ANALYTIC CONTINUATION OF PADé APPROXIMATIONS TO THE UNSTEADY KERNEL FUNCTIONS TO OBTAIN A BETTER UNDERSTANDING OF THE ANALYTIC CONTINUATION OF PADé APPROXIMATIONS TO UNSTEADY PARAMETERS IN GENERAL

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Abstract

The suggestion has been made by Hounjet [1] and Hounjet and Eussen [2] that aerodynamic forces for harmonic motions may be obtained from analytic continuations of approximations to the aerodynamic forces for exponentially diverging motions. The aerodynamic forces are approximated by rational polynomials in the frequency parameter s, based on the assumption that any unsteady parameter of interest can be represented in this way. Although the assumption was substantiated by numerous examples, no physical explanation or justification was given. In this study, the procedure was applied to the unsteady subsonic kernel function, being a more basic unsteady parameter than generalized forces, to gain a better understanding of the approximation and its analytic continuation. The results obtained support the concept of analytic continuation, but also shows that the form of the approximation used here and by Hounjet and Eussen [2] can suffer from numerical instabilities. The techniques of this study can be applied to test the reliability of the application of analytic continuation to other unsteady parameters.

Introduction

Padé approximations to aerodynamic data for harmonic motions and their analytic continuation to oscillations of increasing or decreasing amplitude is routinely used in the design of active control systems [3]. Hounjet [1], Hounjet and Eussen [2] and much earlier Jones [4], suggested that the analytic continuation of the approximations is valid over the entire complex plane and that aerodynamic forces for harmonic motions may be obtained from analytic continuations of approximations to the aerodynamic forces for diverging motions. This would be very useful to extend the applicable frequency range of some iterative methods which show improved convergence for diverging motion as compared to harmonic motion.

The use of the kernel function rather than generalized force in a study of this nature has the following advantages :

1) the kernel function is cheaper to compute, making it possible to generate more data points in a given period of time,
2) the averaging effect of integration over a configuration is avoided,
3) the attenuation by distance in the calculation of influence coefficients is avoided,
4) high frequency behaviour can be investigated free from the influence of panel size.

In any given configuration, the kernel functions that are computed are much more numerous than the generalized forces. It would be a huge task to study a complete range of kernel function parameters. In stead, a set of parameters was chosen that is likely to occur in most configurations. The geometric parameters can be reduced to a radial distance in terms of wave lengths and the ratio of the x-coordinate to the radial distance. Since the wavelength was varied over a wide range, it was only necessary to select the ratio of x-coordinate to radial distance. It was taken as unity to correspond to two of the three kernel function evaluations used according to the method of Rodden et al. in [5] to calculate the influence coefficient of a square box on itself. A moderate Mach number of 0.5 was used throughout.

Nomenclature

K_1	Co–planar part of the kernel numerator
r	Radial distance $\sqrt{y^2+z^2}$
x,y,z	Cartesian coordinates
k	Reduced frequency $i\omega c/V$ (Note that k is imaginary for harmonic motion.)

s	Frequency parameter kr/c
ω	Angular frequency in rad/second
θ	Argument of frequency parameter defined by $s = \lvert s \rvert \exp(i\theta)$
c	Reference length, usually the semi–chord of a wing
V	True air speed
f_k	$F(s_k)$

Analysis

The kernel function was evaluated at 51 equally spaced values of radial distance from zero to $5/2\pi$ wavelengths. This maximum value corresponds to a radial distance of five times the reference length and a reduced frequency of unit magnitude. The approximations were performed not only for exponentially diverging motions, but also for harmonic motions and oscillatory motions of exponentially increasing amplitude. The expressions of [2] for the weighted error and the linear system had to be modified to make this possible. The unknown coefficients were restricted to be real to be compatible with approximations for exponentially diverging motion. The approximation is of the form

$$G(s) = \frac{F(0) + \sum_{i=1}^{n} A_i s^i}{1 + \sum_{j=1}^{m} B_j s^j} \qquad (1)$$

where F(s) is the function to be approximated and G(s) is the approximation. To be able to solve the coefficients from a linear system the error, i.e. the difference between F and G, is weighted by the denominator. The resulting expression for the weighted error is

$$E = \sum_{k=1}^{ndata} \left| (1 + \sum_{j=1}^{m} B_j s_k^j) f_k - f_1 - \sum_{i=1}^{n} A_i s_k^i \right|^2$$

$$= \sum_{k=1}^{ndata} \left\{ (Re(f_k) + \sum_{j=1}^{m} B_j Re(s_k^j f_k) - Re(f_l) - \sum_{i=1}^{n} A_i Re(s_k^i))^2 \right.$$

$$\left. + (Im(f_k) + \sum_{j=1}^{m} B_j Im(s_k^j f_k) - Im(f_l) - \sum_{i=1}^{n} A_i Im(s_k^i))^2 \right\} \qquad (2)$$

From this expression for the error, the linear equations from which the coefficients are solved are derived by differentiation of E with respect to each coefficient.

$$\sum_{i=1}^{n} A_i \sum_{k=1}^{ndata} \left\{ Re(s_k^i) Re(s_k^l) + Im(s_k^i) Im(s_k^l) \right\}$$

$$- \sum_{j=1}^{m} B_j \left\{ Re(s_k^j f_k) Re(s_k^l) + Im(s_k^j f_k) Im(s_k^l) \right\}$$

$$= \sum_{k=1}^{ndata} \left\{ (Re(f_k) - Re(f_l)) Re(s_k^l) + (Im(f_k) - Im(f_l)) Im(s_k^l) \right\} \qquad (3)$$

from the differentiation of E with respect to A_l for $l = 1...n$, and

$$- \sum_{i=1}^{n} A_i \sum_{k=1}^{ndata} \left\{ Re(s_k^i) Re(s_k^l f_k) + Im(s_k^i) Im(s_k^l f_k) \right\}$$

$$+ \sum_{j=1}^{m} B_j \left\{ Re(s_k^j f_k) Re(s_k^l f_k) + Im(s_k^j f_k) Im(s_k^l f_k) \right\}$$

$$= - \sum_{k=1}^{ndata} \left\{ (Re(f_k) - Re(f_l)) Re(s_k^l f_k) \right.$$

$$\left. + (Im(f_k) - Im(f_l)) Im(s_k^l f_k) \right\} \qquad (4)$$

from the differentiation of E with respect to B_l for $l = 1...m$. This formulation will be referred to as the complex formulation and the formulation of [2] for diverging motions as the real formulation. Where and error is shown, it is the root mean square value

$$e = \sqrt{E/ndata} \qquad (5)$$

The selection of the order of the numerator and denominator polynomials is clearly a trade–off between the freedom required to represent the original function and roundoff errors in the calculation of the coefficients. There is no reason why the numerator and the denominator should be of the same order, but it is assumed to be the case here to limit the number of variables that need to be investigated. The roundoff error was investigated by using both formulations where they are both applicable and by using single and double precision. The original data was always calculated using single precision and the 12.1 approximation of Desmarais [6].

Results

First, the methods described above were used to approximate the kernel function for diverging motion by a fourth order approximation as was used by Hounjet and Eussen [2]. The original function and the approximations are presented in figure 1, and the values of the coefficients as determined by the the four calculations in tables 1 and 2. Although the coefficients are very different, the approximations are all very accurate. This indicates that more freedom was allowed than is actually necessary to represent the data. The complex formulation was used to approximate data for harmonic motion, also using a fourth order approximation. The original function and the approximations are presented in figure 2, and the values of the coefficients as determined by the the two calculations in table 3. In this case, the approximations are not as accurate as those presented in figure 1, but there is little difference between the coefficients calculated using single and double precision. The approximation to the harmonic data was also performed using a sixth order approximation. In this case there was an appreciable difference between the single and double precision results as presented in table 4, but both approximations as presented in figure 3 are improvements on the fourth order approximations presented in figure 2. Second order approximations were also made to the diverging motion data. The results of the four calculations are presented in figure 4 and tables 5 and 6. There is little variation in the coefficients, but the approximations are not as accurate as those presented in figure 1.

Analytic continuation of the fourth order approximations from diverging motion to harmonic motion is presented in figures 5 and 6. Although the agreement of all four calculations with the direct calculation is good up to $|kr/c| = 0.7$ and reasonable up to 2, the variation among the calculations is considerable and the overall agreement is poor. The behaviour of the single precision calculation of the complex formulation is due to an almost imaginary root of the denominator at $kr/c = -0.0088 + i3.8707$. The continuation of the fourth order approximations of the harmonic data to diverging motion, presented in figure 7, shows good agreement with the direct calculation up to the maximum value of $|kr/c|$ considered. The continuation of the second order approximations of the data for diverging motion to harmonic motion is presented in figures 8 and 9 and shows that the agreement with directly calculated data is poor, much worse than the original fit to the diverging motion data.

Approximations of three different orders were performed at 31 values of θ equally spaced from 0 to 90 degrees. The variation of the coefficients and the error in the approximation was calculated. Double precision calculation of the complex formulation was used throughout. The variation of the coefficients of the second order approximation is presented in figure 10, of the fourth order approximation in figures 11 and 12 and of the sixth order approximation in figures 13 and 14. The variation of the error of all three approximations is presented in figure 15. The random variation in the coefficients decreases with increasing θ while the error increases. This is consistent with the earlier observations. The ideal situation would be for all the coefficients to have constant values over the whole range of θ.

To verify that the approximations to generalized forces behave in the same way, a simple model of a 45 degree swept wing with aspect ratio five was used. Two modes were considered, pitching about mid–chord and parabolic bending. The generalized force term Q_{21}, i.e. displacement mode two and pressure mode one, was approximated. Fourth order approximations were made using single precision calculations at θ values of 0, 3, 6 and 9 degrees. The coefficients are presented in table 7. The results of using these coefficients to calculate Q_{21} as a function of k for diverging motion are presented in figure 16. Despite the large differences in the coefficients, the approximations are all quite accurate apart from local deviations in the vicinity of denominator roots. The same was done for harmonic motion using approximations at 90, 87, 84 and 81 degrees. The results are presented in figures 17 and 18 and table 8. In this case the variation in coefficients is small and all the approximations are accurate. This suggests that the approximations to generalized forces behave similarly to the approximations to the kernel function.

Conclusions

The assumption that any unsteady parameter can be described by a rational polynomial in s was investigated by studying approximations of the unsteady subsonic kernel function. The analytic continuation of the approximations seems to be valid up to high values of kr/c, based on the satisfactory continuation from harmonic to diverging data.

Relatively low orders of approximations are required to approximate the data for diverging motions. Using higher degrees lead to large roundoff errors. Higher orders are required for harmonic motions, but roundoff errors are much smaller. This implies that it is fundamentally difficult to obtain the coefficients of the higher degree polynomials necessary to represent the data for harmonic motion from data for diverging motion.

The accuracy of the analytic continuations is much better than one would expect from the results presented in figures 10 to 14. No explanation for this arose from the present study.

It was shown that approximations to other unsteady parameters behave in the same way as the approximations to the kernel function. It would be interesting to subject the application of analytic continuation to other parameters to the same tests for reliability as was used here.

References

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[2] Hounjet, M.H.L. and Eussen, B.J.G., "Beyond the frequency limits of time–linearized methods", Paper 91–063, International Forum on Aeroelasticity and Structural Dynamics 1991.

[3] Karpel, M., "Design for Active and Passive Flutter Suppression and Gust Alleviation", NASA CR 3482, 1981.

[4] Jones, W.P., "Aerodynamic Forces on Wings in Non–uniform Motion", R. & M. No. 2117, August 1945.

[5] Rodden, W.P. , Giesing, J.P. and Kalman, T.P., "New developments and applications of the subsonic doublet lattice method for non–planar configurations", AGARD CP–80–71, Part 2, No. 4, 1971.

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Tables 1 to 8

Table 1 Coefficients of a 4th order real formulation approximation to diverging motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-1.476338	-0.918890
A2	0.061581	0.296288
A3	-0.084753	-0.055261
A4	0.019944	0.020295
B0	1.000000	1.000000
B1	1.221243	0.912482
B2	-0.156320	-0.428725
B3	-0.039565	0.029719
B4	0.004647	0.000099

Table 2 Coefficients of a 4th order complex formulation approximation to diverging motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-0.170438	-1.133942
A2	-0.452986	0.193147
A3	0.046697	-0.066940
A4	-0.027296	0.016396
B0	1.000000	1.000000
B1	0.440550	1.035640
B2	0.015315	-0.326791
B3	0.029264	0.013321
B4	-0.003467	0.000754

Table 3 Coefficients of a 4th order complex formulation approximation to harmonic motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-0.577598	-0.584281
A2	-0.122310	-0.123522
A3	-0.012346	-0.012662
A4	0.000140	0.000139
B0	1.000000	1.000000
B1	0.631858	0.635822
B2	-0.106269	-0.104490
B3	-0.003743	-0.004315
B4	0.001352	0.001411

Table 4 Coefficients of a 6th order complex formulation approximation to harmonic motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-1.594918	-0.842063
A2	2.363206	3.181038
A3	0.715709	0.991180
A4	0.228105	0.292891
A5	0.020584	0.027703
A6	0.002149	0.002670
B0	1.000000	1.000000
B1	1.288432	0.852079
B2	-1.479621	-2.091405
B3	-0.908566	-1.081111
B4	0.104160	0.113406
B5	0.004099	0.008369
B6	-0.000887	-0.001435

Table 5 Coefficients of a 2nd order real formulation approximation to diverging motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-0.173792	-0.175097
A2	-0.208869	-0.208970
B0	1.000000	1.000000
B1	0.399781	0.400860
B2	-0.056739	-0.056882

Table 6 Coefficients of a 2nd order complex formulation approximation to diverging motion data

	Single precision	Double precision
A0	-1.755929	-1.755929
A1	-0.172762	-0.175116
A2	-0.208790	-0.208971
B0	1.000000	1.000000
B1	0.398930	0.400875
B2	-0.056625	-0.056884

Table 7 Coefficients of a 4th order approximation to Q_{21} for almost purely diverging motion

θ	0 deg	3 deg	6 deg	9 deg
A0	20.4581	20.4581	20.4581	20.4581
A1	-142.9770	-17.4181	-169.3163	59.6590
A2	-45.7469	57.0835	-26.4926	49.0871
A3	-69.8963	6.5446	-4.7482	-15.8744
A4	54.1931	14.6839	37.0794	10.4362
B0	1.0000	1.0000	1.0000	1.0000
B1	-7.5071	-1.0195	-9.0951	3.0501
B2	0.2407	2.3956	3.3190	-0.7268
B3	0.0878	-0.0901	0.0054	0.3907
B4	0.5832	0.1845	0.4185	0.0478

Table 8 Coefficients of a 4th order approximation to Q_{21} for almost purely harmonic motion

θ	81 deg	84 deg	87 deg	90 deg
A0	20.4581	20.4581	20.4581	20.4581
A1	81.3256	80.3423	80.9829	80.7179
A2	97.4547	94.1549	93.9493	96.9731
A3	24.3588	20.0884	18.4428	24.7064
A4	-9.1733	-11.3806	-13.2105	-8.1168
B0	1.0000	1.0000	1.0000	1.0000
B1	4.1133	4.0583	4.0940	4.0804
B2	1.5903	1.4419	1.4031	1.5849
B3	-0.1022	-0.2161	-0.2943	-0.0613
B4	-0.0976	-0.1147	-0.1297	-0.0890

Figures 1 to 18

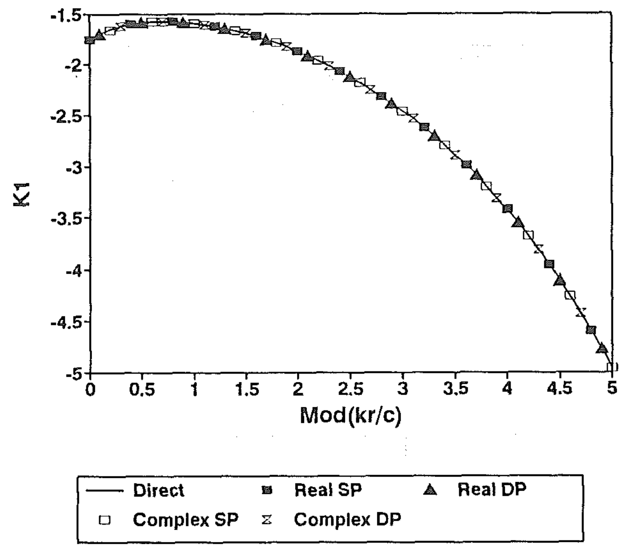

Figure 1 Fourth order approximations to the kernel function for diverging motion using the real and complex formulations and single and double precision.

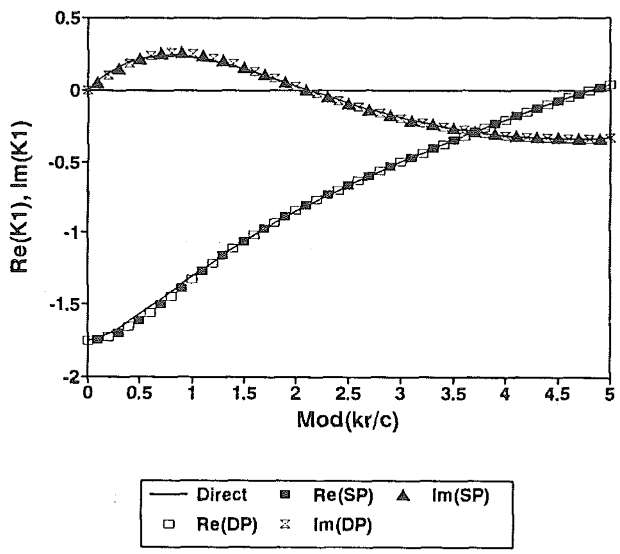

Figure 2 Fourth order approximations to the kernel function for harmonic motion using single and double precision.

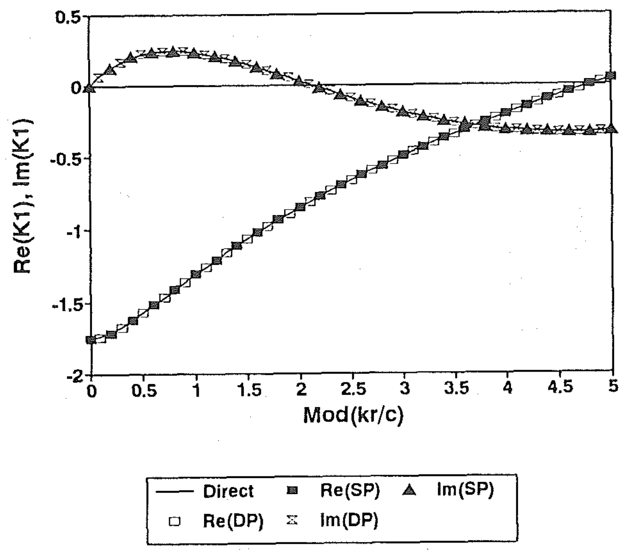

Figure 3 Sixth order approximations to the kernel function for harmonic motion using single and double precision.

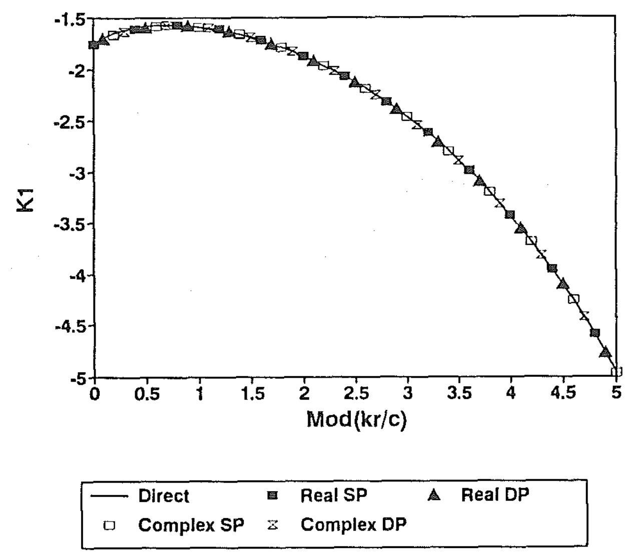

Figure 4 Second order approximations to the kernel function for diverging motion using the real and complex formulations and single and double precision.

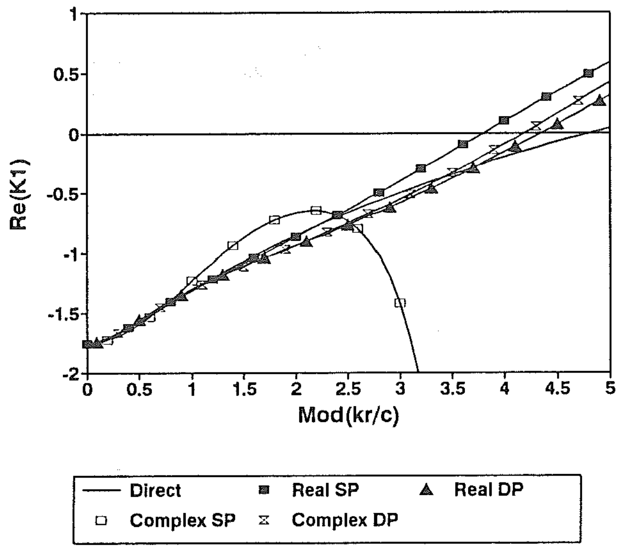

Figure 5 Analytic continuation of the fourth order approximations to the kernel function for diverging motion to harmonic motion, real part.

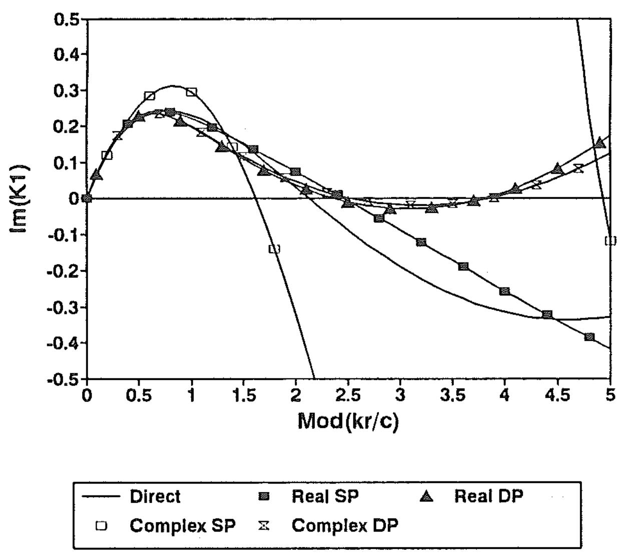

Figure 6 Analytic continuation of the fourth order approximations to the kernel function for diverging motion to harmonic motion, imaginary part.

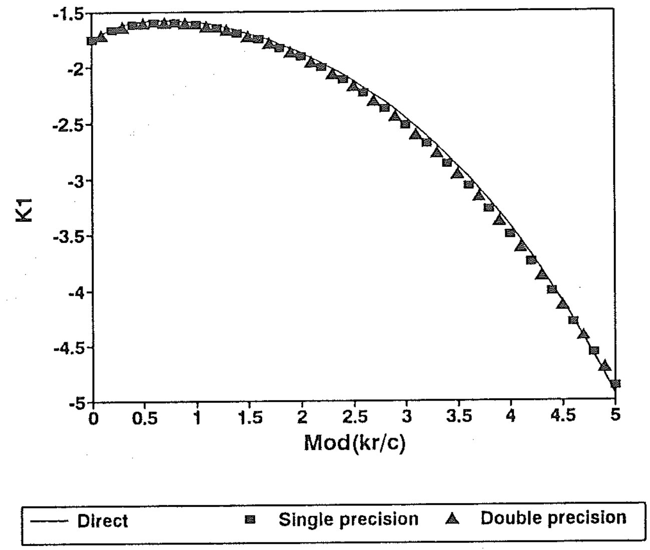

Figure 7 Analytic continuation of the fourth order approximations to the kernel function for harmonic motion to diverging motion.

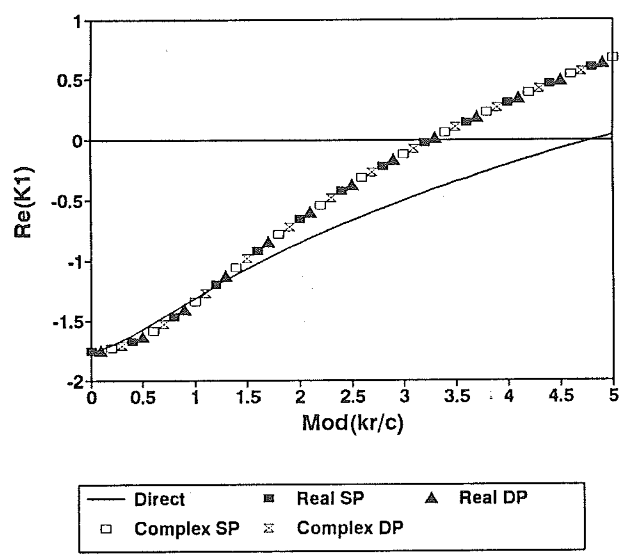

Figure 8 Analytic continuation of the second order approximations to the kernel function for diverging motion to harmonic motion, real part.

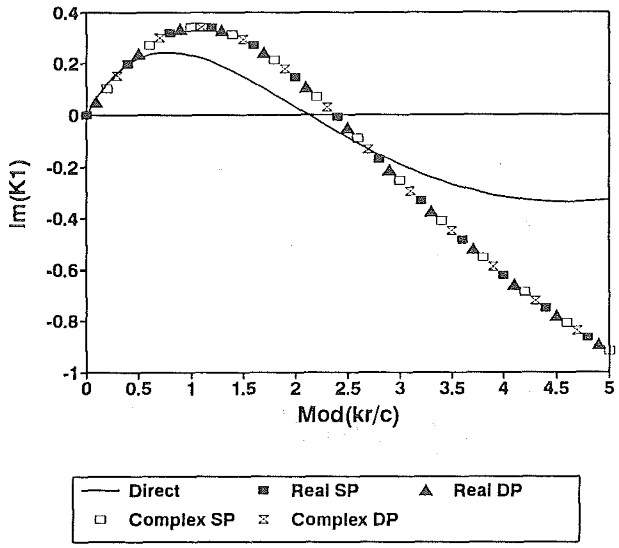

Figure 9 Analytic continuation of the second order approximations to the kernel function for diverging motion to harmonic motion, imaginary part.

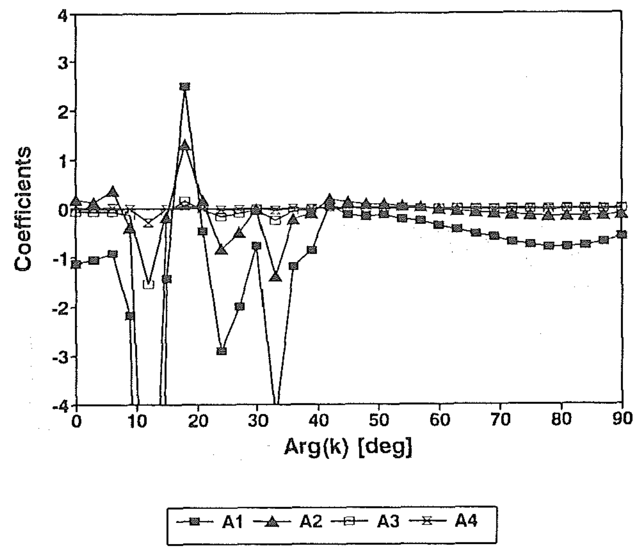

Figure 11 Variation with θ of the numerator coefficients of the fourth order approximation.

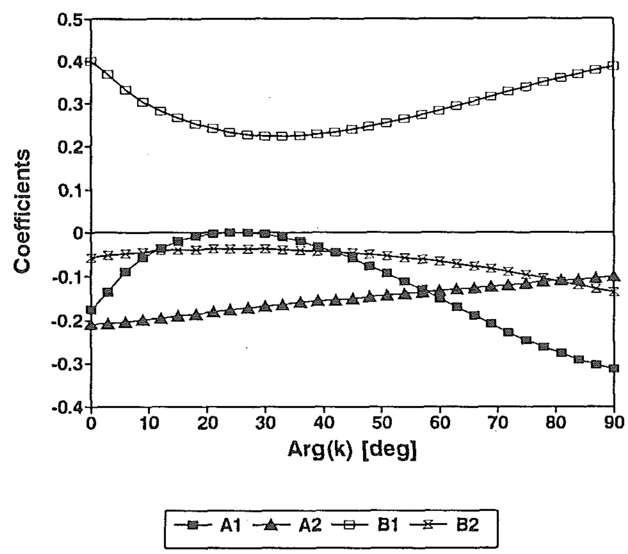

Figure 10 Variation with θ of the coefficients of the second order approximation.

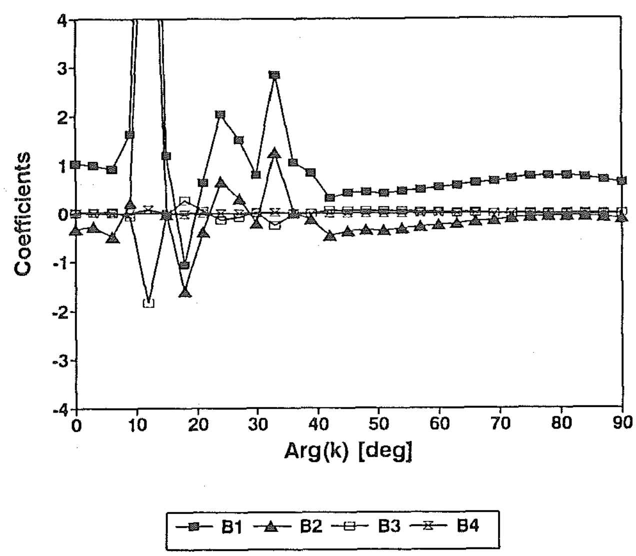

Figure 12 Variation with θ of the denominator coefficients of the fourth order approximation.

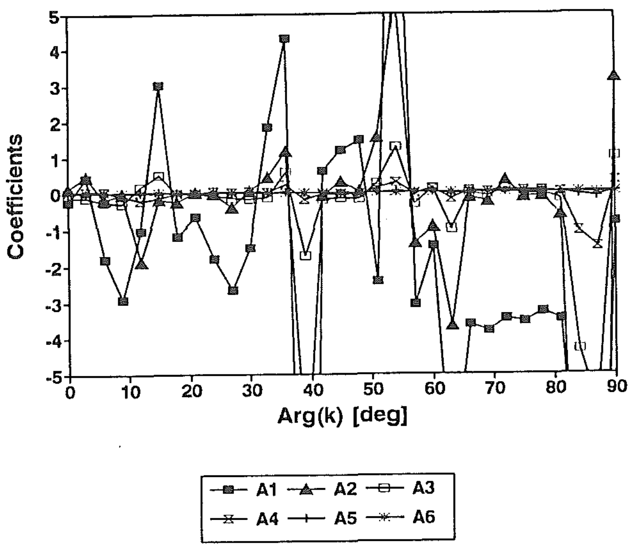

Figure 13 Variation with θ of the numerator coefficients of the sixth order approximation.

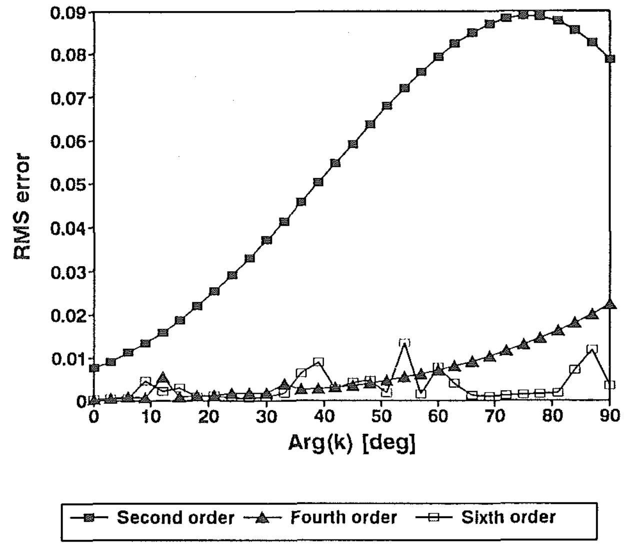

Figure 15 Variation with θ of the error for the second, fourth and sixth order approximations.

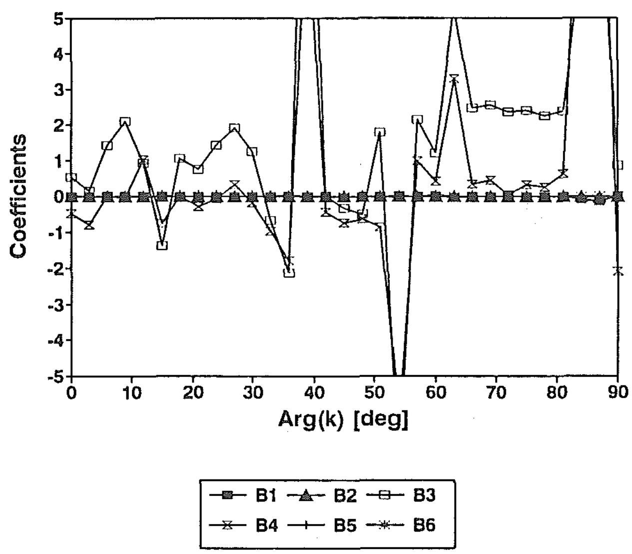

Figure 14 Variation with θ of the denominator coefficients of the sixth order approximation.

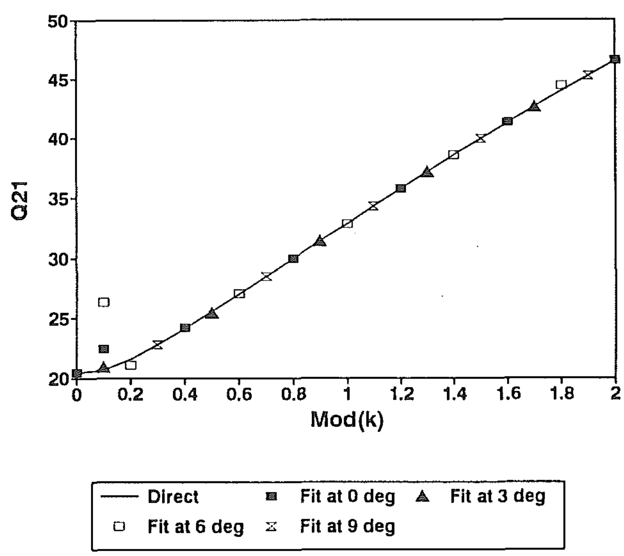

Figure 16 Generalized force Q_{21} for diverging motion calculated from approximations at θ of 0, 3, 6 and 9 degrees.

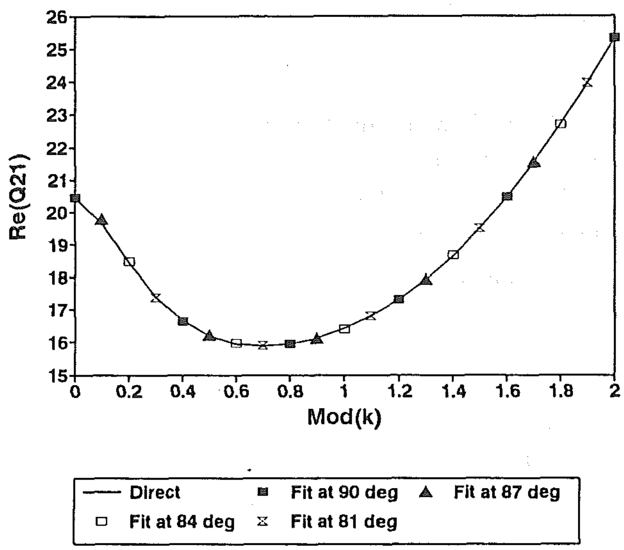

Figure 17 Generalized force Q_{21} for harmonic motion calculated from approximations at θ of 81, 84, 87 and 90 degrees, real part.

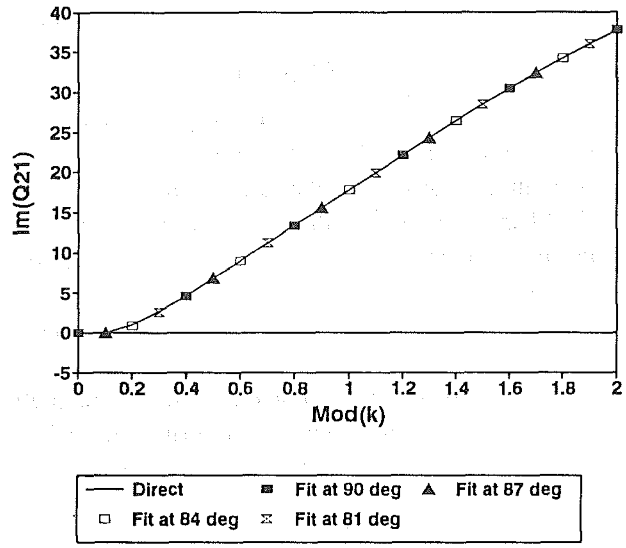

Figure 18 Generalized force Q_{21} for harmonic motion calculated from approximations at θ of 81, 84, 87 and 90 degrees, imaginary part.