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AEROELASTIC STABILITY CHARACTERISTICS OF COMPOSITE CYLINDRICAL SHELLS BY THE FINITE ELEMENT METHOD

Giuseppe Surace
Aerospace Department
Politecnico di Torino

Cecilia Surace
Structural Department
Politecnico di Torino

Summary

In the first part, the stiffness matrix is developed for composite circular cylindrical shells by elastic theories with the application of finite element method. A 14 parameter element is adopted. The results obtained using Novozhilov and Naghdi theory [1], [2] are compared. In the second part, the mass, initial stiffness and aerodynamic matrices are given. The eigenvalue problem is formulated and numerical results are presented.

1 Introduction

By applying the Novozhilov [1] and Naghdi [2] theory, finite elements can be used for circular cylindrical shells and for stepped axially symmetric shells in general. As demonstrated by G. Cantin [3-4], Donnell [5] and Flugge [6], this theory violates Maxwell's principle for linear structures and limits the degrees of freedom of rigid-body motion. With this theory, rigid body motion can never be completely free of deformation. Application of the Reissner-Wang [7-8] theory provides similar results, which coincide with those of Love [9], who first developed the general theory of bending for thin shells. This fact vitiates the particular characteristics of a finite element, which must be valid for general use and must lead to deformation-free modes for, and only for, rigid body motion. In this paper, the finite element method is applied to the aeroelastic stability characteristics of a cylindrical shell subject to internal pressure and axial compression by using the Novozhilov [1] and Naghdi [2] theory. The elements have 14 degrees of freedom and are *conforming* in that the curvatures are guaranteed to be equal along the cylinder axis at the nodes. The material is assumed to be composite and the stiffness matrices, which are also a function of fiber orientation, exhibit extension flexure coupling typical of these materials. These matrices can be used to compare the two theories, given that they can be calculated directly and permit the parameters involved to be varied readily.

In the second part of the study, the cylindrical elements with 14 degrees of freedom employed in the first part are used to calculate the mass matrix, the aerodynamic matrix, and the initial stiffness matrix due to internal pressure p_m and axial compression load P_x. The flutter equation is then written directly using the principle of virtual work. This approach is based on that used for panel flutter [10], and permits parametric analysis of aeroelastic stability of the cylinder. The results agree with those of the analytical and experimental works of Carter-Stearman [11], Olson-Fung [12] [13] and other authors when initial imperfections are not taken into account.

2 Theoretical considerations

One of the fundamental requirements of a finite element is that it must have rigid body motion modes which are free of strain. If the theory used to formulate the finite element is inconsistent, erroneous strain-free modes may be introduced by the displacement field assumed when developing the element. Because several theories have been proposed to formulate the relationship between displacement and strain, it has been necessary to analyse each theory to determine, and then subsequently to disregard, those which are not consistent.

With reference to figure (1), the following notation can be established.

- ξ, η, z : local curvilinear coordinates ;
- u, v, w : displacements of the middle surface of the shell, in the directions ξ, η, z respectively.

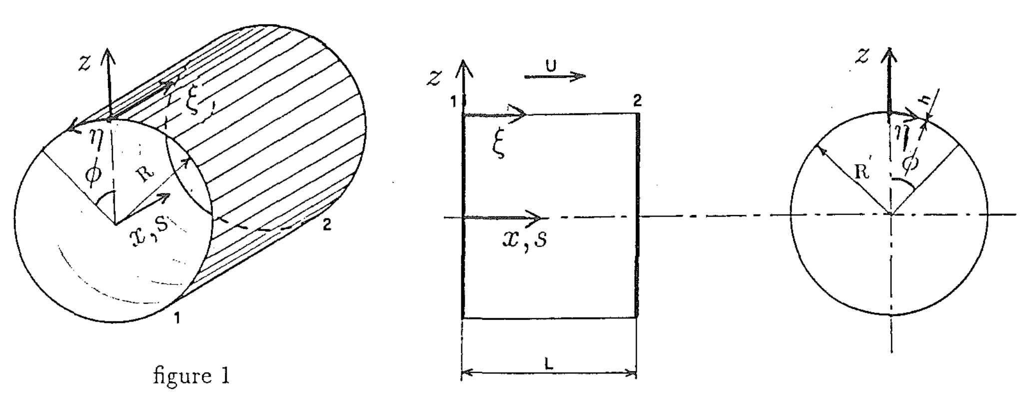

figure 1

In general, the relationship between strain and displacement can be written,

$$\{\epsilon\} = [\bar{O}]\{\delta\} \tag{1}$$

where $\{\delta\}$ is the vector of the displacement relative to the mid-surface, while $[\bar{O}]$ is a matrix which constitutes the specific relationship between strain end displacement. The matrix $[\bar{O}]$ varies according to the theoretical development used to attain the formulation (1).

In the table shown below, the matrix $[\bar{O}]$ is listed for several cases which are commonly presented in the literature on this subject.

(a) Donnell and Flügge			(b) Reissner and Wang		
$\partial/\partial\xi$			$\partial/\partial\xi$		
	$\partial/\partial\eta$	$1/R$		$\partial/\partial\eta$	$1/R$
$\partial/\partial\eta$	$\partial/\partial\xi$		$\partial/\partial\eta$	$\partial/\partial\xi$	
		$-\partial^2/\partial\xi^2$			$-\partial^2/\partial\xi^2$
		$-\partial^2/\partial\eta^2$		$R^{-1}\partial/\partial\eta$	$-\partial^2/\partial\eta^2$
		$-2\partial^2/\partial\xi\partial\eta$		$R^{-1}\partial/\partial\xi$	$-2\partial^2/\partial\xi\partial\eta$
(c) Novozhilov			(d) Naghdi		
$\partial/\partial\xi$			$\partial/\partial\xi$		
	$\partial/\partial\eta$	$1/R$		$\partial/\partial\eta$	$1/R$
$\partial/\partial\eta$	$\partial/\partial\xi$		$\partial/\partial\eta$	$\partial/\partial\xi$	
		$-\partial^2/\partial\xi^2$			$-\partial^2/\partial\xi^2$
	$R^{-1}\partial/\partial\eta$	$-\partial^2/\partial\eta^2$		$2R^{-1}\partial/\partial\eta$	$-\partial^2/\partial\eta^2 + R^{-2}$
	$2R^{-1}\partial/\partial\xi$	$-2\partial^2/\partial\xi\partial\eta$		$2R^{-1}\partial/\partial\xi$	$-2\partial^2/\partial\xi\partial\eta$

table 1

It has been demonstrated in reference [4] that, by using theories (a) and (b), a rigid body motion of the cylinder produce strain which is not consistent. Therefore only (c) and (d), the theories of Novozhilov [1] and of Naghdi [2], are considered to be suitable to be used in formulating the finite element.

3 Finite Element Formulation

In this study, a finite element is considered which has two nodes or, more precisely, two nodal rings. The characteristics are the same for each of the points along the perimeter of the nodal rings.

The displacements consist of a w component orthogonal to the surface, a v component tangential to the surface and an axial u component, as illustrated in fig. 1.
For this analysis, a displacement field has been selected which is function of the axial coordinate x and the circumferential coordinate ϕ. The dependence on x is a polynomial type function and can be written as

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{2}$$
$$v(x) = \alpha_5 + \alpha_6 x + \alpha_7 x^2 + \alpha_8 x^3 \tag{3}$$
$$w(x) = \alpha_9 + \alpha_{10} x + \alpha_{11} x^2 + \alpha_{12} x^3 + \alpha_{13} x^4 + \alpha_{14} x^5 \tag{4}$$

while the dependence on ϕ is represented by harmonic functions.

The properties of orthogonality of the trigonometric functions allows the decoupling of equations of motion relative to the different harmonics. Thus, for the generical harmonic n, the displacement of a point of the surface of the cylindrical element $P(s,\varphi)$ can be written as,

$$\{\delta_e\} = [\Phi]\{\delta(x)\} \tag{5}$$

where,

$$\{\delta_e\} = [u(x,\phi), v(x,\phi), w(x,\phi)]^T \tag{6}$$

are respectively the axial, tangential and radial displacements of $P(x,\phi)$,

$$\{\delta(x)\} = [u(x), v(x), w(x)]^T \tag{7}$$

Thus,

$$[\Phi] = [\cos n\phi, \sin n\phi, \cos n\phi] \tag{8}$$

The determination of the constants α_i is obtained by imposing conditions on (2), (3), (4). The number of conditions is equal to the number of nodal displacements.

The vector of nodal displacement is assumed to be as follows,

$$\{\Delta_e\}_{14\times 1} = [u_1, u_{x1}, v_1, v_{x1}, w_1, w_{x1}, w_{xx1}, -u_2, u_{x2}, v_2, -v_{x2}, w_2, -w_{x2}, w_{xx2}]^T \tag{9}$$

where the first suffix x indicates the derivative of the function with respect to x, and the second suffices 1 and 2 indicate that this is relative to nodes 1 and 2. By writing $x/L = s$ it is possible to show that, in matrix notation this can be written

$$\{\delta(s)\} = [N]\{\Delta_e\} \tag{10}$$

where the non-zero terms of the matrix $[N]_{3\times 14}$ are

$$\begin{aligned}
n_{(1,1)} &= 2s^3 - 3s^2 + 1 \\
n_{(1,2)} &= L(s^3 - 2s^2 + s) \\
n_{(2,3)} &= n_{(1,1)} \\
n_{(2,4)} &= n_{(1,2)} \\
n_{(3,5)} &= -6s^5 + 15s^4 - 10s^3 + 1 \\
n_{(3,6)} &= L(-3s^5 + 8s^4 - 6s^3 + s) \\
n_{(3,7)} &= \frac{L^2}{2}(-s^5 + 3s^4 - 3s^3 + s) \\
n_{(1,8)} &= 2s^3 - 3s^2 \\
n_{(1,9)} &= L(s^3 - s^2) \\
n_{(2,10)} &= -n_{(1,8)} \\
n_{(2,11)} &= -n_{(1,9)} \\
n_{(3,12)} &= 6s^5 - 15s^4 + 10s^3 \\
n_{(3,13)} &= L(3s^5 - 7s^4 + 4s^3) \\
n_{(3,14)} &= \frac{L^2}{2}(s^5 - 2s^4 + s^3)
\end{aligned}$$

Therefore, substituting (10) into (5),

$$\{\delta_e\} = [\Phi][N]\{\Delta_e\} \tag{11}$$

4 The Deformation State

The vector containing strains and curvatures,

$$\epsilon = [\epsilon_s, \epsilon_\theta, \gamma_{s\theta}, \chi_s, \chi_\theta, \chi_{s\theta}]^T \quad (12)$$

is related to the nodal displacement through the relationship,

$$\{\epsilon\} = [\bar{O}]\{\delta\} \quad (13)$$

where $[\bar{O}]$ is a matrix of differential operators, as defined by Novozhilov and Nagdhi theory (fig.1).

In this specific case the reference system is (s, ϕ) instead of (ξ, η), where $\xi = x, \eta = \phi R$. Substituting (10) into (1),

$$\{\epsilon\} = [B]\{\Delta_e\} \quad (14)$$

with,

$$[B] = [\bar{O}][\Phi][N] \quad (15)$$

5 The Elasticity Matrix

The relationship which exists between the membrane and bending forces,

$$\{N\} = [n_x, n_\phi, n_{x\phi}, m_x, m_\phi, m_{x\phi}]^T \quad (16)$$

is shown in ref. [1] to be,

$$\{N\} = [E(z)]\{\epsilon\} \quad (17)$$

The elasticity matrix $\mathbf{E}$ is function of z for a general laminate. It has been shown that,

$$\mathbf{E} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \quad (18)$$

where, as illustrated in figure 2,

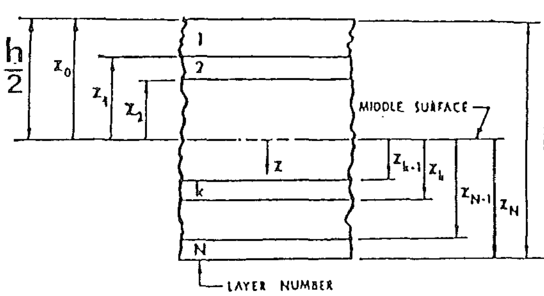

figure 2

$$[\mathcal{A}] = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{bmatrix} \quad (19)$$

$$[\mathcal{B}] = \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{B}_{23} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & \mathcal{B}_{33} \end{bmatrix} \quad (20)$$

$$[\mathcal{D}] = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{D}_{13} \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} \\ \mathcal{D}_{31} & \mathcal{D}_{32} & \mathcal{D}_{33} \end{bmatrix} \quad (21)$$

$$\mathcal{A}_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k - z_{k-1}) \quad (22)$$

$$\mathcal{B}_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2)/2 \quad (23)$$

$$\mathcal{D}_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3)/3 \quad (24)$$

The coefficients Q_{ij} are defined as follows,

$$[\bar{Q}] = [\Theta]^{-1}[\mathrm{Q}][\Theta]^{-\mathrm{T}} \quad (25)$$

with

$$[\Theta]^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \quad (26)$$

and

$$[\mathrm{Q}] = \begin{bmatrix} Q_{1,1} & Q_{1,2} & Q_{1,6} \\ & Q_{2,2} & Q_{2,6} \\ & & Q_{6,6} \end{bmatrix} \quad (27)$$

The terms Q_{ij} are defined for the case of the materials being orthotropic by,

$$Q_{11} = E_1/(1 - \nu_{12}\nu_{21}) \quad (28)$$

$$Q_{12} = \nu_{12}E_1/(1 - \nu_{12}\nu_{21}) \quad (29)$$

$$Q_{22} = E_2/(1 - \nu_{12}\nu_{21}) \quad (30)$$

$$Q_{66} = \tilde{G}_{12} \quad (31)$$

$$Q_{16} = Q_{26} = 0 \quad (32)$$

with $\nu_{12}E_2 = \nu_{21}E_1$

6 The Stiffness Matrix

To construct the stiffness matrix of a single element, the principle of virtual work can be used,

$$L_i = \int_V \{\epsilon^v\}^T \{\sigma\}.dV \quad (33)$$

Taking into account equation (13), the internal work L_i can be expressed as

$$L_i = \int_S \{\epsilon^v\}^T \{N\}.dS \quad (34)$$

Therefore, by substituting into (35) equations (14),(17)

$$L_i = \{\Delta_e^v\}^T \int_0^1 \int_0^{2\pi} [B]^T[E][B]RL.d\phi.dS\{\Delta_e\} \quad (35)$$

By equating the internal work L_i to the work done by the external forces $L_e = \{\Delta_e^v\}^T\{F_e\}$ (positive when they have the same direction as the corresponding displacement), the stiffness matrix can be written,

$$[\tilde{K}] = \int_0^1 \int_0^{2\pi} [B]^T[E][B]RL.d\phi.dS \quad (36)$$

and more simply,

$$[\tilde{K}] = \pi RL[K] \quad (37)$$

The stiffness matrix $[K]$ obtained using Novozilov's theory is presented in Appendix. It is constituted by elements which exhibit coupling extension-bending. The matrix $[_cK]$ represents the correction due to Nagdhi [2]. The terms are small with respect to the corresponding terms of matrix $[{}_cK]$ and are independent of the $[\mathcal{A}]$ matrix. The matrices $[K]$ and $[{}_cK]$ become very simple when the material is either isotropic or one of the following:

- Antisymmetrical Cross-Ply Laminates: An odd number of orthotropic layers with fibers alternatively at 0 and 90 degrees.

$$\mathcal{A}_{16} = \mathcal{A}_{26} = \mathcal{B}_{12} = \mathcal{B}_{26} = \mathcal{B}_{66} = \mathcal{D}_{16} = 0 \tag{38}$$

$$\mathcal{B}_{22} = \mathcal{B}_{11} \tag{39}$$

- Antisymmetrical Angle-Ply Laminates: an orthotropic layers orientated at $-\alpha$ and α degrees with respect to the middle plane.

$$\mathcal{A}_{16} = \mathcal{A}_{26} = \mathcal{B}_{11} = \mathcal{B}_{22} = \mathcal{B}_{66} = \mathcal{D}_{16} = \mathcal{D}_{26} = 0 \tag{40}$$

- Nonsymmetrical Laminates

$$\mathcal{A}_{16} = \mathcal{A}_{26} = \mathcal{B}_{16} = \mathcal{B}_{26} = \mathcal{D}_{16} = \mathcal{D}_{26} = 0 \tag{41}$$

7 The Mass Matrix

The mass matrix is obtained by imposing the condition that the virtual work done by the inertial forces must equalise the virtual work of equivalent inertial forces applied in the nodes of the element. The inertial force is written as,

$$\{\Delta F_m\} = -\rho \frac{\partial^2\{\delta_e(t)\}}{\partial t^2} h.R.L.d\phi.ds \tag{42}$$

If displacement $\{\delta_e\}$ is assumed to be an exponential function of time,

$$\{\delta_e(t)\} = \{\delta_e\}e^{i\omega t} \tag{43}$$

then,

$$\{\Delta F_m\} = \rho\omega^2\{\delta_e\}e^{i\omega t}hRLd\phi ds \tag{44}$$

Using equation (11),

$$[\tilde{m}] = \rho RhL \int_0^1 \int_0^{2\pi} [N]^T[\Phi]^T[\Phi][N]d\phi ds \tag{45}$$

or

$$[\tilde{m}] = \pi\rho RhL[m] \tag{46}$$

The elements of the matrix $[m]$ are presented in the Appendix.

8 The Aerodynamic Matrix

The aerodynamic theory considered here is called the *Piston Theory*. This constitutes a first order approximation of the linear theory relative to the potential flow of a supersonic stream. According to this theory, the aerodynamic force [16] acting on the infinitesimal area $Rd\phi dx$ can be written,

$$\Delta P(x,\phi,t) = -\frac{2qR}{\beta}\left[\frac{\partial\{W\}}{\partial x} + \frac{1}{U}\left(\frac{M^2-2}{M^2-1}\right)\frac{\partial\{W\}}{\partial t} - \frac{\{W\}}{2\beta R}\right]d\phi dx \tag{47}$$

In (47), the first term constitutes the true aerodynamic force, while the second and third terms represent the so called aerodynamic damping. Assuming a harmonic dependence on time,

$$W(x,\phi,t) = w(x,\phi)e^{i\omega t} \tag{48}$$

$$\Delta P(x,\phi,t) = \Delta p(x,\phi)e^{i\omega t} \tag{49}$$

it is possible to write,

$$\Delta p(x,\phi) = -\frac{2qR}{\beta}\left[\frac{\partial\{w(x,\phi)\}}{\partial x} + \frac{i\omega}{U}\left(\frac{M^2-2}{M^2-1}\right)\{w(x,\phi)\} - \frac{\{w(x,\phi)\}}{2\beta R}\right]d\phi dx \tag{50}$$

Also in this case, for the determination of the aerodynamic matrix called $[A]$, the pressure p is reduced to the nodes obtaining

$$[A] = \frac{2qR}{\beta}\int_0^L\int_0^{2\pi}\left[\{w(x,\phi)\}^T\frac{\partial\{w(x,\phi)\}}{\partial x} + \frac{i\omega}{U}\left(\frac{M^2-2}{M^2-1}\right)\{w(x,\phi)\} - \frac{\{w(x,\phi)\}}{2\beta R}\right]d\phi dx \tag{51}$$

That for simplicity can be written with this notation

$$[A] = U^2([a_R] + ik[a_I]) \tag{52}$$

where,

$$\begin{aligned} [a_R] &= c_1[{}_1A] + c_2[{}_2A] \\ [a_I] &= c_3[{}_2A] \\ c_1 &= -\pi R\rho_a/\beta \\ c_2 &= \pi L\rho_a/2\beta^2 \\ c_3 &= -(\pi RL\rho_a/\beta^3 S)(M^2-2) \end{aligned} \tag{53}$$

The elements of matrices $[{}_1A]$ and $[{}_2A]$ are presented in the Appendix.

9 The Initial Stiffness Matrix

This matrix takes into account the application of a compression axial load P_x and the presence of an internal pressure p_m. In the formulation of this matrix only the membrane forces due to P_x and p_m are considered. Thus, the transversal effect of the shear is neglected. The potential energy due to the internal force p_m and the axial compression P_x for the generic harmonic can be written as,

$$V_i = \frac{1}{2}\int_0^L \int_0^{2\pi} \{r\}^T[F]\{r\}R.d\phi.dx \qquad (54)$$

where $\{r\}$ is the vector of rotations so defined,

$$\{r\} = [r_\phi, r_x, r_n]^T = [C]\{\delta\} \qquad (55)$$

with,

$$[C] = \begin{bmatrix} & & \frac{\partial}{\partial x} \\ & \frac{1}{R} & -\frac{1}{R}\frac{\partial}{\partial \phi} \\ \frac{1}{2R}\frac{\partial}{\partial \phi} & -\frac{1}{2}\frac{\partial}{\partial x} & \end{bmatrix} \qquad (56)$$

whilst

$$[F] = [F_x, F_\phi, (F_x + F_\phi)] \qquad (57)$$

with

$$F_x = \sigma_x h = -\frac{P_x}{2\pi R}$$
$$F_\phi = \sigma_\phi = Rp_m$$

By substituting (11) in (55), it is possible to express the vector $\{r\}$ as function of nodal displacement through the matrix $[B_1]$ obtained applying the operator $[C]$ to $\{\delta\}$

$$\{r\} = [B_1]\{\Delta_e\} \qquad (58)$$

Equation (54) can be written as function of nodal displacements $\{\Delta_e\}$ as follows

$$V_i = \frac{1}{2}\{\Delta_e\}^T[K_i]\{\Delta_e\} \qquad (59)$$

Comparing (54) with (55) and taking into account (58) it is possible to write,

$$[\tilde{K}_i] = RL\int_0^1 \int_0^{2\pi} [B_1]^T[F][B_1]d\phi.ds \qquad (60)$$

where, for simplicity,

$$[\tilde{K}_i] = \pi RL[K_i] \qquad (61)$$

The elements of matrix $[K_i]$ are presented in the Appendix.

10 Flutter Equation

Assembling finite elements and applying kinematic boundary condition the instability of the cylinder can be expressed as follows,

$$[[D] - \lambda[I]]\{\Delta\} = \{0\} \qquad (62)$$

where

$$[D] = [K_T]^{-1}\left([M] + (\tfrac{S}{k})^2([A_R] + ik[A_I])\right)$$

$\{\Delta\}$ = matrix of generalized displacement of all the nodes of the structure
$[K_T]$ = global stiffness matrix of the structure
$[M]$ = mass matrix of the structure
$[A_R]$ = real part of the total aerodynamic matrix
$[A_I]$ = imaginary part of the total aerodynamic matrix
$\lambda = 1/\omega^2$
$k = \omega S/U$

11 Concluding Remarks

Equation (62) constitutes a complex eigenvalue problem. The aeroelastic instability of the cylinder is determined by finding the eigenvalues λ. Infact the instability is manifested when the amplitude of oscillation of the cylinder increases exponentially with time. From a mathematical point of view, this can be verified when the imaginary part of ω becomes negative. In those conditions $\{\Delta(t)\} = \{\Delta\}e^{i\omega t}$ corresponds to a harmonic oscillation with an amplitude that grows exponentially. The instability can be determined by fixing the values of the internal pressure p_m of the axial load P_x and of the number of circumferential harmonic n. Then the value of k is varied until the imaginary part of one of the eigenvalues becomes negative. This procedure must be repeated for different values of n in order to determine the minimum critical velocity which is attained in correspondence of the critical number of harmonics, n_{cr}, as illustrated in [15].

The finite element method allows the study of flutter of composite circular cylindrical shells subjected to initial internal pressure and axial compressional load. The results obtained in the case in which the material is omogeneus and isotropic, are comparable with those found analytically [13], [16] and experimentally [14].

As already it has been shown, in reference [10], for a flat panel, also for a composite cylindrical shell the angle of fiber orientation θ influences the value of the natural frequencies and the flutter parameters.

Finally, it is necessary to point out the structure of the matrices for the element (see appendix). These are symmetrical and are constituted of four submatrices that are symmetric, as well with the exception of matrix $[{}_1A]$.

12 Acknowledgments

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13 Notation

- R Cylinder radius ;
- θ Fibre orientation ;
- L Length of finite element ;
- S Length of cylinder shell ;
- N Number of layers ;
- t Time ;
- E_1,E_2 Youngś moduli in the two principal directions ;
- $\{\epsilon^v\}$ Virtual deformation ;

- ν_{12} Major Poisson's ratio ;
- ν_{21} Minor Poisson's ratio ;
- G_{12} Shear moduli in the mid-surface ;
- $\tilde{G}_{12} = G_{12}(1 - \nu_{12}\nu_{21})$;
- ρ Density per surface unit of the cylindrical shell ;
- ρ_a Freestream mass density ;
- U Freestream velocity ;
- $q = \frac{1}{2}\rho_a U^2$ Freestream dynamic pression ;
- M Mach number ;
- $\beta = \sqrt{M^2 - 1}$

The other symbols have been defined within the test.

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Appendix

$$[K] = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$$

$$[X] = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5} & k_{1,6} & k_{1,7} \\ & k_{2,2} & -k_{1,4} & 0 & -\frac{10}{L}k_{1,7} & k_{2,6} & k_{2,7} \\ & & k_{3,3} & k_{3,4} & k_{3,5} & k_{3,6} & k_{3,7} \\ & & & k_{4,4} & k_{4,5} & k_{4,6} & k_{4,7} \\ & & & & k_{5,5} & k_{5,6} & k_{5,7} \\ & & & & & k_{6,6} & k_{6,7} \\ & & & & & & k_{7,7} \end{bmatrix}$$

$$[Y] = \begin{bmatrix} k_{1,8} & k_{1,9} & \frac{5}{L}k_{1,4} & k_{1,4} & k_{1,12} & k_{1,6} & k_{1,7} \\ & k_{2,9} & k_{1,4} & \frac{L}{6}k_{1,4} & \frac{10}{L}k_{1,7} & k_{2,13} & k_{2,14} \\ & & k_{3,10} & k_{3,11} & k_{3,12} & k_{3,13} & k_{3,14} \\ & & & k_{4,11} & k_{4,12} & k_{4,13} & k_{4,14} \\ & & & & k_{5,12} & k_{5,13} & k_{5,14} \\ & & & & & k_{6,13} & k_{6,14} \\ & & & & & & k_{7,14} \end{bmatrix}$$

$$k_{1,1} = \frac{42A_{11}R^2 + 13A_{33}L^2n^2}{35L^2R^2}$$

$$k_{1,2} = \frac{21A_{11}R^2 + 11A_{33}L^2n^2}{210LR^2}$$

$$k_{1,3} = \frac{n(A_{33}R - A_{12}R + 2B_{33} - B_{12})}{2LR^2}$$

$$k_{1,4} = -\frac{n(A_{33}R + A_{12}R + 2B_{33} + B_{12})}{10R^2}$$

$$k_{1,5} = -\frac{A_{12}R - 2B_{33}n^2 + B_{12}n^2)}{2LR^2}$$

$$k_{1,6} = -\frac{168B_{11}R^2 + 17A_{12}L^2R + 34B_{33}L^2n^2 + 17L^2n^2}{140L^2R^2}$$

$$k_{1,7} = -\frac{28B_{11}R^2 + 3A_{12}L^2R + 6B_{33}L^2n^2 + 3B_{12}L^2n^2}{280LR^2}$$

$$k_{1,8} = \frac{3(28A_{11}R^2 - 3A_{33}L^2n^2)}{70L^2R^2}$$

$$k_{1,9} = \frac{42A_{11}R^2 - 13A_{33}L^2n^2}{420LR^2}$$

$$k_{1,12} = -\frac{A_{12}R + 2B_{33}n^2 + B_{12}n^2}{2LR^2}$$

$$k_{2,2} = \frac{14A_{11}R^2 + A_{33}L^2n^2}{105R^2}$$

$$k_{2,6} = \frac{336B_{11}R^2 + A_{12}L^2R + 2B_{33}L^2n^2 + B_{12}L^2n^2}{840LR^2}$$

$$k_{2,7} = -\frac{84B_{11}R^2 + A_{12}L^2R + 2B_{33}L^2n^2 + B_{12}L^2n^2}{1680R^2}$$

$$k_{2,9} = -\frac{14A_{11}R^2 + 3A_{33}L^2n^2}{420R^2}$$

$$k_{2,13} = -\frac{504B_{11}R^2 + 19A_{12}L^2R + 38B_{33}L^2n^2 + 19B_{12}L^2n^2}{840LR^2}$$

$$k_{2,14} = -\frac{28B_{11}R^2 + A_{12}L^2R + 2B_{33}L^2n^2 + B_{12}L^2n^2}{560R^2}$$

$$
\begin{aligned}
k_{3,3} &= \frac{1}{35L^2R^4}(42A_{33}R^4 + 168B_{33}R^3 + 13A_{22}L^2n^2R^2 \\
&+ 168D_{33}R^2 + 26B_{22}L^2n^2R + 13D_{22}L^2n^2) \\
k_{3,4} &= \frac{1}{210LR^4}(21A_{33}R^4 + 84B_{33}R^3 + 11A_{22}L^2n^2R^2 \\
&+ 84D_{33}R^2 + 22B_{22}L^2n^2R + 11D_{22}L^2n^2) \\
k_{3,5} &= \frac{1}{21L^2R^4}n(54B_{33}R^3 + 27B_{21}R^3 + 8A_{22}L^2R^2 \\
&+ 108D_{33}R^2 + 27D_{21}R^2 + 8B_{22}L^2n^2R + 8D_{22}L^2n^2 + 8B_{22}L^2R) \\
k_{3,6} &= \frac{1}{168LR^4}n(48B_{33}R^3 + 192B_{21}R^3 + 11A_{22}L^2R^2 \\
&+ 96D_{33}R^2 + 192D_{21}R^2 + 11B_{22}L^2n^2R + 11B_{22}L^2R + 11D_{22}L^2n^2) \\
k_{3,7} &= \frac{1}{5040R^4}n(72B_{33}R^3 + 36B_{21}R^3 + 25A_{22}L^2R^2 \\
&+ 144D_{33}R^2 + 36D_{21}R^2 + 25B_{22}L^2n^2R + 25B_{22}L^2R + 25D_{22}L^2n^2) \\
k_{3,10} &= -\frac{1}{70L^2R^4}3(28A_{33}R^4 + 112B_{33}R^3 - 3A_{22}L^2n^2a^2 \\
&+ 112D_{33}R^2 - 6B_{22}L^2n^2R - 3D_{22}L^2n^2) \\
k_{3,11} &= -\frac{1}{420LR^4}(42A_{33}R^4 + 168B_{33}R^3 - 13A_{22}L^2n^2R^2 \\
&+ 168D_{33}R^2 - 26B_{22}L^2n^2R - 13D_{22}L^2n^2) \\
k_{3,12} &= -\frac{1}{42L^2R^4}n(108B_{33}R^3 + 54B_{21}R^3 - 5A_{22}L^2R^2 \\
&+ 216D_{33}R^2 + 54D_{21}R^2 - 5B_{22}L^2n^2R - 5B_{22}L^2R - 5D_{22}L^2n^2) \\
k_{3,13} &= -\frac{1}{840LR^4}n(240B_{33}R^3 + 120B_{21}R^3 - 29A_{22}L^2R^2 \\
&+ 480D_{33}R^2 + 120D_{21}R^2 - 29B_{22}L^2n^2R - 29B_{22}L^2R - 29D_{22}L^2n^2) \\
k_{3,14} &= \frac{1}{5040R^4}n(72B_{33}R^3 + 36B_{21}R^3 - 17A_{22}L^2R^2 + 144D_{33}R^2 \\
&+ 36D_{21}R^2 - 17B_{22}L^2n^2R - 17B_{22}L^2R - 17D_{22}L^2n^2) \\
k_{4,4} &= \frac{1}{105R^4}(14A_{33}R^4 + 56B_{33}R^3 + A_{22}L^2n^2R^2 + 56D_{33}R^2 \\
&+ 2B_{22}L^2n^2R + D_{22}L^2n^2) \\
k_{4,5} &= \frac{1}{56LR^4}n(16B_{33}R^3 + 8B_{21}R^3 + 3A_{22}L^2R^2 + 32D_{33}R^2 \\
&+ 8D_{21}R^2 + 3B_{22}L^2n^2R + 3B_{22}L^2R + 3D_{22}L^2n^2) \\
k_{4,6} &= \frac{1}{420R^4}n(144B_{33}R^3 + 72B_{21}R^3 + 5A_{22}L^2R^2 + 288D_{33}R^2 \\
&+ 72D_{21}R^2 + 5B_{22}L^2n^2R + 5B_{22}L^2R + 5D_{22}L^2n^2) \\
k_{4,7} &= \frac{1}{1008R^4}Ln(24B_{33}R^3 + 12B_{21}R^3 + A_{22}L^2R^2 + 48D_{33}R^2 \\
&+ 12D_{21}R^2 + B_{22}L^2n^2R + D_{22}L^2n^2 + B_{22}L^2R) \\
k_{4,11} &= \frac{1}{420R^4}(14A_{33}R^4 + 56B_{33}R^3 + 3A_{22}L^2n^2R^2 + 56D_{33}R^2 \\
&+ 6B_{22}L^2n^2R + 3D_{22}L^2n^2) \\
k_{4,12} &= -\frac{1}{168LR^4}n(48B_{33}R^3 + 24B_{21}R^3 - 5A_{22}L^2R^2 + 96D_{33}R^2 \\
&+ 24D_{21}R^2 - 5B_{22}L^2n^2R - 5B_{22}L^2R - 5D_{22}L^2n^2) \\
k_{4,13} &= \frac{1}{840R^4}n(48B_{33}R^3 + 24B_{21}R^3 + 7A_{22}L^2R^2 + 96D_{33}R^2 \\
&+ 24D_{21}R^2 + 7B_{22}L^2n^2R + 7B_{22}L^2R + 7D_{22}L^2n^2) \\
k_{4,14} &= \frac{1}{1260R^4}Ln(12B_{33}R^3 + 6B_{21}R^3 + A_{22}L^2R^2 + 24D_{33}R^2 \\
&+ 6D_{21}R^2 + B_{22}L^2n^2R + B_{22}L^2R + D_{22}L^2n^2) \\
k_{5,5} &= \frac{1}{462L^4R^4}(7920D_{11}R^4 + 1320B_{12}L^2R^3 + 2640D_{33}L^2n^2R^2 \\
&+ 1320D_{12}L^2n^2R^2 + 181A_{22}L^4R^2 + 362B_{22}L^4n^2R + 181D_{22}L^4n^4) \\
k_{5,6} &= \frac{1}{4260L^3R^4}(39600D_{11}R^4 + 6600B_{12}L^2R^3 + 3960D_{33}L^2n^2R^2 \\
&+ 6600D_{12}L^2n^2R^2 + 311A_{22}L^4R^2 + 622B_{22}L^4n^2R + 311D_{22}L^4n^4) \\
k_{5,7} &= \frac{1}{55440L^2R^4}(23760D_{11}R^4 + 1320B_{12}L^2R^3 + 2640D_{33}L^2n^2R^2 \\
&+ 1320D_{12}L^2n^2R^2 + 281A_{22}L^4R^2 + 562B_{22}L^4n^2R + 281D_{22}L^4n^4) \\
k_{5,12} &= -\frac{5}{231L^4R^4}(792D_{11}R^4 + 132B_{12}L^2R^3 + 264D_{33}L^2n^2R^2 \\
&+ 132D_{12}L^2n^2R^2 - 5A_{22}L^4R^2 - 10B_{22}L^4n^2R - 5D_{22}L^4n^4) \\
k_{5,13} &= -\frac{1}{4620L^3R^4}(39600D_{11}R^4 + 1980B_{12}L^2R^3 + 3960D_{33}L^2n^2R^2 \\
&+ 1980D_{12}L^2n^2R^2 - 151A_{22}L^4R^2 - 302B_{22}L^4n^2R - 151D_{22}L^4n^4) \\
k_{5,14} &= -\frac{1}{55440L^2R^4}(23760D_{11}R^4 + 1320B_{12}L^2R^3 + 2640D_{33}L^2n^2R^2 \\
&+ 1320D_{12}L^2n^2R^2 - 181A_{22}L^4R^2 - 362B_{22}L^4n^2R - 181D_{22}L^4n^4)
\end{aligned}
$$

$$
\begin{aligned}
k_{6,6} &= \frac{1}{3465L^4R^4}4(4752D_{11}R^4 + 396B_{12}L^2R^3 + 792D_{33}L^2n^2R^2 \\
&+ 396D_{12}L^2n^2R^2 + 13A_{22}L^4R^2 + 26B_{22}L^4n^2R + 13D_{22}L^4n^4) \\
k_{6,7} &= \frac{1}{18480LR^4}(5808D_{11}R^4 + 616B_{12}L^2R^3 + 1232D_{33}L^2n^2R^2 \\
&+ 616D_{12}L^2n^2R^2 + 23A_{22}L^4R^2 + 46B_{22}L^4n^2R + 23D_{22}L^4n^4) \\
k_{6,13} &= \frac{1}{13680L^2R^4}(-42768D_{11}R^4 + 396B_{12}L^2R^3 + 792D_{33}L^2n^2R^2 \\
&+ 396D_{12}L^2n^2R^2 + 133A_{22}L^4R^2 + 133B_{22}L^4n^2R + 133D_{22}L^4n^4) \\
k_{6,14} &= -\frac{1}{13860LR^4}(1584D_{11}R^4 - 132B_{12}L^2R^3 - 264D_{33}L^2n^2R^2 \\
&- 132D_{12}L^2n^2R^2 - 13A_{22}L^4R^2 - 26B_{22}L^4n^2R - 13D_{22}L^4n^4) \\
k_{7,7} &= \frac{1}{27720R^4}(2376D_{11}R^4 + 88B_{12}L^2R^3 + 176D_{33}L^2n^2R^2 \\
&+ 88D_{12}L^2n^2R^2 + 3A_{22}L^4R^2 + 6B_{22}L^4n^2R + 3D_{22}L^4n^4) \\
k_{7,14} &= \frac{1}{55440R^4}(792D_{11}R^4 + 88B_{12}L^2R^3 + 176D_{33}L^2n^2R^2 \\
&+ 88D_{12}L^2n^2R^2 + 5A_{22}L^4R^2 + 10B_{22}L^4n^2R + 5D_{22}L^4n^4)
\end{aligned}
$$

$$
[{}_cK] = \begin{bmatrix} {}_cX & {}_cY \\ {}_cY & {}_cX \end{bmatrix}
$$

$$
[{}_cX] = \begin{bmatrix}
0 & 0 & {}_ck_{1,3} & {}_ck_{1,4} & {}_ck_{1,5} & {}_ck_{1,6} & {}_ck_{1,7} \\
 & 0 & -{}_ck_{1,4} & 0 & {}_ck_{2,5} & {}_ck_{2,6} & {}_ck_{2,7} \\
 & & {}_ck_{3,3} & {}_ck_{3,4} & {}_ck_{3,5} & {}_ck_{3,6} & {}_ck_{3,7} \\
 & & & {}_ck_{4,4} & {}_ck_{4,5} & {}_ck_{4,6} & {}_ck_{4,7} \\
 & & & & {}_ck_{5,5} & {}_ck_{5,6} & {}_ck_{5,7} \\
 & & & & & {}_ck_{6,6} & {}_ck_{6,7} \\
 & & & & & & {}_ck_{7,7}
\end{bmatrix}
$$

$$
[{}_cY] = \begin{bmatrix}
0 & 0 & {}_ck_{1,3} & {}_ck_{1,4} & {}_ck_{1,5} & {}_ck_{1,6} & {}_ck_{1,7} \\
 & 0 & {}_ck_{1,4} & {}_ck_{2,11} & -{}_ck_{2,5} & {}_ck_{2,13} & {}_ck_{2,14} \\
 & & {}_ck_{3,10} & {}_ck_{3,11} & {}_ck_{3,12} & {}_ck_{3,13} & {}_ck_{3,14} \\
 & & & {}_ck_{4,11} & {}_ck_{4,12} & {}_ck_{4,13} & {}_ck_{4,14} \\
 & & & & {}_ck_{5,12} & {}_ck_{5,13} & {}_ck_{5,14} \\
 & & & & & {}_ck_{6,13} & {}_ck_{6,14} \\
 & & & & & & {}_ck_{7,14}
\end{bmatrix}
$$

$$
\begin{aligned}
{}_ck_{1,3} &= -\frac{B_{12}n}{2LR^2} \\
{}_ck_{1,4} &= -\frac{B_{12}n}{10R^2} \\
{}_ck_{1,5} &= -\frac{B_{12}n}{2LR^2} \\
{}_ck_{1,6} &= -\frac{17B_{12}}{140R^2} \\
{}_ck_{1,7} &= -\frac{3B_{12}L}{280R^2} \\
{}_ck_{2,5} &= \frac{3B_{12}}{28R^2} \\
{}_ck_{2,6} &= \frac{B_{12}L}{840R^2} \\
{}_ck_{2,7} &= -\frac{B_{12}L^2}{1680R^2} \\
{}_ck_{2,11} &= \frac{B_{12}Ln}{60R^2} \\
{}_ck_{2,13} &= -\frac{19B_{12}L}{840R^2} \\
{}_ck_{2,14} &= -\frac{B_{12}L^2}{560R^2} \\
{}_ck_{3,3} &= \frac{13n^2(2B_{22}R + 3D_{22})}{35R^4} \\
{}_ck_{3,4} &= \frac{11Ln^2(2B_{22}R + 3D_{22})}{210R^4}
\end{aligned}
$$

$$
\begin{aligned}
{}_ck_{3,5} &= \frac{n(27D_{12}R^2 + 16B_{22}L^2R + 8D_{22}L^2n^2 + 16D_{22}L^2)}{21L^2R^4} \\
{}_ck_{3,6} &= -\frac{n(192D_{12}R^2 + 22B_{22}L^2R + 11D_{22}L^2n^2 + 22D_{22}L^2)}{168LR^4} \\
{}_ck_{3,7} &= \frac{n(36D_{12}R^2 + 50B_{22}L^2R + 25D_{22}L^2n^2 + 50D_{22}L^2)}{5040R^4} \\
{}_ck_{3,10} &= \frac{9n^2(2B_{22}R + 3D_{22})}{70R^4} \\
{}_ck_{3,11} &= \frac{13Ln^2(2B_{22}R + 3D_{22})}{420R^4} \\
{}_ck_{3,12} &= -\frac{n(54D_{12}R^2 - 10B_{22}L^2R - 5D_{22}L^2n^2 - 10D_{22}L^2)}{42L^2R^4} \\
{}_ck_{3,13} &= -\frac{n(120D_{12}R^2 - 58B_{22}L^2R - 29D_{22}L^2n^2 - 58D_{22}L^2)}{840LR^4} \\
{}_ck_{3,14} &= -\frac{n(36D_{12}R^2 - 34B_{22}L^2R - 17D_{22}L^2n^2 - 34D_{22}L^2)}{5040R^4} \\
{}_ck_{4,4} &= \frac{L^2n^2(2B_{22}R + 3D_{22})}{105R^4} \\
{}_ck_{4,5} &= \frac{n(8D_{12}R^2 + 6B_{22}L^2R + 3D_{22}L^2n^2 + 6D_{22}L^2)}{56LR^4} \\
{}_ck_{4,6} &= \frac{n(72D_{12}R^2 + 10B_{22}L^2R + 5D_{22}L^2n^2 + 10D_{22}L^2)}{420R^4} \\
{}_ck_{4,7} &= \frac{Ln(12D_{12}R^2 + 2B_{22}L^2R + D_{22}L^2n^2 + 2D_{22}L^2)}{1008R^4} \\
{}_ck_{4,11} &= \frac{L^2n^2(2B_{22}R + 3D_{22})}{140R^4} \\
{}_ck_{4,12} &= \frac{n(24D_{12}R^2 - 10B_{22}L^2R - 5D_{22}L^2n^2 - 10D_{22}L^2)}{168LR^4} \\
{}_ck_{4,13} &= -\frac{n(24D_{12}R^2 + 14B_{22}L^2R + 7D_{22}L^2n^2 + 14D_{22}L^2)}{840R^4} \\
{}_ck_{4,14} &= \frac{Ln(6D_{12}R^2 + 2B_{22}L^2R + D_{22}L^2n^2 + 2D_{22}L^2)}{1260R^4} \\
{}_ck_{5,5} &= \frac{(1320D_{12}R^2 + 362B_{22}L^2R + 362D_{22}L^2n^2 + 181D_{22}L^2)}{462L^2R^4} \\
{}_ck_{5,6} &= \frac{(6600D_{12}R^2 + 622B_{22}L^2R + 622D_{22}L^2n^2 + 311D_{22}L^2)}{4620LR^4} \\
{}_ck_{5,7} &= \frac{1320D_{12}R^2 + 562B_{22}L^2R + 562D_{22}L^2n^2 + 281D_{22}L^2)}{55440R^4} \\
{}_ck_{5,12} &= -\frac{5(132D_{12}R^2 - 10B_{22}L^2R - 10D_{22}L^2n^2 - 5D_{22}L^2)}{231L^2R^4} \\
{}_ck_{5,13} &= -\frac{1980D_{12}R^2 - 302B_{22}L^2R - 302D_{22}L^2n^2 - 151D_{22}L^2}{4260LR^4} \\
{}_ck_{5,14} &= -\frac{1320D_{12}R^2 - 362B_{22}L^2R - 362D_{22}L^2n^2 - 181D_{22}L^2}{55440R^4} \\
{}_ck_{6,6} &= \frac{4(396D_{12}R^2 + 26B_{22}L^2R + 25D_{22}L^2n^2 + 13D_{22}L^2)}{3465R^4} \\
{}_ck_{6,7} &= -\frac{L(88D_{12}R^2 + 26B_{22}L^2R + 26D_{22}L^2n^2 + 13D_{22}L^2)}{3696R^4} \\
{}_ck_{6,13} &= \frac{396D_{12}R^2 + 133B_{22}L^2n^2R + 266B_{22}L^2R + 266D_{22}L^2n^2 + 133D_{22}L^2}{13680\quad R^4} \\
{}_ck_{6,14} &= -\frac{L(132D_{12}R^2 + 26B_{22}L^2R + 26D_{22}L^2n^2 + 13D_{22}L^2)}{13860R^4} \\
{}_ck_{7,7} &= \frac{L^2(88D_{12}R^2 + 6B_{22}L^2R + 6D_{22}L^2n^2 + 3D_{22}L^2)}{27720R^4} \\
{}_ck_{7,14} &= \frac{L^2(88D_{12}R^2 + 10B_{22}L^2R + 10D_{22}L^2n^2 + 5D_{22}L^2)}{55440R^4}
\end{aligned}
$$

$$
[M] = \begin{bmatrix} {}_mX & {}_mY \\ {}_mY & {}_mX \end{bmatrix}
$$

$$
[{}_mX] = \begin{bmatrix}
m_{1,1} & m_{1,2} & 0 & 0 & 0 & 0 & 0 \\
 & m_{1,2} & 0 & 0 & 0 & 0 & 0 \\
 & & m_{1,1} & m_{1,2} & 0 & 0 & 0 \\
 & & & m_{2,2} & 0 & 0 & 0 \\
 & & & & m_{5,5} & m_{5,6} & m_{5,7} \\
 & & & & & m_{6,6} & m_{6,7} \\
 & & & & & & m_{7,7}
\end{bmatrix}
$$

$$
[{}_mY] = \begin{bmatrix}
m_{1,8} & m_{1,9} & 0 & 0 & 0 & 0 & 0 \\
 & m_{2,9} & 0 & 0 & 0 & 0 & 0 \\
 & & -m_{1,8} & m_{1,9} & 0 & 0 & 0 \\
 & & & -m_{2,9} & 0 & 0 & 0 \\
 & & & & m_{5,12} & m_{5,13} & m_{5,14} \\
 & & & & & m_{6,13} & m_{6,14} \\
 & & & & & & m_{7,14}
\end{bmatrix}
$$

$$
\begin{aligned}
m_{1,1} &= \frac{13}{35} & m_{5,6} &= \frac{311L}{4620} & m_{6,13} &= \frac{19L^2}{1980} \\
m_{1,2} &= \frac{11L}{210} & m_{5,7} &= \frac{281L^2}{55440} & m_{6,14} &= \frac{13L^3}{13680} \\
m_{1,8} &= -\frac{9}{70} & m_{5,12} &= \frac{25}{131} & m_{7,7} &= \frac{L^4}{9240} \\
m_{1,9} &= -\frac{13L}{420} & m_{5,13} &= \frac{151L}{4620} & m_{7,14} &= \frac{L^4}{11088} \\
m_{2,2} &= \frac{L^2}{105} & m_{5,14} &= \frac{181L^2}{55440} & m_{12,13} &= \frac{311L}{4620} \\
m_{2,9} &= -\frac{L^2}{140} & m_{6,6} &= \frac{52L^2}{3465} & m_{13,13} &= \frac{52L^2}{3465} \\
m_{5,5} &= \frac{181}{462} & m_{6,7} &= \frac{23L^3}{18480} & m_{13,14} &= \frac{23L^3}{18480}
\end{aligned}
$$

$$
[{}_1A] = \begin{bmatrix} {}_1X & {}_1Y \\ -{}_1Y & -{}_1X \end{bmatrix}
$$

$$
[{}_1X] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & {}_1a_{5,5} & {}_1a_{5,6} & {}_1a_{5,7} \\
0 & 0 & 0 & 0 & -{}_1a_{5,6} & 0 & {}_1a_{6,7} \\
0 & 0 & 0 & 0 & -{}_1a_{5,7} & -{}_1a_{6,7} & 0
\end{bmatrix}
$$

$$
[{}_1Y] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & & -{}_1a_{5,5} & {}_1a_{5,6} & {}_1a_{5,7} \\
 & & & & & {}_1a_{6,13} & {}_1a_{6,14} \\
 & & & & & & {}_1a_{7,14}
\end{bmatrix}
$$

$$
\begin{aligned}
{}_1a_{5,5} &= -\frac{1}{2} & {}_1a_{6,13} &= \frac{13L^2}{420} \\
{}_1a_{5,6} &= \frac{11L}{84} & {}_1a_{6,14} &= \frac{13L^3}{5040} \\
{}_1a_{5,7} &= \frac{L^2}{84} & {}_1a_{7,14} &= \frac{L^4}{5040} \\
{}_1a_{6,7} &= \frac{L^3}{1008}
\end{aligned}
$$

$$[{}_2A] = \begin{bmatrix} {}_2X & {}_2Y \\ {}_2Y & {}_2X \end{bmatrix}$$

$$[{}_2X] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & {}_2a_{5,5} & {}_2a_{5,6} & {}_2a_{5,7} \\ & & & & & {}_2a_{6,6} & {}_2a_{6,7} \\ & & & & & & {}_2a_{7,7} \end{bmatrix}$$

$$[{}_2Y] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & {}_2a_{5,12} & {}_2a_{5,13} & {}_2a_{5,14} \\ & & & & & {}_2a_{6,13} & {}_2a_{6,14} \\ & & & & & & {}_2a_{7,14} \end{bmatrix}$$

$$\begin{aligned} {}_2a_{5,5} &= \frac{181}{462} & {}_2a_{6,6} &= \frac{52}{3465} \\ {}_2a_{5,6} &= \frac{311}{4620} & {}_2a_{6,7} &= \frac{23}{18480} \\ {}_2a_{5,7} &= \frac{281L^2}{55440} & {}_2a_{6,13} &= -\frac{19}{1980} \\ {}_2a_{5,12} &= \frac{25}{231} & {}_2a_{6,14} &= \frac{13L^3}{13860} \\ {}_2a_{5,13} &= \frac{151L}{4620} & {}_2a_{7,7} &= \frac{L^4}{9240} \\ {}_2a_{5,14} &= \frac{181L^2}{55440} & {}_2a_{7,14} &= \frac{L^4}{11088} \end{aligned}$$

$$[{}_iK] = \begin{bmatrix} {}_iX & {}_iY \\ {}_iY & {}_iX \end{bmatrix}$$

$$[{}_iX] = \begin{bmatrix} {}_ik_{1,1} & {}_ik_{1,2} & {}_ik_{1,3} & {}_ik_{1,4} & 0 & 0 & 0 \\ & \frac{2}{11}{}_ik_{1,2} & -{}_ik_{1,4} & 0 & 0 & 0 & 0 \\ & & {}_ik_{3,3} & {}_ik_{3,4} & {}_ik_{3,5} & {}_ik_{3,6} & {}_ik_{3,7} \\ & & & {}_ik_{4,4} & {}_ik_{4,5} & {}_ik_{4,6} & \frac{L}{5}{}_ik_{3,7} \\ & & & & {}_ik_{5,5} & {}_ik_{5,6} & {}_ik_{5,7} \\ & & & & & {}_ik_{6,6} & {}_ik_{6,7} \\ & & & & & & {}_ik_{7,7} \end{bmatrix}$$

$$[{}_iY] = \begin{bmatrix} -\frac{9}{26}{}_ik_{1,1} & -\frac{L}{12}{}_ik_{1,1} & -{}_ik_{1,3} & {}_ik_{1,4} & 0 & 0 & 0 \\ & \frac{L^2}{52}{}_ik_{1,1} & {}_ik_{1,4} & \frac{6}{L}{}_ik_{1,4} & 0 & 0 & 0 \\ & & {}_ik_{3,10} & {}_ik_{3,11} & {}_ik_{3,12} & {}_ik_{3,13} & {}_ik_{3,14} \\ & & & {}_ik_{4,11} & {}_ik_{4,12} & {}_ik_{4,13} & {}_ik_{4,14} \\ & & & & {}_ik_{5,12} & {}_ik_{5,13} & {}_ik_{5,14} \\ & & & & & {}_ik_{6,13} & {}_ik_{6,14} \\ & & & & & & {}_ik_{7,14} \end{bmatrix}$$

$$\begin{aligned} {}_ik_{1,1} &= \frac{13(F_\phi + F_x)n^2}{140R^2} \\ {}_ik_{1,2} &= \frac{11(F_\phi + F_x)Ln^2}{840R^2} \\ {}_ik_{1,3} &= -\frac{(F_\phi + F_x)n}{8LR} \\ {}_ik_{1,4} &= \frac{(F_\phi + F_x)n}{40R} \end{aligned}$$

$$\begin{aligned} {}_ik_{3,3} &= \frac{21F_\phi R^2 + 21F_x R^2 + 26F_\phi L^2}{70L^2R^2} \\ {}_ik_{3,4} &= \frac{21F_\phi R^2 + 21F_x R^2 + 44F_\phi L^2}{840LR^2} \\ {}_ik_{3,5} &= \frac{8F_\phi n}{21R^2} \\ {}_ik_{3,6} &= \frac{11F_\phi Ln}{1680R^2} \\ {}_ik_{3,7} &= \frac{5F_\phi L^2 n}{1008R^2} \\ {}_ik_{3,10} &= -\frac{3(7F_\phi R^2 + 7F_x R^2 - 3F_\phi L^2)}{70L^2R^2} \\ {}_ik_{3,11} &= -\frac{21F_\phi R^2 + 21F_x R^2 - 26F_\phi L^2}{840LR^2} \\ {}_ik_{3,12} &= \frac{5F_\phi n}{42LR^2} \\ {}_ik_{3,13} &= \frac{29F_\phi Ln}{840R^2} \\ {}_ik_{3,14} &= \frac{17F_\phi L^2 n}{5040R^2} \\ {}_ik_{4,4} &= \frac{7F_\phi R^2 + 7F_x R^2 + 2F_\phi L^2}{210R^2} \\ {}_ik_{4,5} &= \frac{3F_\phi Ln}{56R^2} \\ {}_ik_{4,6} &= \frac{F_\phi L^2 n}{84R^2} \\ {}_ik_{4,11} &= \frac{7F_\phi R^2 + 7F_x R^2 + 6F_\phi L^2}{840R^2} \\ {}_ik_{4,12} &= \frac{5F_\phi Ln}{168R^2} \\ {}_ik_{4,13} &= \frac{F_\phi L^2 n}{120R^2} \\ {}_ik_{4,14} &= \frac{F_\phi L^3 n}{1260R^2} \\ {}_ik_{5,5} &= \frac{660F_x R^2 + 181F_\phi L^2 n^2}{462L^2R^2} \\ {}_ik_{5,6} &= \frac{990F_x R^2 + 311F_\phi L^2 n^2}{4620LR^2} \\ {}_ik_{5,7} &= \frac{660F_x R^2 + 281F_\phi L^2 n^2}{55440R^2} \\ {}_ik_{5,12} &= -\frac{5(66F_x R^2 - 5F_\phi L^2 n^2)}{231L^2R^2} \\ {}_ik_{5,13} &= -\frac{990F_x R^2 - 151F_\phi L^2 n^2}{4620LR^2} \\ {}_ik_{5,14} &= -\frac{660F_x R^2 - 181F_\phi L^2 n^2}{55440L^2R^2} \\ {}_ik_{6,6} &= \frac{4(198F_x R^2 + 13F_\phi L^2 n^2)}{3465R^2} \\ {}_ik_{6,7} &= \frac{L(308F_x R^2 + 23F_\phi L^2 n^2)}{18480R^2} \\ {}_ik_{6,13} &= \frac{198F_x R^2 + 133F_\phi L^2 n^2}{13860R^2} \\ {}_ik_{6,14} &= \frac{L(66F_x R^2 + 13F_\phi L^2 n^2)}{13860R^2} \\ {}_ik_{7,7} &= \frac{L^2(44F_x R^2 + 3F_\phi L^2 n^2)}{27720R^2} \\ {}_ik_{7,14} &= \frac{L^2(44F_x R^2 + 5F_\phi L^2 n^2)}{55440R^2} \end{aligned}$$