AN ENGINEERING-ORIENTED APPROACH TO THE ANALYSIS OF GEOMETRICALLY NONLINEAR ELASTIC STRUCTURES SUBJECTED TO RANDOM EXCITATION

by

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Abstract

This study describes an Engineering-oriented approach to the calculation of the response of a geometrically nonlinear elastic structure to random excitation loads. It also outlines a systematic computational process, which incorporates the use of static finite element computer code with the statistical linearization method. A simple though general numerical example, solved by the computational process described, demonstrates the applicability of this approach. The computational routine described yields designs which are less conservative.

1.Introduction

Nonlinear elastic effects often control the behavior of structures, especially ones like aeronautical structures which contain plates, shells and stringers. The linear analysis of these structures is usually based on bending behavior. Nevertheless, during real loadings, large deflections may be obtained which introduce membrane stretching. As a result, the structure tends to "harden" and introduce nonlinear effects. An engineering design which considers the "hardening" effects will be less conservative, and therefore lighter and more efficient.

Usually, linear analysis is done when deflections are of the same order of magnitude as the thickness. Investigation of simple models shows that nonlinear effects are introduced even for deflection lower than half the thickness, and the necessity for nonlinear analysis is therefore enhanced.

The nonlinear analysis of structures under static loadings is performed in the industry by an extensive use of nonlinear options of the large numeric computer codes (such as NASTRAN, ADINA, ANSYS, and others), which are available to most engineers. The nonlinear analysis of structures subjected to deterministic dynamic loads is less frequently used. Solutions for such cases were presented extensively in the literature, mainly for simple structures or for systems with a small, discrete number of degrees of freedom, but they are seldom used by engineers. The nonlinear response analysis of structures subjected to random dynamic loads is even more rare in the daily life of engineers, although this kind of problem is much more actual and realistic in the life cycle of a real structure. Most of the literature in these cases relates to very simple structures.

Numerous publications on the subject of random vibration of nonlinear structures were published in the last decade. Nevertheless, the practical applications of random analysis of nonlinear structures in the industry are few. It seems that, for some reason, engineers avoid the dynamic random analysis of structures. Only in recent years has a major effort been made by NASA to develope and implement a routine, practical tool for this purpose (e.g. Cruse et al.,1988), which will, in the near future, make random analysis a daily routine.

In the work described herein, an effort was done to define and construct the computational process for nonlinear elastic structure under stationary, random excitation. This enables the design engineers to routinly use the analytical and numerical tools available to them.

The nonlinear problem is solved by the statistical linearization method, which is well documented in the literature (e.g. Roberts, 1981; Spanos,1981). The functions of elastic nonlinearity of the structure are calculated by assuming cubic expressions for the nonlinear stiffnesses and by using static, finite element code (ANSYS). An iterative computational process, which incorporates the statistical linearization formulation and the finite element static computer code, results in the power spectral density functions, variances and covariances of the response.

Numerical example is presented. This example, although simple, is general enough to demonstrate the applicability of the method. The two main nonlinear effects - (a) a decrease in the power spectral density functions and rms values of the response, and (b) a significant shift of the frequencies governing the response, are clearly demonstrated.

2. Statistical Linearization

2.1 General Formulation

The formulation of the method of statistical linearization is repeated in this section to enhance understanding of the process described in this work.

The equations of motion of a system with many degrees of freedom are expressed by the general matrix equations *

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{Q}(t) \qquad (1)$$

where M is the nxn mass matrix, C is the nxn

* In the following description, bold face letters represent matrices

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damping matrix, K is the nxn stiffness matrix, q is the nx1 general coordinate matrix, Q is the nx1 excitation matrix, and Φ is a nx1 matrix which contains the nonlinear part of the relationship between forces and displacements, velocities and accelerations ($q, \dot{q}, \ddot{q}$, respectively).

Equation (1) is approximated by an "equivalent" linear equation

$$M^{eq}\ddot{q} + C^{eq}\dot{q} + K^{eq}q = Q(t) \tag{2}$$

where

$$\begin{aligned} M^{eq} &= M + M_e \\ C^{eq} &= C + C_e \\ K^{eq} &= K + K_e \end{aligned} \tag{3}$$

are the equivalent mass, damping and stiffness matrices. The difference between the original equations of (1) and the approximation of (2) is

$$\epsilon = \Phi - M_e\ddot{q} - C_e\dot{q} - K_e q \tag{4}$$

and a set of M_e, C_e and K_e matrices is sought so that the expected value of the quantity A has a minimum:

$$E[A] = E[\epsilon^T \cdot \epsilon] = \text{minimum} \tag{5}$$

Equatin (5) can be expressed as

$$E[\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2] = \text{minimum} \tag{6}$$

where ϵ_i are the terms of the matrix ϵ.

Roberts & Spanos,1990, showed that equation (6) yields the following conditions for the terms in matrices M_e, C_e, and K_e:

$$\begin{aligned} \frac{\partial}{\partial m_{ij}^e}(E[\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2]) &= 0 \\ \frac{\partial}{\partial c_{ij}^e}(E[\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2]) &= 0 \\ \frac{\partial}{\partial k_{ij}^e}(E[\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2]) &= 0 \end{aligned} \tag{7}$$

where $m_{ij}^e, c_{ij}^e, k_{ij}^e$ are the i,j term of the matrices M_e, C_e and K_e, respectively.

Roberts & Spanos,1990, also showed that the value of $E[\epsilon^T \cdot \epsilon]$, which corresponds to M_e, C_e and K_e obtained by equations (7), is an absolute minimum if the components of the vector $[q, \dot{q}, \ddot{q}]$ are independant. In this case, a unique set of equations exists for M_e, C_e, and K_e.

2.2 Gaussian Approximation

Many random excitations which act on real structures have Gaussian distributions. In these cases, the response of the linear system is also Gaussian, and it is also logical to assume Gaussian approximation for the nonlinear response. By virtue of the Central Limit Theorem, many real processes in engineering applications can be idealized as. Gaussian processes, as an approximation. Use of the Gaussian distribution significantly simplifies the process of calculating the equivalent terms in equation (7). Kazakov,1965;Atalik & Utku,1976, showed that, for a stationary Gaussian process with zero mean, equations (7) and the properties of the Gaussian distributed random variables yield the following equations:

$$\begin{aligned} m_{ij}^e &= E[\frac{\partial \Phi_i}{\partial \ddot{q}_j}] \\ c_{ij}^e &= E[\frac{\partial \Phi_i}{\partial \dot{q}_j}] \\ k_{ij}^e &= E[\frac{\partial \Phi_i}{\partial q_j}] \end{aligned} \tag{8}$$

Equations (8) give a direct relationship between the terms of the equivalent matrices and the nonlinear functions, Φ.

We may assume that the nonlinear response problem can be solved by introducing equations (8) into equations (2), and solving them for q. This is, however, not so. Another relationship must be found between the expected values of combinations of q, (represented in equation (8) by the differentiation of Φ).

If the power spectral density functions of the excitations is given by the nxn matrix S_Q, then the PSD functions of the response, S_q, may be expressed as:

$$S_q(\omega) = \alpha(\omega) S_Q(\omega) \alpha^{T*}(\omega) \tag{9}$$

where

$$\alpha(\omega) = [-\omega^2(M + M_e) + i\omega(C + C_e) + (K + K_e)]^{-1} \tag{10}$$

then, by using

$$E[q_i(t)q_j(t)] = \int_{-\infty}^{+\infty} S_{q_i q_j}(\omega) d\omega \tag{11}$$

the expected values of ixj combinations of q are obtained. The main practical problem in following this procedure is the integral shown on the right hand side of equation (11). To simplify this integration, a normal mode approximation for $\alpha(\omega)$ is used.

2.3 Normal Modes Approximation

For lightly damped structures, $[\alpha(\omega)]$ can be expanded in the following series:

$$\alpha(\omega) = \sum_{r=1}^{n} \frac{K^{(r)}}{\omega_{req}^2 - \omega^2 + 2i\xi_{req}\omega_{req}\omega} \tag{12}$$

where

$$K^{(r)} = \frac{\lambda^{(r)}\lambda^{(r)T}}{M_r} \tag{13}$$

$\lambda^{(r)}$ is the r-th normal mode of the system, ω_{req} is the resonance frequency of the undamped equivalent linear system, ξ_{req} is the modal damping coefficient, and M_r is the generalized mass of the r-th mode. $\lambda^{(r)}$ and ω_{req} are calculated from the undamped equivalent equation

$$(\mathrm{M}+\mathrm{M_e})\ddot{\mathrm{q}}+(\mathrm{K}+\mathrm{K_e})\mathrm{q}=0 \tag{14}$$

Using the expansion of (12), we can show that

$$E[q_r q_s]=\int_{-\infty}^{+\infty}\frac{S_{Q_{r,s}}\,d\omega}{(\omega_{req}^2-\omega^2+2i\xi_{req}\omega_{req}\omega)(\omega_{seq}^2-\omega^2-2i\xi_{seq}\omega_{seq}\omega)} \tag{15}$$

This integral can be evaluated to a closed form solution for rational functions of $S_{Q_{r,s}}$.

Note that, when equations (9) and (10) are introduced into equation (11), the denominator of the latter is at least in the order of 2n in $(i\omega)$, with the exact order determined by the order of the denominator of the excitation $S_{Q_{r,s}}$. When the modal approximation is used, the denominator of (15) is of the order of 4 (plus the order of the denominator of $S_{Q_{r,s}}$). This significantly simplifies the integration of equation (15).

The modal approximation (12) yields excellant results for damping coefficients as high as 10%, and good results for values as high as 20%. These values are usually much higher than the values of damping coefficients of realistic structures, and therefore the use of the modal approximation is generally justified for practical cases.

3. Calculation of K and Φ

3.1 General Description

The nonlinear structure is characterized by the M , C , K and Φ matrices. Generally, a nonlinear behavior in mass, damping and stiffness can be handled, although practically, nonlinearity in mass is rare.

The nonlinearity in damping cannot, usually, be calculated either theoretically or numerically. In most cases of nonlinear damping, an assumption on the dependence of C on q must be made, based on experience gained by the design engineer. Nevertheless, even in cases where there is no nonlinear effects on the terms of matrix C , the damping coefficient changes with the frequency, based on the definition of the modal damping coefficient:

$$\xi_{req}=\frac{C}{2m\omega_{req}} \tag{16}$$

This means that, for a system with a linear damping (terms of C depend only on $\dot{q}$), the product, $\omega_{req}\xi_{req}$, is constant.

3.2 Nonlinear Rigidity

Assume an elastic system, in which the only nonlinearity is caused by geometric effects, such as those described in the introduction (Section 1 above).

In this case, the static behavior of the structure will be described by

$$\mathrm{Kq}+\Phi(\mathrm{q})=0 \tag{17}$$

Assume further that the geometrical nonlinearity behaviour of the structure is cubic:

$$\mathrm{Kq}+\mathrm{Bq}^3=0 \tag{18}$$

or

$$\begin{array}{c} K_{1,1}q_1+K_{1,2}q_2+\cdots+K_{1,n}q_n+B_{1,1}q_1^3+B_{1,2}q_2^3+\cdots+B_{1,n}q_n^3=F_1\\ \vdots\\ K_{n,1}q_1+K_{n,2}q_2+\cdots+K_{n,n}q_n+B_{n,1}q_1+B_{n,2}q_2^3+\cdots+B_{n,n}q_n^3=F_n \end{array} \tag{19}$$

K in (18) is the stiffness matrix of the linear structure, and therefore is symmetric. B is a matrix of coefficients of nonlinearity, which has to be calculated for the specific system under consideration.

From (17) and (18):

$$\begin{array}{c} \Phi_1=B_{1,1}q_1^3+B_{1,2}q_2^3+\cdots+B_{1,n}q_n^3\\ \vdots\\ \Phi_n=B_{n,1}q_1^3+B_{n,2}q_2^3+\cdots+B_{n,n}q_n^3 \end{array} \tag{20}$$

and, by using equation (8) and the definition of variance

$$\sigma_{i,i}^2=E[q_i(t)q_i(t)] \tag{21}$$

one obtains the following matrix for K

$$\mathrm{K_e}=\begin{bmatrix} 3B_{1,1}\sigma_{1,1}^2 & 3B_{1,2}\sigma_{2,2}^2\cdots\cdots 3B_{1,n}\sigma_{n,n}^2\\ \vdots & \vdots \qquad\qquad \vdots\\ 3B_{n,1}\sigma_{1,1}^2 & 3B_{n,2}\sigma_{2,2}^2\cdots\cdots 3B_{n,n}\sigma_{n,n}^2 \end{bmatrix} \tag{22}$$

In equation (22) only variances appear, due to the nature of the problem. If the q is not the normal modes vector, co-variances will be also included in the K matrix.

The assumption of cubic relationship for the stiffness nonlinearities (equation (18)) results in a simple relationship between the terms of the equivalent stiffness matrix and the variances (or standard deviations) of the response. This assumption was found to be an excellent approximation for many geometrical nonlinearities. In fact, even when the structure does not behave excactly according to a cubic rule, it would be a good reccommendation to approximate the true behavior with a cubic equation. This way, much simpler linear equivalent equations are obtained for the original nonlinear structure. The cubic assumption means that the equations of motion are of the Duffing type. The Duffing oscillator was investigated extensively in the past, both analytically (e.g. Stoker, 1950, and many others) and experimentally (e.g. Maymon, 1978;Maymon & Rehfield 1984, and many others), and shows many interesting features, such as "overhang" of the response curve, and "jump" phenomena.

3.3 The K and B matrices

The linear stiffness matrix, K, can be calculated for a given structure either theoretically (wherever possible) or numerically, with any static linear finite element program. For numerical calculation, the number of degrees of freedom taken into account is not necessarily the number of nodes into which the structure is divided. One may wish to do the numerical static analysis with many nodes, to increase accuracy, while the dynamic analysis will be done with much fewer modes. The stiffness matrix calculated during the run of a finite element program is not the rquired K matrix.

The principle of superposition can be used for the solution of the equation

$$\mathrm{Kq} = \mathrm{F} \tag{23}$$

Numerical loading of n sets of F, in which all terms except one (for each set) are zero, and solution of the equation for q, by the finite element code, yield nxn equations for nxn terms of K .

Once the linear stiffness matrix, K, is known, the terms of B can be found by writing equation (18) in the form

$$\mathrm{Bq}^3 = \mathrm{F} - \mathrm{Kq} \tag{24}$$

This is done through the nonlinear static options contained in most of the large stuctural analysis computer codes. Loading n sets of F vector, and solution of the equations for q by the finite element code, yield nxn linear equations, like equation (24), for the nxn terms of B . The sets of forcing vector F are required to be such that static deflection similar to the n normal modes of the structure will be obtained. Therefore, a prior knowledge of the approximate modal deflections is required, and would be a good practice to solve the dynamic linear eigenvalues and eigenvectors problem numerically, using the same structural computer code.

4. The Iterative Solotion Procedure

The terms m^e_{ij} c^e_{ij} k^e_{ij} are functions of the nonlinear functions Φ, and, for the cases studied in this work, they are functions of σ^2_{ij} (equations (8), (20), (22)). The variances and covariances (equation (11)) are functions of the equivalent response frequencies, equivalent mode shapes and equivalent damping coefficients (equations (9),(12),(13) and (15)).

In order to solve for the variances (which are the standard deviations of the responses at a certain point, for the Gaussian, zero mean process), an iterative solution procedure is adopted.

The first iteration is done on the linear equation, neglecting the nonlinearities. A result for the σ^2_{ij} terms is obtained, $\mathrm{K_e}$ terms, new equivalent resonance frequencies, new modal shapes (due to modal approximation) and new generalized masses are calculated, and new values for σ^2_{ij} are generated. The process is repeated until convergence in all σ^2_{ij} and equivalent resonance frequencies is obtained. A relatively small number of iterations is required for convergence, as shown by the numerical example.

5. Computational Process

In many practical cases, the dynamic response of a system with an infinite number of degrees of freedom contains a finite, usually small, number of modes. There are two major reasons for this observation:

(a) The magnitude of the maximum values of the power spectral density function of the displacement response is inversly proportional to ω^4, thus the contribution of higher frequencies to the variance of the displacement can be neglected.

(b) For many cases of random, uncorrelated excitation, the generalized forces of excitation vanish in antisymmetric modes. As a result, antisymmetric modes are not excited and therefore have no influence on the total random response.

The number of degrees of freedom to be included in the calculation has a major influence on the complexity of the process of the numerical solution. Too many degrees of freedom will result in calculations containing much more cumbersome expressions, and, consequently, longer calculation time and more iterations required before convergence of the solution is obtained. Deterministic, linear, dynamic analysis may help decide how many degrees of freedom are to be considered.

Once the number of modes participating in the response has been determined, the linear matrix, K, and the nonlinear functions, Φ, can be calculated by static, nonlinear numerical computer codes which will take those modes into account. Expressions for the relationship between K and Φ are formulated. If nonlinear damping terms are expected, the dependence of damping forces on the displacements and velocities (matrix C) should be defined. The iterative computational process is then applied.

Figure 1 illustrates a schematic diagram of the computational process.

6. Numerical Example

The numerical example presented below is relatively simple yet general, since it uses all the steps required for the solution. The only difference between this simple example and a more general realistic structure is the number of degrees of freedom applied in the solution.

The nonlinear structure calculated is a simply supported beam, where the supports can not come closer when lateral loads are applied. The span of the beam is 60 cm (23.62 in.). Its width is 8 cm (3.15 in.), and its thickness is 0.5 cm (0.192 in.). The beam is made of steel (E=2.1x10^6 Kg/cm^2 =29.82x10^6 psi) and totals 1.872 Kg (4.12 Lbs) in weight. All the modal damping coefficients of the linear beam are ξ_i = 2%. The beam is excited by a random force, with Gaussian, zero mean distribution acting on mid-span. The power spectral density of the exciting force is

$$\begin{aligned} S_Q(\omega) &= 0 \qquad & -\infty < \omega < 0 \\ S_Q(\omega) &= S \qquad & 0 \le \omega < +\infty \end{aligned} \tag{25}$$

with S taking different values.

The following results are obtained for the

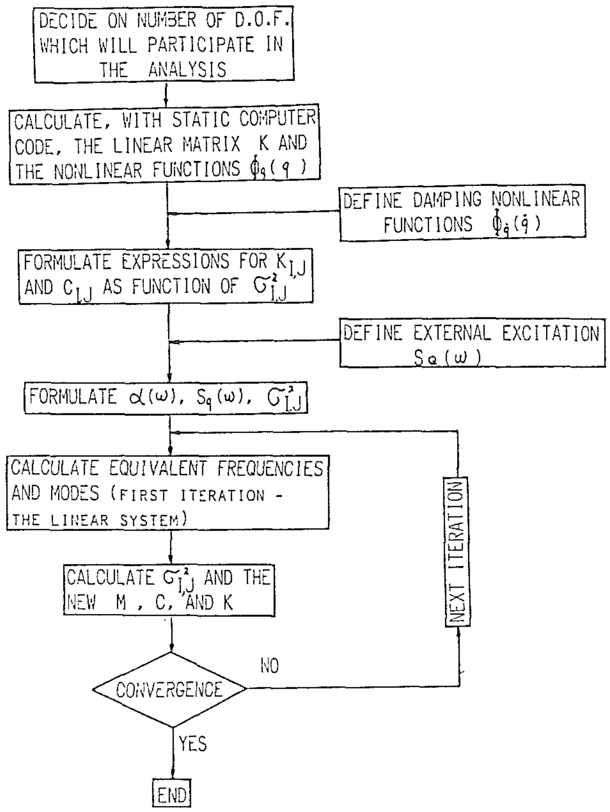

Figure 1: Schematic Diagram of the Computational Process

first three resonance frequencies and mode shapes of the linear beam:

$$\omega_1 = 206.2 \text{ rad/sec}$$
$$\omega_2 = 807.1 \text{ rad/sec}$$
$$\omega_3 = 1713.4 \text{ rad/sec} \qquad (26)$$
$$\lambda = \begin{bmatrix} 1 & 1 & 1 \\ 1.4142 & 0 & -1.4142 \\ 1 & -1 & 1 \end{bmatrix}$$

We have decided to solve the response problem with three degrees of freedom (three normal modes). σ^2_{y11} is the mean square of the response at 1/4-span, and σ^2_{y22} is the mean square of the response at mid-span. Using the procedure described in section 3.3 above, matrices K and B were calculated with the ANSYS finite element program. The beam was divided for this purpose into 24 beam elements. The following values were obtained for the K and B matrices:

$$K = \begin{bmatrix} 511.0922 & -488.8714 & 199.9934 \\ -488.8714 & 711.0856 & -488.8714 \\ 199.9934 & -488.8714 & 511.0922 \end{bmatrix} \text{Kg/cm} \quad (27)$$

$$B = \begin{bmatrix} 640.1012 & -75.3566 & -7.1450 \\ -91.6179 & 333.4502 & -91.6179 \\ -7.1450 & -75.3566 & 640.1012 \end{bmatrix} \text{Kg/cm}^3 \quad (28)$$

K^{eq}_{ij} is obtained by equations (3), (22), (27) and (28).

The iterative computation process yields results for all nine values of σ^2_{ij} (some of which are equal, because of the symmetry of the structure and the loading), of which σ^2_{y11} and σ^2_{y22}

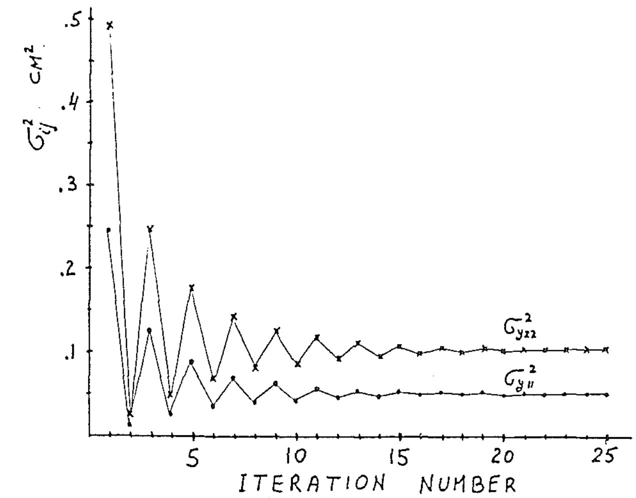

Figure 2: Convergence of Standard Deviations

are shown in Figure 2 for one level of S. Figure 2 shows the convergence of the solution.

The power spectral density function (PSD) of the response is shown in Figure 3 for seven values of S. Also shown are results obtained for an analysis done with only one degree of freedom (an equivalent mass centered at mid-span, with a massless beam). As expected, there is no response in the second, antisymmetric mode. Two points are shown (for S=0.01 and S=0.025 Kg^2/rad/sec) for the linear analysis of the one degree of freedom problem.

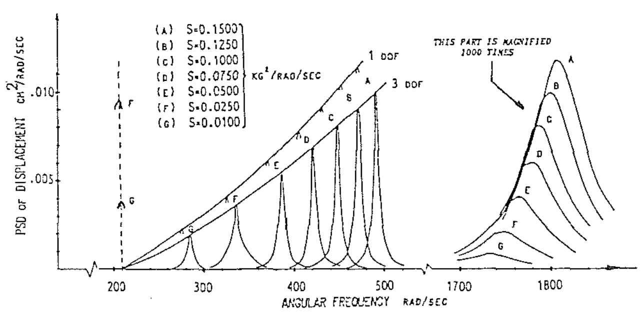

Figure 3: Power Spectral Density Functions of the Rsponse (Note the different scale for the right hand side)

Note the reduction in the maximum value of the PSD function of the response, relative to the linear solution. The ratio between the maximum value of the response PSD and that of the linear response PSD is shown in Figure 4. Values of this ratio drop to approximately 20% of the linear value for higher values of S.

Figure 5 shows different values of response displacements:

1. The amplitude of the response of the nonlinear beam to harmonic excitation (backbone curve),
2. The rms deflection of the nonlinear beam to harmonic excitation (peak/$\sqrt{2}$),
3. The rms deflection of the nonlinear beam, excited by the random force, calculated with 1 and 3 degrees of freedom,

4. The 3σ deflection (99.97% of all amplitudes) of the nonlinear beam, calculated with three degrees of freedom.

In both Figures 3 and 5, the shift in resonance frequency (frequency of maximum response), namely, the "hardening" effect, is clearly demonstrated. Also, nonlinear effects are clearly shown to influence the behavior of the beam for deflections of the same order of magnitude as the thickness (and smaller).

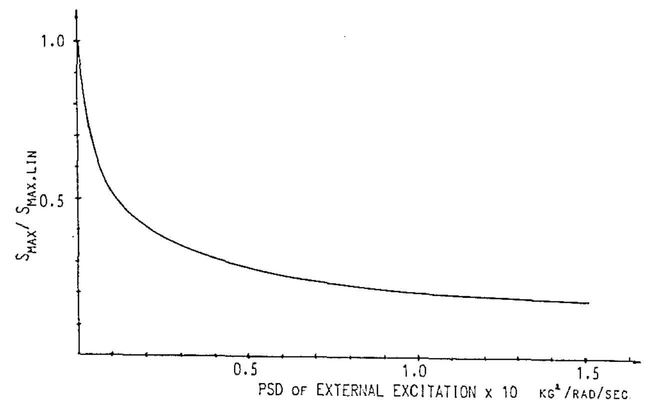

Figure 4: Maximum Value of Nonlinear PSD Relative to Linear PSD

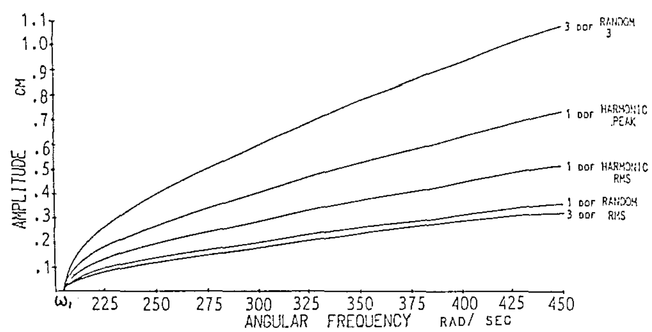

Figure 5: Expected Values of Amplitudes

7. Conclusion

Examination of the nonlinear response of a geometrically nonlinear structure, subjected to random, Gaussian distributed excitation loads, has led to the following observations:

(a) The influence of higher modes of the structure is small, therefore a behavior similar to a one degree of freedom system is observed. As a result, analysis of the first mode response is reccommended for the first engineering approximation. This conclusion will not be adequate if, because of some design criterion, the acceleration response is required.

(b) The maximum value of the response PSD is significantly lower than that of a linear system. Therefore, the rms values of the response amplitudes, and hence the strains and stresses, are lower. This implies that nonlinear analysis in the design process will yield a less conservative design. Engineers are encouraged to use nonlinear analysis even for relatively small deflections.

(c) A shift in the resonance frequencies is observed, similar to that of one degree of freedom system. Therefore, a nonlinear response contains a range of resonance frequencies, the values of which depend on the magnitude of the excitation.

(d) The results demonstrated in this work are applicable for wide band random excitation. Because of the "overhang" of the response curve, which causes "jump" phenomena in the response to periodic excitations, instability of response behavior is possible when narrow band excitation is applied.

(e) The statistical linearization method, together with existing static computer codes, can provide a systematic tool for the analysis of nonlinear structures.

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