with the same amount. So over a given time T, for γ augmenting in time, more upgoing fronts pass than downgoing fronts. The resultant contribution of the fronts to the integral is

$$(\bar{u}_t - u_l)\frac{\partial \gamma}{\partial t}.$$

So the time derivative rule is like the space derivative rule

$$\overline{I\frac{\partial u}{\partial t}} = \gamma\frac{\partial \bar{u}_t}{\partial t} + (\bar{u}_t - u_l)\frac{\partial \gamma}{\partial t}. \tag{4}$$

Further we need the conditioned average of the product of a quantity and the space derivative of an other quantity $\overline{Ia\frac{\partial b}{\partial x}}$. The contribution of the turbulent phase to the conditioned average is

$$\gamma\left[a_t\frac{\partial b_t}{\partial x} + \overline{a'_t\frac{\partial b'_t}{\partial x}}\right].$$

During the passage of an upgoing front, the term is seen as

$$\frac{a_t + a'_t + a_l}{2}\;\frac{b_l - b_t - b'_t}{\delta x_1}.$$

The contribution to the integral is

$$\left(\frac{a_t + a_l}{2}(b_l - b_t) - \frac{a'_t b'_t}{2}\right)\frac{\delta t_1}{\delta x_1}.$$

Taking into account the contribution from the downgoing front, the final result is

$$\begin{aligned}\overline{Ia\frac{\partial b}{\partial x}} &= \gamma\left[a_t\frac{\partial b_t}{\partial x} + \overline{a'_t\frac{\partial b'_t}{\partial x}}\right]\\ &\quad + \frac{a_t + a_l}{2}(b_t - b_l)\frac{\partial \gamma}{\partial x} + \frac{\overline{a'_t b'_t}}{2}\frac{\partial \gamma}{\partial x}.\end{aligned} \tag{5}$$

Finally, global mean values of quadratic terms like Reynolds stresses $-\overline{(u'v')}$ must be deduced.

For I=1, we have:

$$\begin{aligned} u' = \bar{u} - u &= \bar{u} - \bar{u}_t - u'_t\\ &= \gamma\bar{u}_t - \bar{u}_t + (1-\gamma)u_l - u'_t\\ &= -(1-\gamma)(\bar{u}_t - u_l) - u'_t.\end{aligned}$$

For I=0, we have:

$$\begin{aligned} u' = \bar{u} - u &= \bar{u} - u_l\\ &= \gamma\bar{u}_t + (1-\gamma)u_l - u_l\\ &= \gamma(\bar{u}_t - u_l).\end{aligned}$$

Hence

$$\begin{aligned}(u')^2 &= (1-\gamma)^2(\bar{u}_t - u_l)^2 + (u'_t)^2\\ &\quad + 2(1-\gamma)(\bar{u}_t - u_l)u'_t \quad \text{for } I=1\\ &= \gamma^2(\bar{u}_t - u_l)^2 \quad \text{for } I=0.\end{aligned}$$

$$\begin{aligned}\overline{(u')^2} &= \gamma(1-\gamma)^2(\bar{u}_t - u_l)^2\\ &\quad + \gamma\overline{(u'_t)^2} + (1-\gamma)\gamma^2(\bar{u}_t - u_l)^2\\ &= \gamma\overline{(u'_t)^2} + \gamma(1-\gamma)(\bar{u}_t - u_l)^2.\end{aligned} \tag{6}$$

Similarly

$$\overline{u'v'} = \gamma\overline{u'_t v'_t} + \gamma(1-\gamma)(\bar{u}_t - u_l)(\bar{v}_t - v_l).$$

Conditioned Navier-Stokes equations

It is immediatelly clear that the rules for conditioned mean values and derivatives go over to Favre-averages. We define mean and fluctuating parts of density:

$$\begin{aligned}\rho &= \bar{\rho}_t + \rho'_t \quad \text{for } I=1, \quad \text{where } \overline{I\rho} = \gamma\bar{\rho}_t,\\ \rho &= \rho_l \quad \text{for } I=0.\end{aligned}$$

Hence

$$\bar{\rho} = \gamma\bar{\rho}_t + (1-\gamma)\rho_l.$$

Further, a turbulent Favre-average for velocity is defined by

$$\overline{I\rho u} = \gamma\bar{\rho}_t\,\tilde{u}_t.$$

The global Favre-average follows from

$$\overline{\rho u} = \bar{\rho}\,\tilde{u} = \gamma\bar{\rho}_t\,\tilde{u}_t + (1-\gamma)\rho_l\,u_l.$$

We derive now the conditioned turbulent mean mass equation.

The unaveraged equation is

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0.$$

According to the rules for derivatives, we obtain as turbulent conditioned mean equation

$$\gamma\frac{\partial \overline{\rho}_t}{\partial t}+(\overline{\rho}_t-\rho_l)\frac{\partial \gamma}{\partial t}+\gamma\frac{\partial \overline{\rho}_t\tilde{u}_t}{\partial x}+(\overline{\rho}_t\tilde{u}_t-\rho_l u_l)\frac{\partial \gamma}{\partial x}$$
$$+\gamma\frac{\partial \overline{\rho}_t\tilde{v}_t}{\partial y}+(\overline{\rho}_t\tilde{v}_t-\rho_l v_l)\frac{\partial \gamma}{\partial y}=0.$$

This equation can also be put into conservative form as

$$\frac{\partial \gamma\overline{\rho}_t}{\partial t}+\frac{\partial(\gamma\overline{\rho}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\gamma\overline{\rho}_t\tilde{v}_t)}{\partial y}=$$
$$\rho_l\frac{\partial \gamma}{\partial t}+\rho_l u_l\frac{\partial \gamma}{\partial x}+\rho_l v_l\frac{\partial \gamma}{\partial y}. \tag{7}$$

The laminar equation simply is

$$\frac{\partial \rho_l}{\partial t}+\frac{\partial \rho_l u_l}{\partial x}+\frac{\partial \rho_l v_l}{\partial y}=0. \tag{8}$$

By summing (7) and (8) multiplied by $(1-\gamma)$, we obtain

$$\frac{\partial(\gamma\overline{\rho}_t+(1-\gamma)\rho_l)}{\partial t}+\frac{\partial(\gamma\overline{\rho}_t\tilde{u}_t+(1-\gamma)\rho_l u_l)}{\partial x}$$
$$+\frac{\partial(\gamma\overline{\rho}_t\tilde{v}_t+(1-\gamma)\rho_l v_l)}{\partial y}=0.$$

This equation represents a global mean mass equation.

The conditioned turbulent mass equation used in the calculation is

$$\frac{\partial \overline{\rho}_t}{\partial t}+\frac{\partial \overline{\rho}_t\tilde{u}_t}{\partial x}+\frac{\partial \overline{\rho}_t\tilde{v}_t}{\partial y}=(\rho_l-\overline{\rho}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial t}$$
$$+(\rho_l u_l-\overline{\rho}_t\tilde{u}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}+(\rho_l v_l-\overline{\rho}_t\tilde{v}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial y}. \tag{9}$$

Similarly, the momentum equations can be treated. We write the momentum equations in compact form as

$$\frac{\partial \rho u_i}{\partial t}+\frac{\partial \rho u_i u_j}{\partial x_j}+\frac{\partial p}{\partial x_i}=\frac{\partial \tau_{ij}}{\partial x_j},$$

where the summation convention is used. The terms τ_{ij} denote the molecular stress components. During the turbulent phase, the Favre- and Reynolds-decompositions are

$$\begin{aligned}\rho u_i &= \rho(\tilde{u}_{ti}+u''_{ti})\\ \rho u_i u_j &= \rho(\tilde{u}_{ti}+u''_{ti})(\tilde{u}_{tj}+u''_{tj})\\ p &= \overline{p}_t+p'_t\\ \tau_{ij} &= \overline{\tau_{tij}}+\tau'_{tij}.\end{aligned}$$

The turbulent conditioned equations are

$$\gamma\frac{\partial \overline{\rho}_t\tilde{u}_{ti}}{\partial t}+(\overline{\rho}_t\tilde{u}_{ti}-\rho_l u_{li})\frac{\partial \gamma}{\partial t}+\gamma\frac{\partial \overline{\rho}_t\tilde{u}_{ti}\tilde{u}_{tj}}{\partial x_j}$$
$$+\gamma\frac{\partial \overline{\rho u''_{ti}u''_{tj}}}{\partial x_j}+(\overline{\rho}_t\tilde{u}_{ti}\tilde{u}_{tj}+\overline{\rho u''_{ti}u''_{tj}}-\rho_l u_{li}u_{lj})\frac{\partial \gamma}{\partial x_j}$$
$$+\gamma\frac{\partial \overline{p}_t}{\partial x_i}+(\overline{p}_t-p_l)\frac{\partial \gamma}{\partial x_i}=\gamma\frac{\partial \overline{\tau_{tij}}}{\partial x_j}+(\overline{\tau_{tij}}-\tau_{lij})\frac{\partial \gamma}{\partial x_j}.$$

The usual eddy viscosity modelling approximations are now introduced:

$$\begin{aligned}\overline{\tau}^R_{ij} = -\overline{\rho u''_{ti}u''_{tj}} &= -\frac{2}{3}\overline{\rho}_t\tilde{k}_t\delta_{ij}+2\mu_t\tilde{S}_{tij}\\ \overline{\tau_{tij}} &= 2\tilde{\mu}\tilde{S}_{tij}\\ \overline{\rho}_t\tilde{k}_t &= \frac{1}{2}\overline{\rho u''^{2}_{ti}}\\ \tilde{S}_{tij} &= \frac{1}{2}\left(\frac{\partial \tilde{u}_{ti}}{\partial x_j}+\frac{\partial \tilde{u}_{tj}}{\partial x_i}\right)-\frac{1}{3}\delta_{ij}\frac{\partial \tilde{u}_{tk}}{\partial x_k},\end{aligned}$$

where $\overline{\tau}^R_{ij}$ are the Reynolds stress components, $\tilde{k}_t$ is the turbulence kinetic energy during the turbulent phase, μ_t is the eddy viscosity and $\tilde{S}_{tij}$ is the rate of shear tensor based on Favre-averages during the turbulent phase. We neglect here front contributions to the turbulent mean shear stress.

For instance, the resulting momentum-x equation is

$$\frac{\partial(\overline{\rho}_t\tilde{u}_t)}{\partial t}+\frac{\partial(\overline{\rho}_t\tilde{u}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\overline{\rho}_t\tilde{u}_t\tilde{v}_t)}{\partial y}$$
$$+\frac{\partial(\overline{p}_t+\frac{2}{3}\overline{\rho}_t\tilde{k}_t)}{\partial x}=\frac{\partial(\tilde{\mu}+\mu_t)\tilde{S}_{txx}}{\partial x}+\frac{\partial(\tilde{\mu}+\mu_t)\tilde{S}_{txy}}{\partial y}$$
$$+(\rho_l u_l-\overline{\rho}_t\tilde{u}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial t}+(\rho_l u_l u_l-\overline{\rho}_t\tilde{u}_t\tilde{u}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}$$
$$+(\rho_l u_l v_l-\overline{\rho}_t\tilde{u}_t\tilde{v}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial y}+(p_l-\overline{p}_t-\frac{2}{3}\overline{\rho}_t\tilde{k}_t)\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}$$
$$-\left[\mu_l S_{lxx}-(\tilde{\mu}+\mu_t)\tilde{S}_{txx}\right]\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}$$
$$-\left[\mu_l S_{lxy}-(\tilde{\mu}+\mu_t)\tilde{S}_{txy}\right]\frac{1}{\gamma}\frac{\partial \gamma}{\partial y}. \tag{10}$$

The momentum y-equation is similar. The energy equation can be treated in the same way. The result is as for the other equations an equation which is similar to the global averaged equation supplemented with source terms due to the front passages. The resulting energy equation is

$$\frac{\partial(\overline{\rho}_t\tilde{E}_t)}{\partial t}+\frac{\partial(\overline{\rho}_t\tilde{H}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\overline{\rho}_t\tilde{H}_t\tilde{v}_t)}{\partial y}=$$
$$\frac{\partial(\overline{\tau}^{tot}_{txx}\tilde{u}_t+\overline{\tau}^{tot}_{txy}\tilde{v}_t-\overline{q}^{tot}_{tx})}{\partial x}$$

$$+\frac{\partial(\overline{\tau}_{txy}^{tot}\tilde{u}_t+\overline{\tau}_{tyy}^{tot}\tilde{v}_t-\overline{q}_{ty}^{tot})}{\partial y}$$

$$+(\rho_l E_l-\overline{\rho}_t\tilde{E}_t)\frac{1}{\gamma}\frac{\partial\gamma}{\partial t}+(\rho_l H_l u_l-\overline{\rho}_t\tilde{H}_t\tilde{u}_t)\frac{1}{\gamma}\frac{\partial\gamma}{\partial x}$$

$$+(\rho_l H_l v_l-\overline{\rho}_t\tilde{H}_t\tilde{v}_t)\frac{1}{\gamma}\frac{\partial\gamma}{\partial y}$$

$$-\left[\tau_{lxx}u_l+\tau_{lxy}v_l-\overline{\tau}_{txx}^{tot}\tilde{u}_t-\overline{\tau}_{txy}^{tot}\tilde{v}_t\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x}$$

$$-\left[\tau_{lxy}u_l+\tau_{lyy}v_l-\overline{\tau}_{txy}^{tot}\tilde{u}_t-\overline{\tau}_{tyy}^{tot}\tilde{v}_t\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial y}$$

$$+\left[q_{lx}-\overline{q}_{tx}^{tot}\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x}+\left[q_{ly}-\overline{q}_{ty}^{tot}\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial y}. \quad (11)$$

The mean total energy $\tilde{E}_t$ and mean total enthalpy $\tilde{H}_t$ during the turbulent phase are given by

$$\tilde{E}_t = \tilde{e}_t+\frac{1}{2}(\tilde{u}_t^2+\tilde{v}_t^2)+\tilde{k}_t$$

$$\tilde{H}_t = \tilde{E}_t+\frac{\overline{p}_t}{\overline{\rho}_t}+\frac{2}{3}\tilde{k}_t,$$

where $\tilde{e}_t$ is the mean internal energy. $\overline{\tau}_{tij}^{tot}$ are stress components formed by the sum of the Reynolds stress components and the mean molecular stress components during the turbulent phase. In the same way $\overline{q}_{ti}^{tot}$ are total heat flux components during the turbulent phase.

Conditioned turbulence equations

We derive here the equation for the turbulence kinetic energy during the turbulent phase $\tilde{k}_t$. From a combination of the conditioned mass equation and the conditioned momentum equations, an equation for the mean flow kinetic energy during the turbulent phase can be derived. This equation is

$$\frac{\partial\overline{\rho}_t\frac{1}{2}\tilde{u}_{ti}^2}{\partial t}+\frac{\partial\overline{\rho}_t\frac{1}{2}\tilde{u}_{ti}^2\tilde{u}_{tj}}{\partial x_j}+\tilde{u}_{ti}\frac{\partial\overline{p}_t}{\partial x_i}-\tilde{u}_{ti}\frac{\partial\overline{\tau}_{ij}^{tot}}{\partial x_j}$$

$$=\overline{u}_{ti}B_i-\frac{1}{2}\tilde{u}_{ti}^2A, \quad (12)$$

where A is the source term in the conditioned mass equation (9) and B_i are the source terms in the conditioned momentum equations due to intermittency. The term B_x can be seen in equation (10). The equation (12) is similar to the equation for the global mean flow kinetic energy but differs from this equation by the source terms due to intermittency. From the unaveraged mass equation and the momentum equations, similar to (12) the equation for the unaveraged kinetic energy is found as

$$\frac{\partial\rho\frac{1}{2}u_i^2}{\partial t}+\frac{\partial\rho\frac{1}{2}u_i^2u_j}{\partial x_j}+u_i\frac{\partial p}{\partial x_i}-u_i\frac{\partial\tau_{ij}}{\partial x_j}=0. \quad (13)$$

The conditioned averaged equation during the turbulent phase corresponding to (12) is

$$\frac{\partial(\overline{\rho}_t\frac{1}{2}\tilde{u}_{ti}^2+\overline{\rho}_t\tilde{k}_t)}{\partial t}$$

$$+\frac{\partial(\overline{\rho}_t\frac{1}{2}\overline{u}_{ti}^2\tilde{u}_{tj}+\overline{\rho}_t\tilde{k}_t\tilde{u}_{tj}+\tilde{u}_{ti}\overline{\rho u_{ti}''u_{tj}''}+\frac{1}{2}\overline{\rho u_{ti}''^2u_{tj}''})}{\partial x_j}$$

$$+\tilde{u}_{ti}\frac{\partial\overline{p}_t}{\partial x_i}+\overline{u_i''}\frac{\partial\overline{p}_t}{\partial x_i}+\overline{u_i''\frac{\partial p_t'}{\partial x_i}}$$

$$-\tilde{u}_{ti}\frac{\partial\overline{\tau_{tij}}}{\partial x_j}-\overline{u_{ti}''}\frac{\partial\overline{\tau_{tij}}}{\partial x_j}-\overline{u_{ti}''\frac{\partial\tau_{tij}'}{\partial x_j}}$$

$$=(\frac{1}{2}\rho_l u_{li}^2-\frac{1}{2}\overline{\rho}_t\tilde{u}_{ti}^2-\overline{\rho}_t\tilde{k}_t)\frac{1}{\gamma}\frac{\partial\gamma}{\partial t}$$

$$+(\frac{1}{2}\rho_l u_{li}^2u_{lj}-\frac{1}{2}\overline{\rho}_t\tilde{u}_{ti}^2\tilde{u}_{tj}-\overline{\rho}_t\tilde{k}_t\tilde{u}_{tj}$$

$$-\tilde{u}_{ti}\overline{\rho u_{ti}''u_{tj}''}-\frac{1}{2}\overline{\rho u_{ti}''^2u_{tj}''})\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}$$

$$+\frac{\tilde{u}_{ti}+\overline{u_{ti}''}+u_{li}}{2}(p_l-\overline{p}_t)\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_i}-\frac{\overline{u_{ti}''p_t'}}{2}\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_i}$$

$$-\frac{\tilde{u}_{ti}+\overline{u_{ti}''}+u_{li}}{2}(\tau_{lij}-\overline{\tau_{tij}})\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}+\frac{\overline{u_{ti}''\tau_{tij}'}}{2}\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}. \quad (14)$$

We denote the source term due to intermittency in (14) by C. By combining (12) and (14), the equation for the turbulence kinetic energy during the turbulent phase is found as

$$\frac{\partial\overline{\rho}_t\tilde{k}_t}{\partial t}+\frac{\partial\overline{\rho}_t\tilde{k}_t\tilde{u}_{tj}}{\partial x_j}+\frac{\partial(-\tilde{u}_{ti}\overline{\tau}_{ij}^R+\frac{1}{2}\overline{\rho u_{ti}''^2u_{tj}''})}{\partial x_j}$$

$$+\overline{u_{ti}''}\frac{\partial\overline{p}_t}{\partial x_i}+\overline{u_{ti}''\frac{\partial p_t'}{\partial x_i}}+\tilde{u}_{ti}\frac{\partial\overline{\tau}_{ij}^R}{\partial x_j}-\overline{u_{ti}''}\frac{\partial\overline{\tau_{tij}}}{\partial x_j}$$

$$-\overline{u_{ti}''\frac{\partial\tau_{tij}'}{\partial x_j}}=C-\tilde{u}_{ti}B_i+\frac{1}{2}\tilde{u}_{ti}^2A. \quad (15)$$

The left hand side of this turbulence kinetic energy equation has the same form as the global turbulence kinetic energy equation. We recognize the following terms:

$$\text{Production}: P_k=\overline{\tau}_{ij}^R\frac{\partial\tilde{u}_{ti}}{\partial x_j}$$

$$\text{Diffusion}: \frac{\partial(\overline{u_{ti}''\tau_{tij}'}-\overline{u_{tj}''p_t'}-\frac{1}{2}\overline{\rho u_{ti}''^2u_{tj}''})}{\partial x_j}$$

$$\text{Dissipation}: -\overline{\tau_{tij}'\frac{\partial u_{ti}''}{\partial x_j}}$$

$$\text{Compressibility}: \overline{p'\frac{\partial u_{tj}''}{\partial x_j}}-\overline{u_{ti}''}\frac{\partial\overline{p}_t}{\partial x_i}-\overline{u_{ti}''}\frac{\partial\overline{\tau_{tij}}}{\partial x_j}$$

The source term due to intermittency in the $\tilde{k}_t$ equation can be worked out into:

$$\left[\frac{1}{2}\rho_l(u_{li}-\tilde{u}_{ti})^2-\overline{\rho}_t\tilde{k}_t\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial t}$$

$$+\left[\frac{1}{2}\rho_l(u_{li}-\tilde{u}_{ti})^2u_{lj}-\frac{1}{3}\overline{\rho}_t\,\tilde{k}_t\,\tilde{u}_{tj}-\frac{1}{2}\overline{\rho u_{ti}''^2u_{tj}''}\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}$$

$$+\left[\frac{\overline{u_{ti}''}+u_{li}-\tilde{u}_{ti}}{2}(p_l-\overline{p}_t)-\frac{\overline{u_{ti}''p_t'}}{2}\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_i}$$

$$-\left[\frac{\overline{u_{ti}''}+u_{li}-\tilde{u}_{ti}}{2}(\tau_{lij}-\overline{\tau_{tij}})-\frac{\overline{u_{ti}''\tau_{tij}'}}{2}\right]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}. \quad (16)$$

In each of the four parts in this source term, the components which are grouped into the square brackets compensate more or less each other. We consider as an example the first group of components. In a wall bounded flow, the turbulent mean velocity component in the direction of the wall is larger than the laminar velocity component near the wall, while the reverse is true far away from the wall. So near to the wall, the considered coefficient is positive; further away from the wall where laminar and turbulent velocities are approximately equal, the coefficient is negative; far away from the wall it is again positive. So, the mean influence on the generation of turbulence kinetic energy is very low. The same can be said from the three other parts. The conclusion is that the source term in the $\tilde{k}_t$ equation has only a second order effect in the sense that it can alter the distribution of $\tilde{k}_t$ in a wall bounded flow, but not the mean level. Therefore taking into account the modelling which anyhow has to be done in the left hand side of the equation (15), it seems appropriate to neglect the right hand side. We verified the influence of the source term by bringing in the first and second group of terms, but leaving out $\overline{\rho u_{ti}''^2u_{tj}''}$, into a calculation. These introduced terms are free of any modelling. The influence on the results was completely negligible. The source terms in the Navier-Stokes equation (9)-(11) are much more significant and cannot be deleted.

It is very important to come to the conclusion that the source terms in the turbulence equations can be neglected. For the $\tilde{k}_t$-equation it would be not difficult to introduce models for the terms in (16) that need closure since these terms are linked to the diffusion process. It would however be almost impossible to construct the second equation for turbulent quantities since, for instance, the ϵ-equation has not at all the same rational basis as the k-equation.

We model the turbulence by the classical (low Reynolds number) k- and ε-equations, but written for the turbulent conditioned averaged values. These equations, written in the Yang-Shih variant[1] are

$$\frac{\partial\overline{\rho}_t\tilde{k}_t}{\partial t}+\frac{\partial\overline{\rho}_t\tilde{k}_t\tilde{u}_t}{\partial x}+\frac{\partial\overline{\rho}_t\tilde{k}_t\tilde{v}_t}{\partial y}=$$
$$\frac{\partial}{\partial x}(\tilde{\mu}+\frac{\mu_t}{\sigma_k})\frac{\partial\tilde{k}_t}{\partial x}+\frac{\partial}{\partial y}(\tilde{\mu}+\frac{\mu_t}{\sigma_k})\frac{\partial\tilde{k}_t}{\partial y}+\tilde{P}_k-\overline{\rho}_t\tilde{\varepsilon}_t,$$

$$\frac{\partial\overline{\rho}_t\tilde{\varepsilon}_t}{\partial t}+\frac{\partial\overline{\rho}_t\tilde{\varepsilon}_t\tilde{u}_t}{\partial x}+\frac{\partial\overline{\rho}_t\tilde{\varepsilon}_t\tilde{v}_t}{\partial y}=$$
$$\frac{\partial}{\partial x}(\tilde{\mu}+\frac{\mu_t}{\sigma_\varepsilon})\frac{\partial\tilde{\varepsilon}_t}{\partial x}+\frac{\partial}{\partial y}(\tilde{\mu}+\frac{\mu_t}{\sigma_\varepsilon})\frac{\partial\tilde{\varepsilon}_t}{\partial y}$$
$$+\left[C_{\varepsilon_1}\tilde{P}_k-C_{\varepsilon_2}f_2\overline{\rho}_t\tilde{\varepsilon}_t\right]\frac{1}{\mathcal{T}}+\mathcal{E},$$

where

$$\tilde{P}_k = \left\{\mu_t\left[\frac{\partial\tilde{u}_{ti}}{\partial x_j}+\frac{\partial\tilde{u}_{tj}}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial\tilde{u}_{tk}}{\partial x_k}\right]-\frac{2}{3}\delta_{ij}\overline{\rho}_t\tilde{k}_t\right\}\frac{\partial\tilde{u}_{ti}}{\partial x_j},$$

$$\mu_t = C_\mu f_\mu\overline{\rho}_t\tilde{k}_t\mathcal{T} \qquad \mathcal{T}=\frac{\tilde{k}_t}{\tilde{\varepsilon}_t}+\sqrt{\frac{\mu}{\overline{\rho}_t\tilde{\varepsilon}_t}}$$

$$\mathcal{E} = \frac{\mu\tilde{\mu}_t}{\overline{\rho}_t}\frac{\partial^2\tilde{u}_{ti}}{\partial x_jx_k}\frac{\partial^2\tilde{u}_{ti}}{\partial x_jx_k} \qquad R_y=\frac{\overline{\rho}_t\sqrt{\tilde{k}_t}\,y}{\mu}$$

$$f_\mu = 1-\exp(a_1R_y+a_3R_y^3+a_5R_y^5)$$

$$f_2 = 1-\exp(-\frac{{R_t}^2}{36}) \qquad R_t = \frac{\overline{\rho}_t\tilde{k}_t\mathcal{T}}{\mu}.$$

The following model constants are used : $C_{\varepsilon_1}=1.44$, $C_{\varepsilon_2}=1.92$, $\sigma_k=1$, $\sigma_\epsilon=1.3$, $C_\mu=.09$, $a_1=-1.5\ 10^{-6}$, $a_3=-5.10^{-7}$, $a_5=-1.10^{-10}$. At the wall, $\tilde{k}_t=0$ and $\overline{\rho}_t\tilde{\varepsilon}_{tw}=2\mu\left(\frac{\partial\sqrt{\tilde{k}_t}}{\partial y}\right)^2$ are imposed.

Intermittency modelling

The intermittency γ can be described algebraically according to Dhawan and Narasimha by

$$\gamma(x)=1-\exp\left[-\hat{n}\sigma(Re_x-Re_{xtr})^2\right], \quad (17)$$

where $\hat{n}$ is the nondimensional turbulence spot production rate and σ the turbulent spot propagation parameter. This law is valid for concentrated breakdown at x_{tr}, which is typical for natural transition. The parameter $\hat{n}\sigma$ in (17) has been correlated by Mayle[2] based on intermittency measurements for zero pressure gradient flow as

$$\hat{n}\sigma = 1.25\ 10^{-11}Tu^{\frac{7}{4}}. \quad (18)$$

The position of transition x_{tr} has been correlated by Mayle[2] and Hourmouziadis[3] as

$$Re_{\theta tr} = 420\ Tu^{-.69},$$
$$Re_{\theta tr} = 460\ Tu^{-.65}. \quad (19)$$

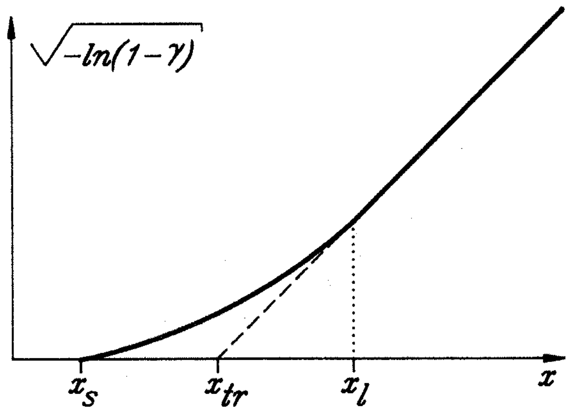

Figure 2: Intermittency in distributed breakdown.

There are slight differences between both correlations. We found the best agreement with the experiments for the Hourmouziadis formula.

According to Mayle[2] and Gostelow *et al.*[4], in by-pass transition a Gaussian distribution of the spot production at the onset of transition is more realistic. To take account of this distributed breakdown, the growth parameter $\hat{n}\sigma$ cannot be seen as constant in the beginning of the transition zone. Fig. (2) shows schematically the evolution of $F(\gamma) = \sqrt{-\ln(1-\gamma)}$ in function of distance for distributed breakdown[4]. In such a diagram a concentrated breakdown would correspond to a linear curve. In actual experiments, the linear growth is obtained at about the level $\gamma = 20\%$. To model the initial behaviour of the growth, we draw a polynomial through the points x_s, x_{tr} and x_l with γ-levels 1%, 2,5% and 20%, with $x_l - x_{tr} = x_{tr} - x_s$ and boundary conditions $\frac{d\gamma}{dx} = 0$ at $x = x_s$ and $\frac{d\gamma}{dx}$ equal to the value of the linear law at $x = x_l$.

To the best of our knowledge, there are no criteria available for start of transition with distributed breakdown. Therefore, in the calculations, we use the Houmouziadis formula (19) but applied to the start of transition x_s instead of x_{tr}. There is clearly a need for more adapted correlations for the start of transition x_s.

Results

The above equations have been used to calculate an intermittent flow on an adiabatic flat plate with no pressure gradient ($\frac{dp}{dx} = 0$). Two test cases with different turbulence levels have been chosen to compare computational results with experimental data. The test cases are described by Savill[5] and are indicated by T3A and T3B. The first test case corresponds with turbulence level $Tu = 3\%$ and freestream velocity $U = 5.4 m/s$. The second one has $Tu = 6\%$ and $U = 9.6 m/s$.

The equations are solved in their steady state form by a relaxation procedure. A vertex-centered finite volume discretization combined with an upwind TVD formulation is used. Full details of the numerical method are given by Steelant and Dick[6]. To obtain stability of the relaxation method, a careful treatment of the source terms is necessary. The negative source terms have to be linearized and put on the left hand side of the equations.

A stretched grid of 385 x 97 points was used. The grid extends upstream of the plate, with the sharp leading edge at station 97. The first grid point in the direction normal to the plate lies at about $y^+ = yu_\tau/\nu = 1$, where u_τ is the friction velocity. Stretching was applied normal to the plate and in the flow direction near the leading edge. A detail of the grid in the leading edge region is shown in figure 3

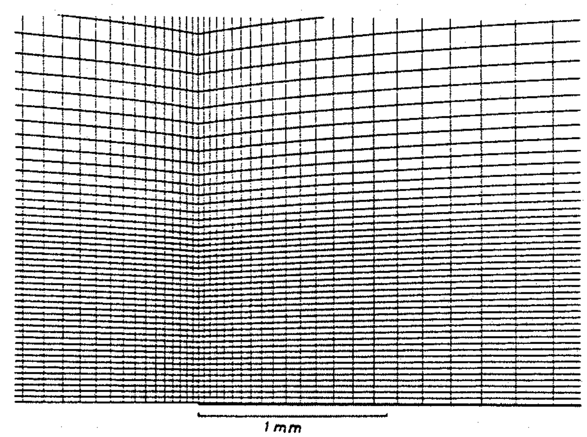

Figure 3: Detail of the grid near the leading edge.

Uniform inlet profiles for total temperature, total pressure, k and ϵ were specified. At inlet, Mach number was extrapolated from the flow field. The values of k and ϵ at the inlet were calculated with the equations for k and ϵ for uniform flow with velocity U:

$$U\frac{\partial k}{\partial x} = -\epsilon,$$
$$U\frac{\partial \epsilon}{\partial x} = -C_{\epsilon_2}\frac{\epsilon^2}{k},$$

where at the leading edge the following values were matched to be in accordance with the experiments, for L = 1m :

$$k = .03(\frac{3}{2}U^2), \quad \frac{\epsilon}{U^3/L} = 2.86\ 10^{-3}, \quad (T3A),$$
$$k = .06(\frac{3}{2}U^2), \quad \frac{\epsilon}{U^3/L} = 1.22\ 10^{-2} \quad (T3B).$$

The upper and right boundaries are outlet boundaries. There, pressure was imposed. Velocity components, temperature and turbulent quantities were extrapolated. The part of the lower boundary upstream of the leading edge was treated as a symmetry line. At the plate, no-slip and adiabatic boundary conditions were imposed. Density and pressure were obtained by characteristic combinations of the equations.

The test cases T3A and T3B are in principal meant to have zero pressure gradient. In the actual experiments there is globally a slight favourable pressure gradient. The evolution of the free stream velocity for the test cases is shown in figure 4. For the T3B case the free stream velocity has a small oscillation. In the calculations, the oscillation has been filtered out (dotted curve in fig. 4). The non-uniform free stream velocity profile is imposed by the corresponding pressure distribution on the upper boundary. In principle, a non-zero pressure gradient influences the spot growth parameter. We did not take into account this effect.

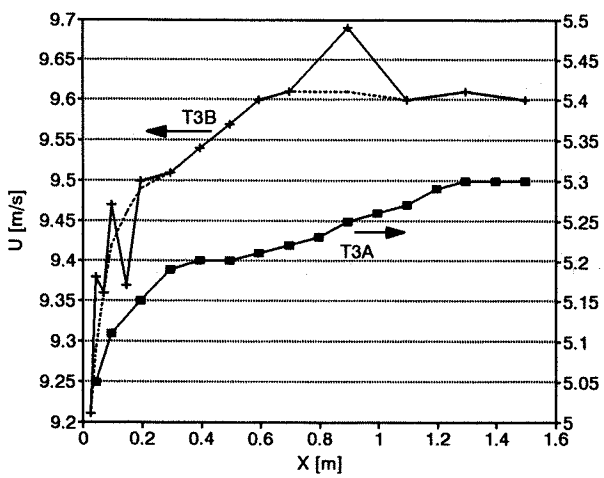

Figure 4: Free stream velocity along the flat plate.

In the turbulent flow equations, only the source due to $\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}$ is used. This term is zero before the start of transition (x_s) and is activated after this point. Before the start of transition the γ-level is taken to be 1%. The term $\frac{1}{\gamma}\frac{\partial \gamma}{\partial x}$ is smooth everywhere, has a maximum between x_s and x_l and tends to zero for large x.

Figure 5 shows the skin friction coefficient in funtion of Re_x, obtained for the T3A case, where the upper and the lower lines represent the laminar and turbulent values. Curve (a) represents the experimental data. Curve (b) is the result obtained with global averaged Navier-Stokes equations and the $k-\varepsilon$ model, without taking into account the intermittency. As is well known, this method gives a too early and too rapid transition. Curve (c) is the result obtained with the present method. The accordance with the experiments is very good. The numerical transition is somewhat faster than given by the experimental data.

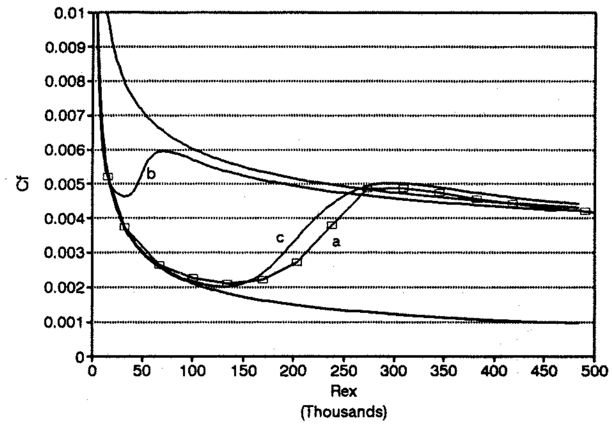

Figure 5: Skin friction coefficient (T3A).

Figure 6 shows the evolution of the profile of the global streamwise velocity fluctuation u′ during transition for different positions along the plate. The global streamwise Reynolds normal stress is, with the usual approximation, given by (6):

$$\widetilde{u'^2} \approx \tilde{k} = \gamma \tilde{k}_t + \frac{1}{2}\gamma(1-\gamma)\left[(\tilde{u}_t - u_l)^2 + (\tilde{v}_t - v_l)^2\right],$$

where $\tilde{k}_t$ is the turbulence kinetic energy during the turbulent phase. Experiments are represented by square boxes. In the beginning of transition the experimental data show already apreciable levels of $u' = \sqrt{\widetilde{u'^2}}$. The numerically predicted level is much lower. This is due to the neglect of the laminar fluctuations in the calculation. The laminar contribution to u′ is important since it is multiplied with $(1-\gamma)$. Further in the transition phase the peak is well represented and corresponds well with the experimental data. Velocity profiles at onset, in the middle and at the end of transition are shown in figure 7. As transition is predicted a bit too fast (fig. 5), the profile in the middle of the transition tends more to a turbulent profile than given by the experiments.

Figure 8 shows the skin friction coefficient in funtion of Re_x for the T3B case. In contrast with the T3A case, the numerical transition is somewhat later than given by the experimental data.

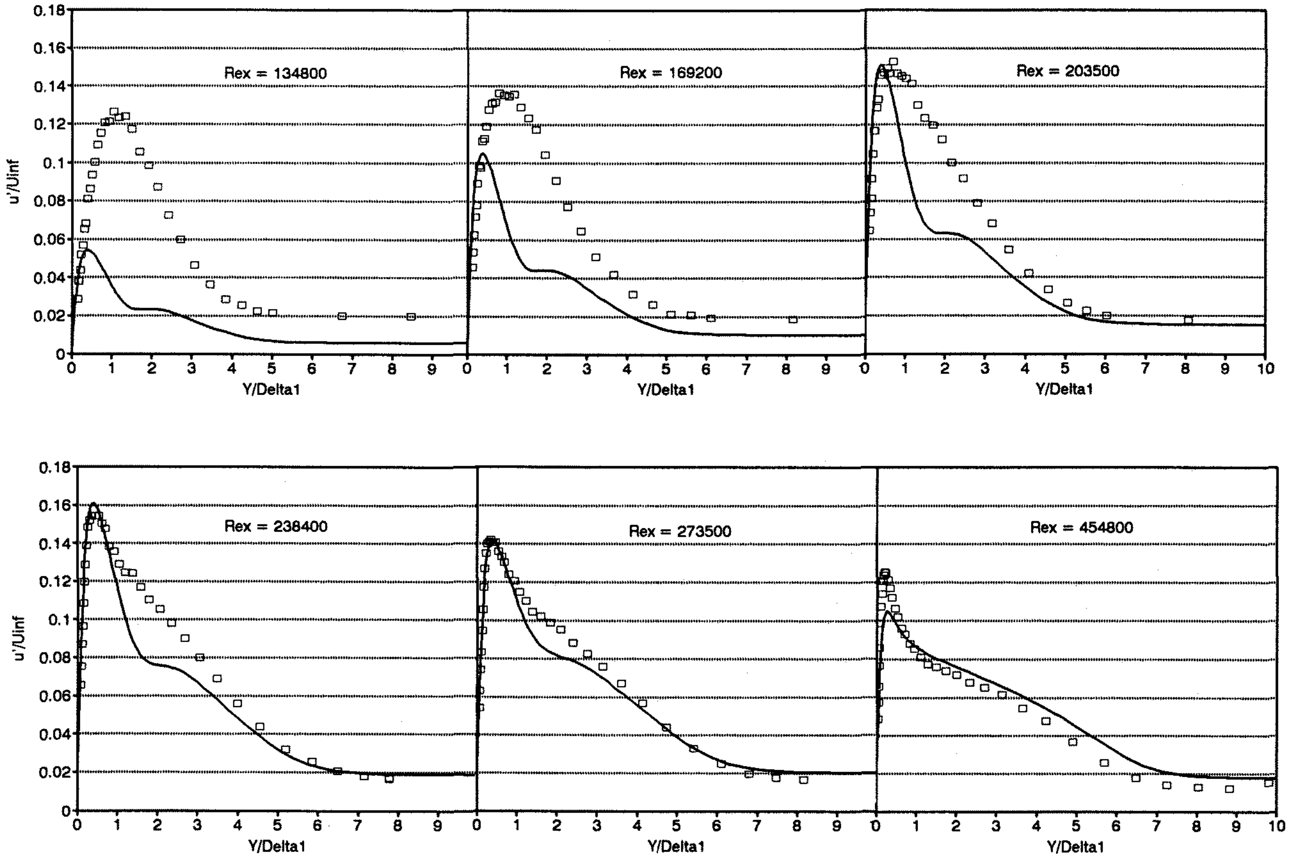

Figure 6: Streamwise fluctuations for different positions on the plate (T3A).

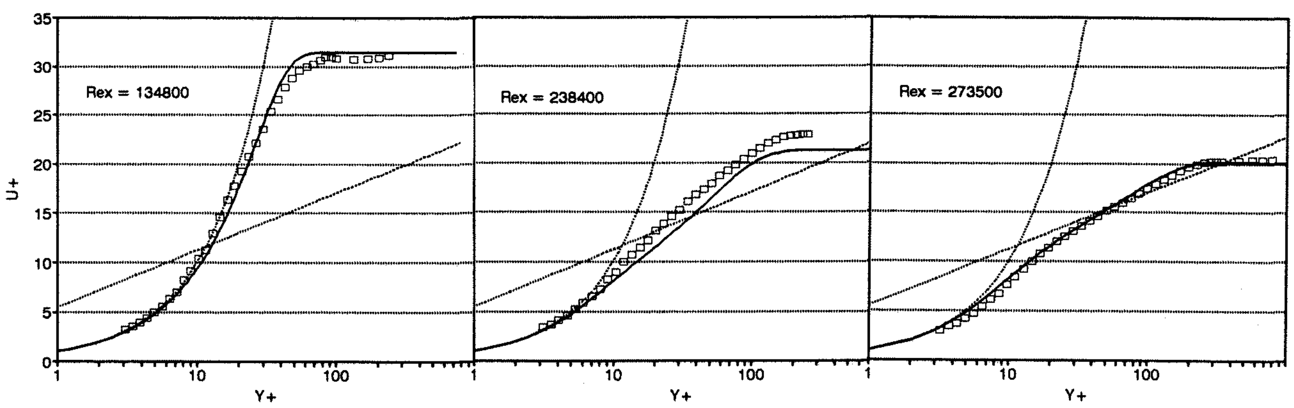

Figure 7: Velocity profiles for different positions on the plate (T3A).

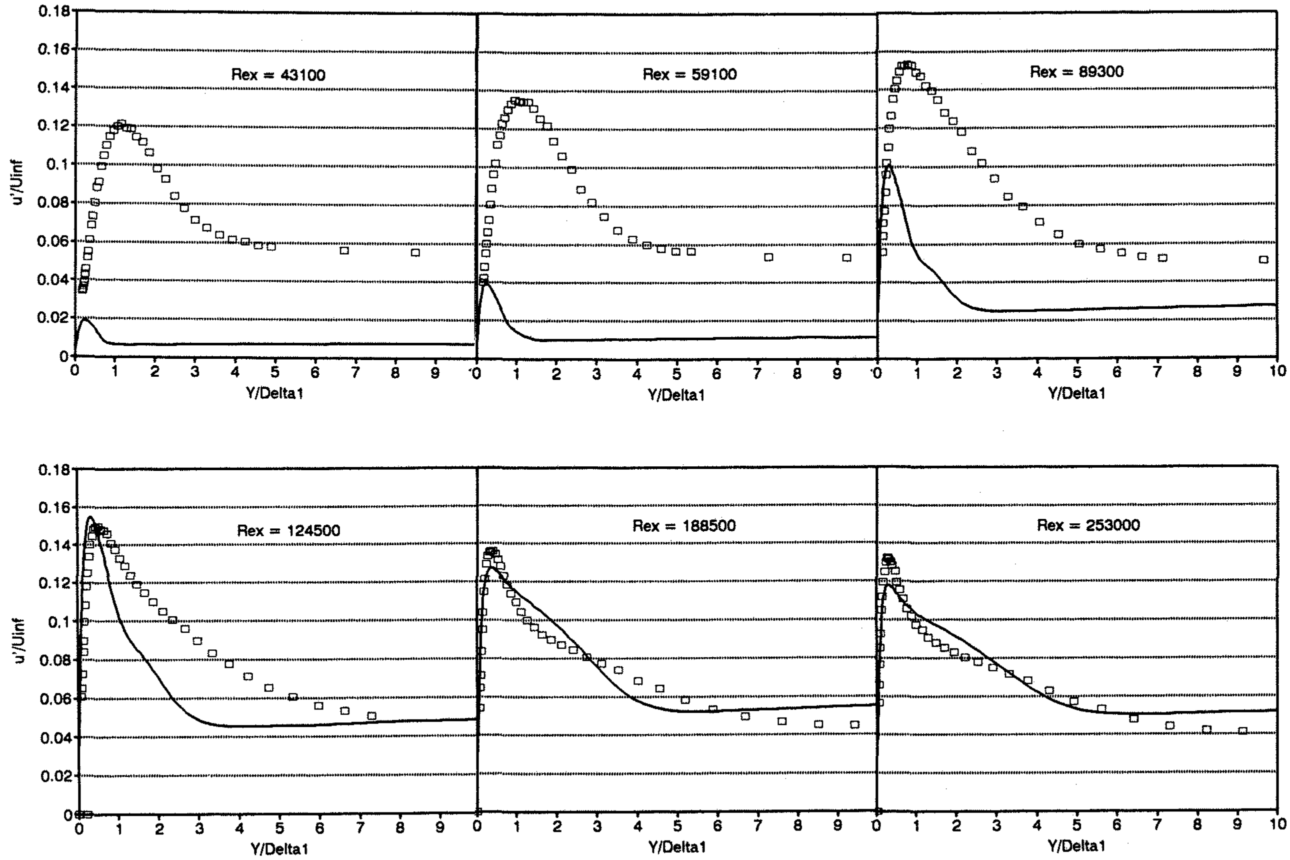

Figure 9: Streamwise fluctuations for different positions on the plate (T3B).

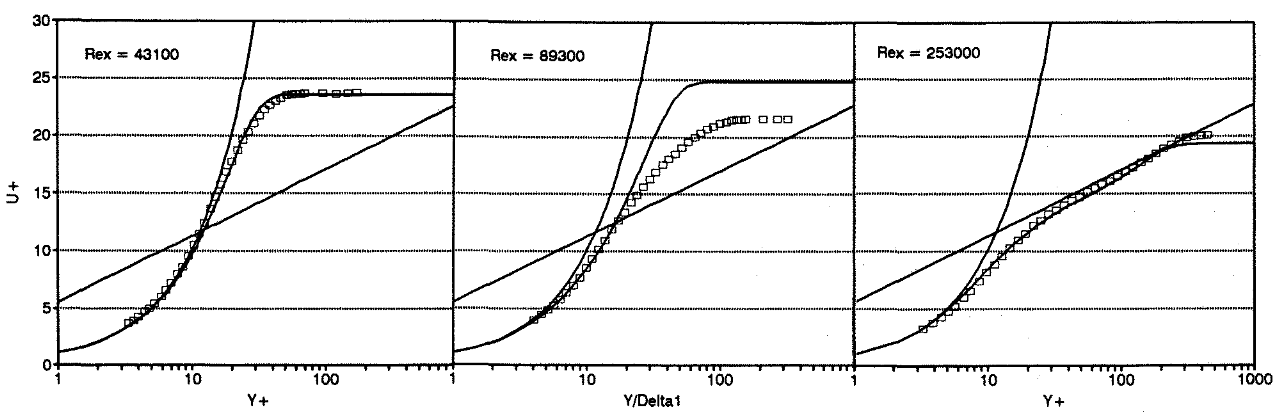

Figure 10: Velocity profiles for different positions on the plate (T3B).

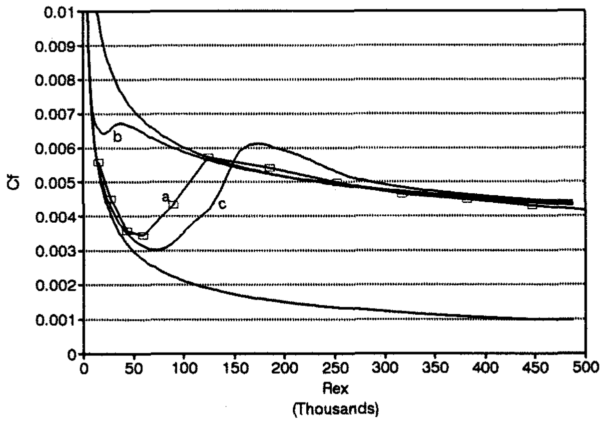

Figure 8: Skin friction coefficient (T3B).

Figure 9 shows the evolution of the profile of the global streamwise velocity fluctuation u′ during transition for different positions along the plate. Concerning the streamwise fluctuation, the same remarks can be made as in the T3A case: the fluctuations are underpredicted in the beginning of the intermittency zone while the levels correspond better further downstream. Velocity profiles at onset, in the middle and at the end of transition are shown in figure 10. As transition is predicted a bit too late (fig. 8), the profile in the middle of the transition tends more to a laminar profile than given by the experiments. The quality of the predictions of the T3B case is lower than in the T3A case. It is however difficult to draw a definite conclusion about the T3B case due to the low quality of the pressure distribution in the experimental set-up. In particular, the oscillation in the pressure profile is a disturbing factor.

Conclusions

Conditioned averaged Navier-Stokes equations have been derived to model the transition zone. An algebraic law for the intermittency factor γ has been proposed to simulate distributed breakdown. Two flat plate flows with high turbulence levels (Tu = 3% and 6%) were calculated and compared with experiments.

The present method predicts the transitional behaviour much better than what can be obtained with global averaged equations.

The general distribution of the skin friction in the boundary layer is very well predicted. Except for the beginning of the transition, the turbulence level and the profiles of turbulent fluctuations are well predicted. Velocity profiles are in good agreement with measured profiles.

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