A PROPOSAL CONCERNING THE DYNAMIC ANALYSIS METHOD OF CONTINUOUS GUST DESIGN RULES

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Abstract

This paper proposes that the Worst Deterministic Input (WDI) analysis can be used as the dynamic basis of continuous gust design rules to replace the current Power Spectral Density (PSD) analysis. The WDI analysis studies the stochastic response of a system via searhing the worst deterministic input from an equiprobable family. For linear time invariant system, the WDI method can give results agreeing exactly with that of PSD method. But the new method give the worst case with time informations, it can deal with linear time variant and non-linear problems. Adopting WDI analysis instead of PSD analysis can greatly expand the application extent of design rules without changing criteria and parameters even compute procedures used currently. So this proposal not only has great value for development of aviation industry but also is easy to implement.

Foreword

Gust loads are highly important for the structure design of a civil transport aircraft. In Civil Aviation Regulation the continuous gust design rules are described specifically besides that the limit loads produced by discrete gusts are regulated.[1,2]

Nevertheless, the dynamic analysis method currently used by the rules is the "Power Spectral Density" method (simply PSD method) which is based on the frequency domain analysis of a linear time invariant system. The PSD method was developed forty years ago. Although the reliability of PSD method has been demonstrated by long practice of aviation industry, it can only be applied to linear time invariant system, so it can not satisfy the demand of quickly developing modern civil aviation industry. For instance, non-linear problems arise more and more often in modern aircraft design. As pointed out by the researchers of NASA, the usefulness of a gust load analysis method depends on the ability to consider the non-linearity in aerodynamic force, structure, and control system.[3] But based on current theory, it is already very difficult to deal with linear time variant system, it is more difficult to solve non-linear problems.

Facing this challenge, aviation industry has been actively searching for new method to analyse gust response of an aircraft for many years. From the end of sixties, J.G.Jones has developed the so-called "Statistical Discrete Gust" method (simply SDG method) with the attempt to absorb the advantages of both the discrete gust method and PSD method. In the SDG model, firstly it is assumed that the gusts encountered by an aircraft can be represented by a sequence of discrete gusts whose profile has some prescribed shape (e.g. 1-cosin, ramp, etc.). Secondly a numerical searching procedure relative to parameters such as gust length and distance between gusts is carried out to find the worst gust from an equiprobable gust family and the maximum response peak of system to this gust. finally the value of this peak is used to study system stochastic response characteristics such as threshold exceedance statistics.[4] The most attractive point is that The SDG method can compute the maximum gust load with its time history and the worst gust profile, and that it can be used to study time variant and non-linear problems.

But, the technique used by SDG to find the worst case adopts the assumption that gusts have fixed shape and uses a numerical searching procedure, this not only imposes extra constraint which will bring extra error to the computation, and also makes the problem complicated and the compute time expensive. The investigation by NASA indicates that for rigid aircraft motion model the error of SDG method may be ±5%, for flexible model reachs ±10%. As to compute time, even the simplified SDG method needs 30 times more than that of PSD method.[3]

This paper proposes a method which can be called as the "Worst Deterministic Input" method (simply WDI method). The new method follows the basic idea of SDG method, that is, the stochastic response of a system can be studied through the worst deterministic input from an equiprobable family. At the same time the new method has the advantage of PSD method, that is, compute is simple and precise. For linear time invariant system the results obtained by both method WDI and PSD are exactly consistent. Also, WDI method like SDG method can deal with time variant and non-linear problems. Therefore the WDI method can be regarded as the improvement of SDG method, and also can be regarded as the extension of PSD method. The following will particularly explain the main points of WDI method and its application to various systems. Before doing this it is necessary to introduce briefly the current PSD method.

The Current Method

The dynamic analysis means used currently by civil aviation regulation for continuous gust design rules is the PSD method that is based on spectrum theory. The main points of this analysis are as follows.

(1) Choose power spectral density function. For

atmospheric turbulence, von – ka'rm'an spectra or Dryden spectra are used usually. As an example for vertical gusts the von – ka'rm'an spectrum can be written as

$$\phi(\omega) = \frac{\sigma^2 T}{\pi} \frac{1 + \frac{8}{3}(1.339T\omega)^2}{[1 + (1.339T\omega)^2]^{11/6}} \quad (1)$$

and the Dryden spectrum as

$$\phi(\omega) = \frac{\sigma^2 T}{\pi} \frac{1 + 3(T\omega)^2}{[1 + (T\omega)^2]^2} \quad (2)$$

where T is the characteristic time that is related to turbulence scale (L) by equation

$$T = \frac{L}{V} \quad (3)$$

here V is the flight speed.

(2) Compute $\bar{A}$ and No. The factor $\bar{A}$ is the ratio of rms σ_y of response y to rms σ of gust velocity (turbulence velocity),

$$\bar{A} = \frac{\sigma_y}{\sigma} \quad (4)$$

And the characteristic frequency No can be expressed as

$$No = \frac{1}{2\pi} \frac{\sigma_{\dot{y}}}{\sigma_y} \quad (5)$$

where $\sigma_{\dot{y}}$ is rms of the rate $\dot{y}$ of response y ($\dot{y} = \frac{dy}{dt}$).

For linear time invariant system, according to the spectrum theory we have

$$\sigma^2 = \int_0^\infty \phi(\omega)\, d\omega \quad (6)$$

$$\sigma_y^2 = \int_0^\infty \phi(\omega)\, |G(i\omega)|^2\, d\omega \quad (7)$$

$$\sigma_{\dot{y}}^2 = \int_0^\infty \omega^2 \phi(\omega)\, |G(i\omega)|^2\, d\omega \quad (8)$$

here $G(i\omega)$ is the frequency response function of system.

(3) Compute design limits. If design envelope analysis is adopted, the design limit of gust load y can be determined by the following formula

$$y_{lim} = \bar{A} U\sigma \quad (9)$$

The values of design gust velocity $U\sigma$ are specified in Civil Aviation Regulation Part 25[1, 2].

If mission analysis is adopted, the threshold exceedance rate can be calculated as follows,

$$N(y_{th}) = \sum tNo\, [p_1 \exp(-\frac{|y_{th}|}{b_1 \bar{A}}) + p_2 \exp(-\frac{|y_{th}|}{b_2 \bar{A}})] \quad (10)$$

where $N(y_{th})$ is the average number of load peaks greater than a prescribed threshold level (y_{th}) per unit time, t is the percentage of flight time of selected mission section to the total flight time, and p_1, p_2, b_1, b_2 are given parameters (see ref. (1) or (2)). The limit load can be read out from the threshold exceedance rate curve by that the exceedance rate equals 2×10^{-5} per hour[1, 2].

The above method requires simple compute and is based on sound theory, so it is widely adopted. But it has following disadvantages:

(1) The PSD method is based on spectrum theory, its formulas are derived under conditions that the system is linear time invariant system, and the atmospheric turbulence is homogeneous and isotropic. Therefore it will be very difficult to apply this method to linear time variant system or inhomogeneous turbulence. And this method is incapable of dealing with non – linear problems.

(2) What computed by PSD method is the average characteristics of response such as σ_y, $\sigma_{\dot{y}}$, etc, but from the view point of flight safety the extreme cases are more interesting. Moreover the power spectral density function has lost the phase information of gust input, so the spectrum analysis can not determine the time characteristic of the worst response. These may results in many inconveniences for design and certification.

Improvement Proposal

Since the limitation of the current method results from its dynamic basis, the proposed improvement will mainly concern dynamic analysis. For linear time invariant system this means that only the computing of $\bar{A}$ and No will be improved.

Proposal: use the worst deterministic input (WDI) analysis as the dynamic basis for continuous gust design rules to replace the power spectral density (PSD) analysis.

WDI analysis follows the basic idea of SDG method, that is, study the stochastic response of a system by the effects of the worst deterministic input on the system under equiprobability condition. The main points of this analysis can be described as follows.

(1) In atmospheric turbulence with power spectral density $\phi(\omega)$, the gusts satisfying the spectral energy constraint will constitute an equiprobable family. Let a sample gust be $x_d(t)$ (a deterministic function of time t), its Fourier transform be $x_d(\omega)$, the equation

$$\pi \int_0^\infty \frac{|x_d(\omega)|^2}{\phi(\omega)}\, d\omega = U_0^2 \quad (11)$$

is called as spectral energy constraint equation. Here U_0 is intensity parameter. To analyze linear system the only case need to be discussed is $U_0 = 1$. In Appendix 1, it is explained that at least for Gaussian turbulence all gusts satisfying equation (11) will have equal probability to occur, that is, those gusts will constitute an equi probable family.

(2) The worst gust input that results in the maximum system response peak can be found from an equi probable gust family. In Appendix 2, it is shown that for a general linear system (include linear time variant system) the worst gust profile and the system response peak can be exactly

determined by variational principle. Let $h(t, t_m - t)$ be the value of the impulse response of a linear system at instant t_m to the unit impulse input at instant t and $G(\omega, t_m)$ be its Fourier transform relative to t, then the maximum response peak of a general linear system at instant t_m under spectral energy constraint will be

$$y_{max}(t_m) = [\pi \int_0^\infty \phi(\omega) \,|\, G(\omega, t_m) \,|^2 \, d\omega]^{\frac{1}{2}} \quad \cdots\cdots\cdots (12)$$

The Fourier transform of the worst gust profile $x_d(t, t_m)$ (a deterministic function of t with parameter t_m) will be

$$x_d(\omega, t_m) = \frac{1}{y_{max}(t_m)} \phi(\omega)\, G(\omega, t_m) \quad \cdots\cdots\cdots\cdots (13)$$

In Appendix 3, it is further shown that the worst case of a non-linear system may be found through the iteration procedure based on the above results for linear system.

(3) In linear case, the response peak of a system to the worst deterministic input is directly proportional to rms of stochastic response of that system. That is,

$$|\, y_{max}(t_m) \,|^2 = \frac{\langle y^2(t_m) \rangle}{\pi} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (14)$$

here $\langle y^2(t_m) \rangle$ is the variance of system stochastic response at t_m, considered as an ensemble average of $y^2(t_m)$.

The above WDI analysis is based on variational principle, hence need'nt hypotheses such as linearity and steadiness. So the limitation of current PSD analysis will be overcome, carving out a way for dealing with time variant and non-linear problems.

At the same time, since $\bar{A}$ and No are determined by rms, according to the point (3) above it would not be difficult to see that the same expression as current used can be derived by the new method for linear time invariant system. Hence the values of $\bar{A}$ and No can be computed by WDI method with the exactly equal precision and simplicity as that of PSD method.

Therefore, adopting the proposal of this paper can widely extend the application of the continuous gust design rules under such condition that the criteria and parameters even the calculating procedures (for linear time invariant system) used currently can be without change. So the developing continuity of design and certification system will be retained.

The Implementation

1. Linear Time Invariant System

According to WDI analysis, there are following results about the response of a linear time invariant system to homogeneous and isotropic atmospheric turbulence in frequency domain:

The worst response peak,

$$y_m = \frac{\sigma_y}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} [\int_0^\infty \phi(\omega) |\, G(i\omega) \,|^2 \, d\omega]^{\frac{1}{2}} \quad \cdots\cdots\cdots\cdots\cdots (15)$$

The worst deterministic input (with $t_m = 0$).

$$x_d(\omega) = \frac{1}{\pi y_m} \phi(\omega)\, G^*(i\omega) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (16)$$

where $G^*(i\omega)$ is the conjugate of frequency response function $G(i\omega)$ of the system. Equation (15) and (16) are, in fact, the special case of equation (12) and (13). Due to definition, in the case of linear time invariant system $G(\omega, t_m)$ is related to $G(i\omega)$ by equation,

$$G(\omega, t_m) = \frac{G^*(i\omega)}{\pi} e^{-i\omega t_m} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (17)$$

Substituting $G(\omega, t_m)$ by $\frac{G^*(i\omega)}{\pi}$, we get equation (15) and (16) from equation (12) and (13). The time profile $x_d(t)$ of the worst deterministic input can be obtained from $x_d(\omega)$ via inverse Fourier transform. FFT technique makes this very fast.

Obviously, for linear time invariant system the worst peak y_m is independent of t_m.

The results can also be expressed directly in time domain (See Appendix 2):

The worst response peak,

$$y_m = \frac{1}{\sqrt{\pi}} [\int_{-\infty}^{t_m} \int_{-\infty}^{t_m} h(t_m - t)\, h(t_m - v)\, R(t-v)\, dv\, dt]^{\frac{1}{2}} \quad \cdots\cdots (18)$$

or more simple

$$y_m = \frac{1}{\sqrt{\pi}} [\int_0^\infty \int_0^\infty h(\xi) h(\eta) R(\eta - \xi)\, d\eta\, d\xi]^{\frac{1}{2}} \quad \cdots\cdots\cdots\cdots\cdots\cdots (18a)$$

The worst deterministic input,

$$x_d(t) = \frac{1}{\pi y_m} \int_{-\infty}^{t_m} h(t_m - v)\, R(t-v)\, dv \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19)$$

or

$$x_d'(\xi) = \frac{1}{\pi y_m} \int_0^\infty h(\eta) R(\eta - \xi)\, d\eta \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19a)$$

where $h(\xi)$ is the impulse response function of the system, $R(\xi)$ is the auto-correlation function of the turbulence.

Noting the relationship between y_m and σ_y, it is not difficult to get

$$\bar{A} = \frac{\sqrt{\pi}\; y_m}{\sigma} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)$$

since the rate $\dot{y}$ of y ($\dot{y} = \frac{dy}{dt}$) has transfer function $sG(s)$, by the same reason one can obtain,

$$(\dot{y})_m = \frac{1}{\sqrt{\pi}} [\int_0^\infty \omega^2 \phi(\omega) \,|G(i\omega)\,|^2 \, d\omega]^{\frac{1}{2}} \quad \cdots\cdots\cdots\cdots\cdots (21)$$

here $(\dot{y})_m$ is the worst peak of the rate $\dot{y}$. Finally,

$$No = \frac{1}{2\pi} \frac{(\dot{y})_m}{y_m} \quad \cdots\cdots (22)$$

Then the design limit can be determined as before (see equation (9) and (10)).

Comparing the frequency domain formulas of WDI method for $\bar{A}$ and No with those of PSD method, it is obvious that the results obtained by both new method and current method agree exactly. Therefore in the extent of linear time invariant system, the WDI method not only has the same precision, but also is equally convenient as the PSD method.

Nevertheless, it should be emphasized that the WDI method has many advantages over the PSD method even for linear time invariant system.

Firstly, the WDI analysis focuses attention on the worst case, this is particularly meaningful for flight safety concerns. For instance, if a measured turbulence sample is inputted to a simulator for test or training, the simulation may not be very reliable and is time expensive, since the finite sample of a stochastic process may not contain the worst case and the establishment of steady response needs time. Directly inputting the worst gust can overcome those difficulties, it will produce more reliable results and be very fast.

Secondly, the proposed method not only can calculate important design parameters such as $\bar{A}$ and No, but also can determine the time history of the worst gust input and system response. This kind of time information is very useful in practice. For example, while a particular load reaches its maximum, the varying of all other related quantities can be computed by WDI method. If a particular load is critical for design, aircraft manufacturer can obtain a group of design loads by the proposed method, and loads the test aircraft with them in reality. The determinacy of time history of various loads makes it very convenient to consider the effects produced by structure deformation, fluid motion, etc. The consideration of those effects is required by civil aviation regulation, but it is very difficult to do this by the traditional PSD method.

2. Linear Time Variant System

The more important advantage of WDI method is that its application does not be restricted to linear time invariant system. In fact, the deduction in Appendix 2 is based on general linear system. Let $h(t, t_m-t)$ be the impulse response at instant t_m to the unit impulse input at instant t, and $G(\omega, t_m)$ be the Fourier transform of $h(t, t_m-t)$ with respect to t, then the worst case under spectral energy constraint can be expressed by equation (12) and (13) in frequency domain. The counterpart in time domain can be written as,

$$y_{max}(t_m) = \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{t_m} \int_{-\infty}^{t_m} h(t, t_m-t)\, h(v, t_m-v) R(t-v) dv dt \right]^{\frac{1}{2}} \quad \cdots\cdots (25)$$

and

$$x_d(t, t_m) = \frac{1}{\pi\, y_{max}(t_m)} \int_{-\infty}^{t_m} h(v, t_m-v) R(t-v) dv \quad \cdots\cdots (26)$$

For linear time variant system, the worst deterministic input and response peak are related to some selected critical instant t_m. So the mission analysis is not convenient in this case. But the design limit of response at t_m can still be determined by the design envelope analysis. It is not difficult to see that $\bar{A}$ is also a function of t_m

$$\bar{A}(t_m) = \frac{\sqrt{\pi}\; y_{max}(t_m)}{\sigma} \quad \cdots\cdots (27)$$

The design limit will vary with t_m too

$$y_{lim}(t_m) = U\sigma\; \bar{A}(t_m) \quad \cdots\cdots (28)$$

Obviously, the selection of t_m will depend on the problem in hand.

3. Non-Linear System

The WDI method can also be applied to non-linear system. In Appendix 3, an iteration method is proposed. If the iteration procedure converges, the worst input gust and response peak of a non-linear system can be obtained under the spectral energy constraint with intensity parameter Uo.

In the case of linearity according to superposition theorem, one can let Uo = 1, compute y_m, then takes

$$y_{lim} = U_{od}\; y_m \quad \cdots\cdots (29)$$

But superposition theorem can'nt be apply to a non-linear system, So direct compute using U_{od} should be carried out,

$$y_{lim} = y_{max}(U_{od}) \quad \cdots\cdots (30)$$

As to U_{od}, refering to equation (9), (20), and (29), it is not difficult to see,

$$U_{od} = \sqrt{\pi}\; \frac{U_\sigma}{\sigma} \quad \cdots\cdots (31)$$

Afterword

This paper proposes that the worst deterministic input (WDI) analysis can be used as the dynamic basis of continuous gust design rules to replace the current power spectral density (PSD) analysis. It has been explained that for linear time invariant system the WDI method can give results agreeing exactly with that of PSD method. In fact, WDI analysis covers PSD analysis. So the current criteria and parameters of regulation can be used by new method, and the continuity of development of design and certification system is retained.

Also it has been pointed out that the new method has many advantages, it gives the worst case with time informations, its application is greatly expanded, it can deal with linear time variant and non-linear problems which are the challenge faced by aviation industry today.

Therefore this proposal not only is easy to implement, but also has great value for development of aviation industry.

Although the SDG mathod can deal with linear time variant and non-linear problems, prescribed gust profile and a numerical searching procedure are used to find the worst gust. This makes not only the application complicated, but also time expensive. The WDI method follows the basic idea of SDG method, i. e. finding the worst gust from an equiprobable gust family to study the stochastic response of a system, but has exactly deduced the analytical formulas for the worst case. Those formulas can directly compute the worst gust profile and system response peak, making the compute as precise and fast as PSD method does. Therefore WDI analysis, in fact, covers the SDG analysis too, only more precise and convenient. In the case that direct computing is difficult, i. g. when the iteration procedure for a non-linear system does'nt converge, the numerical searching technique of the original SDG method can be used as an approximate means of WDI analysis.

In current design and certification system, the continuous gust loads and the discrete gust loads should be computed respectively, and check each other. The discrete gust method also prescribes gust profile (e. g. 1-cosin)[1, 2]. In the United States, 25 MAC is taken as gust length, the "resonance" effect isn't considered. In the United Kingdom, a "tuned" gust length is chosen among gusts with the same amplitude, "resonance" is only partly considered. The WDI method chooses the exactly "tuned" worst gust from a more reasonable equiprobable family, not only considers the "resonance" effect perfectly, but also carries out a discrete gust load analysis in fact. Therefore it is hopeful that the WDI analysis would combine the continuous gust load analysis and the discrete gust load analysis into an unified procedure, making the design and certification process greatly simplified. Of course, to change design rules is a serious matter, many works should be done before doing this. So although there is such unifying prospect, this paper concerns only continuous gust design rules first.

The proposed dynamic analysis can also be expanded to deal with inhomogeneous turbulence problems, for instance, the gust encounter during final approach. If a corresponding Dryden spectral density function is adopted for each altitude, an analytical expression of the autocorrelation function along flight path can be obtained (see reference (5) for detail). And the rms of system response at any given point of the flight path can be calculated.

Essentially, WDI method is an analytical method for stochastic system. Its application, of course, does'nt be limited to aircraft gust load analysis. In fact, the idea of the worst deterministic analysis came originally from Drenick's study on asseimic structures[6]. While knowing the spectrum or autocorrelation function of a stochastic input, WDI method can be applied to analyse all stochastic response. Therefore, it is worthy to study WDI method further to make it more sophisticated.

Appendix 1. Equiprobable Family and the Spectral Energy Constraint Equation

If the turbulence with power spectral density function $\emptyset(\omega)$ is Gaussian, then all gusts satisfying equation,

$$\pi \int_0^{\infty} \frac{|x_d(\omega)|^2}{\emptyset(\omega)} \, d\omega = U_0^2 \quad \cdots\cdots \quad (A1,1)$$

constitute an equiprobable family. Equation (A1,1) is called as spectral energy constraint equation with intensity parameter U_0 , and $x_d(\omega)$ is the Fourier transform of $x_d(t)$ that is a gust sample of turbulence.

To prove this result is not a trivial matter, it concerns path integral. What follows is an explanation which is intuitive and, hopefully, easier to understand.

(1) Gaussian White Noise

Suppose $x_w(t)$ is Gaussian white noise with zero mean value and autocorrelation function,

$$<x_w(t)\, x_w(t')> = K\delta(t-t') \quad \cdots\cdots \quad (A1,2)$$

Dividing the time axis by equi-spaced points $t_1, \cdots, t_{N+1}$ where $t_{j+1} - t_j = \Delta$, let the mean values be

$$x_{wj} = \frac{1}{\Delta} \int_{t_j}^{t_{j+1}} x_w(t)\, dt \quad \cdots\cdots \quad (A1,3)$$

which satisfy

$$<x_{wj}^2> = \frac{K}{\Delta} \quad \cdots\cdots \quad (A1,4)$$

Then x_{wj} is a Gaussian random variable with probability density

$$P\,(x_{wj}) = (2\pi \frac{K}{\Delta})^{-\frac{1}{2}} \exp\,[-\frac{x_{wj}^2}{2K}\Delta\,] \quad \cdots\cdots \quad (A1,5)$$

The joint probability density may be expressed as a product:

$$P(x_{w1}, x_{w2}, \cdots, x_{wN}) = P\,(x_{w1}) \cdot P\,(x_{w2}) \cdots P\,(x_{wN})$$

$$= (2\pi \frac{K}{\Delta})^{-\frac{N}{2}} \exp[-\frac{1}{2K} \sum_{j=1}^{N} x_{wj}^2 \Delta] \quad \cdots\cdots \quad (A1,6)$$

Writing the summation as an integral, it is infered from equation (A1,6) that the required probability functional is of the form

$$P[x_W(t)] = Z^{-1} \exp\,[-\frac{1}{2K} \int_0^{v} x_w^2(t)\, dt] \quad \cdots\cdots \quad (A1,7)$$

where v is the duration of the selected time interval and Z is introduced as a normalising factor. In order to satisfy the equation

$$\int P[x_W(t)]\, d\,[x_w(t)] = 1 \quad \cdots\cdots \quad (A1,8)$$

Z can be expressed as

$$Z = \int \exp\left[-\frac{1}{2K}\int_0^v x_W^2(t)\right] d[x_w(t)] \quad \cdots\cdots\cdots\cdots (A1,9)$$

The symbol $d[x_w(t)]$is analogous to the expression dx in the usual probability expression and has replace the volume expression $dx_{w1}dx_{w2}\cdots dx_{wN}$ which is implicit in the definition of probability density in equation (A1,6). The $d[x_w(t)]$ is obtained by applying a limiting process to the standard (Lebesque) measure $dx_{w1}dx_{w2}\cdots dx_{wN}$, and may be refered to as an infinite dimensional Lebesque measure.

Note that the integral in equation (A1,7) is the energy of $x_w(t)$ over the interval [0,v]. The sample function of interest concentrates their energy in finite time interval, i.e. $x_w(t) \to o$ as $t \to \infty$. Thus for a Gaussian white noise, the sample functions with equal energy have equal probability of occurrence with respect to the measure $d[x_w(t)]$.

So for Gaussian white noise the equiprobable family constraint equation can be written as follows,

$$\int_{-\infty}^{\infty} |x_{wd}(t)|^2 \, dt = C \quad \cdots\cdots\cdots\cdots (A1,10)$$

with C is a constant, $x_{wd}(t)$ is a particular realisation – a sample function of white noise, a subscript d is added to emphasize that this function is deterministic.
Specifically if

$$C = U_0^2 \, \emptyset x_w \, x_w \quad \cdots\cdots\cdots\cdots (A1,11)$$

where $\emptyset x_w x_w$ is the power spectral density of white noise, then the constraint equation in frequency domain can be written as follows by Parseval's theorem,

$$\pi \int_0^{\infty} \frac{|x_{wd}(\omega)|^2}{\emptyset x_w x_w} \, d\omega = U_0^2 \quad \cdots\cdots\cdots\cdots (A1,12)$$

here $x_{wd}(\omega)$ is the Fourier transform of $x_{wd}(t)$.

Equation (A1,12) is the special case of equation (A1,1). In consequence, for Gaussian white noise, samples satisfying the spectral energy constraint equation (A1,1) constitute an equiprobable family.

(2) Gaussian Coloured Noise

Now let x(t) be a Gaussian stochastic signal having power spectral density function $\emptyset(\omega)$. First consider the Dryden spectra composed of rational functions. It is well known that such a "coloured" noise signal can be generated physically by feeding a white noise signal $x_w(t)$ to a constant coefficient linear filter with transfer function $G_f(s)$. The parameters of the linear filter can be so chosen such that,

$$|G_f(i\omega)|^2 = \frac{\emptyset(\omega)}{\emptyset x_w x_w} \quad \cdots\cdots\cdots\cdots (A1,13)$$

Through such filter, there is a sample function $x_d(t)$ in the output "coloured" noise corresponding to the sample function $x_{wd}(t)$ in the input white noise, and

$$x_d(\omega) = G_f(i\omega) \, x_{wd}(\omega) \quad \cdots\cdots\cdots\cdots (A1,14)$$

where $x_{wd}(\omega)$ and $x_d(\omega)$ is the Fourier transform of $x_{wd}(t)$ and $x_d(t)$, respectively. Then

$$|x_{wd}(\omega)|^2 = \emptyset x_w x_w \frac{|x_d(\omega)|^2}{\emptyset(\omega)} \quad \cdots\cdots\cdots\cdots (A1,15)$$

Substituting this equation into equation (A1,12),

$$\pi \int_0^{\infty} \frac{|x_d(\omega)|^2}{\emptyset(\omega)} \, d\omega = U_0^2 \quad \cdots\cdots\cdots\cdots (A1,16)$$

So, if any sample function $x_{wd}(t)$ in input white noise to the filter satisfies the spectral energy constraint equation (A1,1), then its corresponding sample function $x_d(t)$ in the output "coloured" noise satisfies the same equation as well.

Since the outputs of such constant coefficient linear filter has a one to one corresponding with inputs, two patterns of $x_{wd}(t)$ in input white noise have equal probability to occur, then the two counterparts of $x_d(t)$ in output "coloured" noise have equal probability too. That is, an equiprobable family in input white noise will correspond an equiprobable family in output "coloured" noise.

It has been discussed that all sample functions $x_{wd}(t)$ satisfying spectral energy constraint equation constitute an equiprobable family in Gaussian white noise. It is not difficult to infer that all sample functions $x_d(t)$ satisfying spectral energy constraint equation also constitute an equiprobable family in Gaussian "coloured" noise.

As to von-Ka'rma'n spectra, it is possible to approximate a von-Ka'rma'n spectrum by rational function to any required precision.[7] For the rational spectrum obtained, it is possible to design a linear filter so that the "coloured" noise can be generated from Gaussian white noise. Therefore the argument presented above can be applied to such a rational spectrum. So spectral energy constraint equation (A1,1) defines an equiprobable family not only for Dryden spectra, but also for the approximate von-Ka'rma'n spectra, finally for von-Ka'rma'n spectra.

Appendix 2. The Worst Deterministic Input (WDI) for Linear System

This Appendix will explain how to deduce the worst case for linear system under spectral energy constraint by variational principle.

Consider a general linear system with input x(t) and output y(t). Assume that the input x(t) is a stationary stochastic signal having power spectral density function $\emptyset(\omega)$. The spectral energy constraint equation can be written as

$$\pi \int_0^{\infty} \frac{|x_d(\omega)|^2}{\emptyset(\omega)} \, d\omega = 1 \quad \cdots\cdots\cdots\cdots (A2,1)$$

where $x_d(\omega)$ is the Fourier transform of input $x_d(t)$, here the subscript "d" is added to emphasize that such a sample of x (t) is a deterministic function. According to definition equation (A2,1) can be rewritten as

$$\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}[\frac{1}{\pi}\int_{-\infty}^{\infty}x_d(t')e^{-i\omega t'}dt'][\frac{1}{\pi}\int_{-\infty}^{\infty}x_d(t)e^{i\omega t}dt]d\omega = 1$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}e^{-i\omega t'}e^{i\omega t}d\omega]x_d(t')x_d(t)dt'dt = 1$$

On the other hand, the zero initial value response of a linear system at instant t_m to an input x (t) can be written as

$$y(t_m) = \int_{-\infty}^{t_m} h(t, t_m - t)\, x(t)\, dt \quad \cdots\cdots (A2,3)$$

where $h(t, t_m - t)$ is the value of the impulse response at instant t_m to the unit impulse input at instant t. Since

$$h(t, t_m - t) = 0 \quad \text{for } t > t_m, \quad \cdots\cdots (A2,4)$$

(A2,3) can be rewritten as

$$y_d(t_m) = \int_{-\infty}^{\infty} h(t, t_m - t)\, x_d(t)\, dt \quad \cdots\cdots (A2,5)$$

Only sample functions of which energy is concentrated in a finite time region (i.e. they go to zero quickly as time goes to infinity) are to be considered. From the view of variational approach, the objective function (A2,5) and the constraint equation (A2,2) compose an isoperimetrical problem. The Hamiltonian function of this problem is as follows,

$$F = \int_{-\infty}^{\infty} h(t, t_m - t)\, x_d(t)\, dt - \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}e^{-i\omega t'}e^{i\omega t}d\omega]x_d(t')x_d(t)dt'\,dt \quad \cdots\cdots (A2,6)$$

where λ is a Lagrange multiplier which is a constant for isoperimetrical problem.

Noting that the second term of F is, in fact, a quadratic function of $x_d(t)$, then

$$\frac{\partial}{\partial x_d(t)}\{\frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}e^{-i\omega t'}e^{i\omega t}d\omega]x_d(t')x_d(t)dt'dt\} = \frac{\lambda}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}x_d(t')e^{-i\omega t'}e^{i\omega t}d\omega\, dt'dt$$

Therefore the Euler equation for this problem can be written as

$$h(t, t_m - t) - \frac{\lambda}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varnothing(\omega)}x_d(t')e^{-i\omega t'}e^{i\omega t}dt'd\omega = 0 \quad \cdots\cdots (A2,7)$$

or more simply

$$h(t, t_m - t) - \lambda\int_{-\infty}^{\infty}\frac{x_d(\omega)}{\varnothing(\omega)}e^{i\omega t}d\omega = 0 \quad \cdots\cdots (A2,8)$$

here $x_d(\omega)$ is the Fourier transform of $x_d(t')$. Let $G(\omega, t_m)$ be the Fourier transform of $h(t, t_m - t)$ with respect to t. Take Fourier transform of equation (A2,8), then

$$G(\omega, t_m) - 2\lambda\frac{x_d(\omega)}{\varnothing(\omega)} = 0 \quad \cdots\cdots (A2,9)$$

Hence the worst deterministic input obtained is

$$x_d(\omega, t_m) = \frac{1}{2\lambda}\varnothing(\omega)G(\omega, t_m) \quad \cdots\cdots (A2,10)$$

Since the worst input is a function of both argumets ω and t_m, $x_d(\omega)$ in (A2,9) is replaced by $x_d(\omega, t_m)$ in (A2,10).

Substituting (A2, 10) into constraint equation (A2,1), it is obtained that

$$\frac{1}{2\lambda} = \frac{1}{[\pi\int_0^{\infty}\varnothing(\omega)|G(\omega, t_m)|^2 d\omega]^{\frac{1}{2}}} \quad \cdots\cdots (A2,11)$$

And the worst response peak will be

$$y_{max}(t_m) = \int_{-\infty}^{\infty} h(t, t_m - t)\, x_d(t, t_m)\, dt$$

$$= \int_{-\infty}^{\infty}[\frac{1}{2}\int_{-\infty}^{\infty}G(\omega', t_m)e^{i\omega' t}d\omega'][\frac{1}{2}\int_{-\infty}^{\infty}x_d(\omega, t_m)e^{i\omega t}d\omega]dt$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\omega', t_m)x_d(\omega, t_m)[\int_{-\infty}^{\infty}e^{i(\omega'+\omega)t}dt]d\omega'\,d\omega$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\omega', t_m)x_d(\omega, t_m)\delta(\omega' + \omega)d\omega'\,d\omega$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty}G(-\omega, t_m)\, x_d(\omega, t_m)\, d\omega$$

Since $h(t, t_m - t)$ is real, so

$$G(-\omega, t_m) = G^*(\omega, t_m)$$

here $G^*(\omega, t_m)$ is the conjugate of $G(\omega, t_m)$. Therefore if the worst response at t_m exists, then

$$y_{max}(t_m) = \frac{\pi}{2}\int_{-\infty}^{\infty}G^*(\omega, t_m)\frac{1}{2\lambda}\varnothing(\omega)G(\omega, t_m)\, d\omega$$

That is

$$y_{max}(t_m) = [\pi\int_0^{\infty}\varnothing(\omega)|G(\omega, t_m)|^2\, d\omega]^{\frac{1}{2}} \quad \cdots\cdots (A2,12)$$

and

$$x_d(\omega, t_m) = \frac{1}{y_{max}(t_m)}\varnothing(\omega)G(\omega, t_m) \quad \cdots\cdots (A2,13)$$

Equations (A2,12) and (A2,13) describe the worst case under spectral energy constraint for linear system at instant t_m in the form of frequency domain.

In time domain, the worst deterministic input $x_d(t, t_m)$ can be obtained by inverse Fourier transform from $x_d(\omega, t_m)$. Also an analytical expression of $x_d(t, t_m)$ can obtained. Noting that $\varnothing(\omega)$ is the Fourier transform of autocorrelation function $R(\xi)$, consider the following expression,

$$f(t, t_m) = \frac{1}{\pi}\int_{-\infty}^{\infty} h(v, t_m - v) R(t-v)\, dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}\int_{-\infty}^{\infty} G(\omega, t_m) e^{i\omega v} d\omega\right]\left[\frac{1}{2}\int_{-\infty}^{\infty} \Phi(\omega') e^{i\omega'(t-v)} d\omega'\right] dv$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega, t_m)\Phi(\omega')\left[\int_{-\infty}^{\infty} e^{i(\omega-\omega')v} dv\right] e^{i\omega' t}\, d\omega'\, d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \Phi(\omega) G(\omega, t_m)\, e^{i\omega t}\, d\omega$$

It is obvious that $f(t, t_m)$ is the inverse Fourier transform of $\Phi(\omega)G(\omega,t_m)$. So the worst deterministic input $x_d(t,t_m)$ can be directly written as (noting condition (A2,4))

$$x_d(t,t_m) = \frac{1}{\pi y_{max}(t_m)}\int_{-\infty}^{t_m} h(v, t_m - v) R(t-v)\, dv \quad \cdots\cdots \quad (A2,14)$$

Also the worst response peak can be writlen as

$$y_{max}(t_m) = \int_{-\infty}^{t_m} h(t, t_m - t)\, x_d(t, t_m)\, dt$$

or

$$y_{max}(t_m) = \frac{1}{\pi y_{max}(t_m)}\int_{-\infty}^{t_m}\int_{-\infty}^{t_m} h(t, t_m - t)\, h(v, t_m - v)\, R(t-v)\, dv\, dt \quad \cdots\cdots \quad (A2,15)$$

Consider the variance of response at t_m ,

$$\langle y^2(t_m)\rangle = \left\langle \int_{-\infty}^{t_m} h(t, t_m - t)\, x(t)\, dt \int_{-\infty}^{t_m} h(v, t_m - v)\, x(v)\, dv \right\rangle$$

$$= \int_{-\infty}^{t_m}\int_{-\infty}^{t_m} h(t, t_m - t) h(v, t_m - v) \langle x(t) x(v)\rangle\, dv\, dt$$

That is

$$\langle y^2(t_m)\rangle = \int_{-\infty}^{t_m}\int_{-\infty}^{t_m} h(t, t_m - t) h(v, t_m - v) R(t-v)\, dv\, dt \cdots\cdots (A2,16)$$

Comparing (A2,15) and (A2,16) , we have

$$y_{max}^2(t_m) = \frac{\langle y^2(t_m)\rangle}{\pi} \quad \cdots\cdots \quad (A2,17)$$

The results obtained above can be deduced by different approach – Filter Method, see reference (9) for detail. Also in reference (9) , some examples are given and it is demostrated that the WDI method described above can determine the worst case precisely.

Appendix 3. The Worst Deterministic Input (WDI) for Non – Linear System

The worst deteministic input analysis for a general linear system presented in Appendix 2 provides a basis for an iteration method to obtain the worst case for non – linear problem.

Suppose a non – linear system is described by an ordinary differential equation

$$\dot{Y} = F(Y, x, t) \quad \cdots\cdots \quad (A3,1)$$

where y is the state output vector, x is a single input which can be regarded as a function of time with a group of parameters $a_K (K = 1, 2, \cdots)$. The zero initial value solutions of problem (A3,1) are functions of both x (t) and t,

$$y = y[x(t), t] = Y[a_1, a_2, \cdots, t] \quad \cdots\cdots \quad (A3,2)$$

Now assume that the functions $F(Y, x, t)$ are continuous and that the derivatives F_y and F_x exist. By formally differentiating equation (A3,1) with respect to any one parameter, say a_i , and reversing the order of differentiation, the derivatives Y_{ai} satisfy the following linear differential equation

$$Y_{ai} = F_y[Y(t), x(t), t]\, Y_{ai} + F_x[Y(t), x(t), t]\, x_{ai} \quad \cdots\cdots \quad (A3,3)$$

A linear equation from a non – linear one in this manner is refered as a variational equation of the original non – linear equation (A3,1)[10].

Let

$$A(t) = F_y[Y(t), x(t), t] \quad \cdots\cdots \quad (A3,4)$$

and

$$B(t) = F_x[Y(t), x(t), t] \quad \cdots\cdots \quad (A3,5)$$

then (A3,3) can be written as

$$Y_{ai} = A(t)\, Y_{ai} + B(t)\, x_{ai} \quad \cdots\cdots \quad (A3,6)$$

On the other hand, consider an associated linear system described by equation

$$Z = A(t) Z + B(t)\, x(t) \quad \cdots\cdots \quad (A3,7)$$

where Z is the state vector for this associated linear system. Its variational equation is

$$Z_{ai} = A(t)\, Z_{ai} + B(t)\, x_{ai} \quad \cdots\cdots \quad (A3,8)$$

which is the same as equation (A3,6). Since the solution of ordinary linear differential equations exists and is unique, hence

$$Y_{ai} \equiv Z_{ai} \quad (i = 1, 2, \cdots) \quad \cdots\cdots \quad (A3,9)$$

Thus it can be said that the variationals of both system relative to input x are the same, *i. e.*

$$\delta Y \equiv \delta Z \quad \cdots\cdots \quad (A3,10)$$

If an input x(t) and its corresponding trajectory Y(t) could be found such that the variation of the associated linear equation (A3,7) satisfies the Euler equation (A2,7), then the variation of the original non – linear equation (A3,1) along this trajectory satisfies the Euler equation (A2,7) as well. Therefore this x(t) is a possible worst deterministic input to the non – linear system.

Such an input can be found by the following iteration procedure:

(1) Assign some initial values to matrices A(t) and B (t), either give some constant values to $A_0(t)$ and $B_0(t)$, or assume an initial input $x_0(t)$ and then calculate the corresponding trajectory $Y_0(t)$ via numerically integrating the original non – linear equation (A3,1), then according to equation (A3,4) and (A3,5) determine the values of functional matrices $A_0(t)$ and

$B_o(t)$.

(2) The values of the impulse response of the associated linear system (A3,7) at time t_m are calculated for a sequence of instants before t_m, at those instants the unit impulse is input to the system. The sample points are chosen to get the required precision. The time extent covered should be such that outside it the impulse input has negligible effect on the response at t_m. And the interval between neighbouring points should be so small that the Fourier fransform will be precise enough.

(3) In the i^{th} $(i=1, 2, \cdots)$ circle of the iteration, the i^{th} approximate worst deterministic input $x_i(t)$ is determined by the technique presented in Appendix2 for the associated linear system (A3,7).

(4) The i^{th} approximate worst response trajectory $Y_i(t)$ of original non-linear system produced by $x_i(t)$ is calculated via numerical integration of the original non-linear equation (A3,1), and the i^{th} approximate maximum peak value of the objetive function $y_i(t_m)$ is obtained.

(5) Using $x_i(t)$ and $Y_i(t)$ obtained in steps (3) and (4), calculate $A_i(t)$ and $B_i(t)$ from equation (A3, 3) and (A3, 4), and repeat steps (2), (3), (4) until the final approximate maximum peak of the objective function satisfies

$$|y_i(t_m) - y_{i-1}(t_m)| \leqslant \varepsilon \qquad \text{(A3, 11)}$$

where ε is a specified tolerance.

If the above procedure converges, then the input x(t) and response trajectory Y(t) found in this way for the original non-linear system will give a variation of Y(t) relative to x(t) satisfying Euler equation (A2,7). Thus this x(t) is possibly the worst deterministic input for the original non-linear system.

In reference (9), an example of non-linear system was given. For the most cases calculated the iteration procedure do converge quickly and interesting results are obtained.

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