# ON AN EXTENSION OF THE KUTTA-JOUKOWSKI THEOREM TO THE SUPERSONIC REGIME

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## Abstract

In this paper an analysis is made in the small perturbations approximation regarding the aerodynamic force that acts upon 2-D vortex distributions in uniform supersonic flow (i.e. the equivalent of the Kutta-Joukowski theorem for subsonic flows).

It is shown that in the case of finite strength vortex lines the induced velocity tends to infinity in the wave front and is zero elsewhere, thus infringing the perfect fluid hypothesis. It results that only vortices of infinitely small strength and distributions of these can be accounted for, by means of the formula :

$$d\overline{F} = \rho_\infty \overline{V}_t \times d\overline{\Gamma}$$

This result is proved to be completely compatible with the small perturbations theory of Ackeret.

## 1. Introduction

The force induced on an infinite vortex line by an uniform incompressible flow is given by the Kutta-Joukowski theorem :

$$\overline{F} = \rho_\infty \overline{V}_\infty \times \overline{\Gamma}$$

The aim of the present paper is to provide a formula of the Kutta-Joukowski type that could enable the direct calculation of the force that acts upon 2-D vortex distributions in supersonic flow. The approach is based on the small perturbations assumption and the results are therefore limited to the supersonic linearized regime.

The most commonly used method for calculating the loading on vortex distributions in supersonic regime within the small perturbations assumption takes into account that :

$$C_p = -\frac{2u}{V_\infty}$$

$$\gamma = \Delta u = 2u$$

and thus

$$C_p = -\frac{\gamma}{V_\infty}$$

Linkowski[1] used this approach in order to evaluate the lift on supersonic airfoils. The result is similar to the subsonic flow one :

$$L = \rho_\infty V_\infty \Gamma$$

The present paper intends to deal with the force in its entirety, i.e. with both its lift and drag components.

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## 2. General

In the supersonic linearized flow conditions the velocity induced by a vortex distribution can be calculated according to the formula[2] :

$$\overline{V}(M) = -\frac{1}{2\pi}\,\mathrm{curl^H} \int_{D'(M)} \frac{\overline{\Omega}(P)}{r^H}\, d\tau_P \qquad (1)$$

where $\overline{V}$ is the induced velocity : $\overline{V} = u\overline{i} + v\overline{j} + w\overline{k}$

M is the point of coordinates $(x_M, 0, z_M)$ where the induced velocity is calculated

P is the current point, of coordinates $(x_P, 0, z_P)$

$\overline{\Omega}$ is the vortex vector, given for a domain D (D'(M) is that part of domain D lying in the interior of the upstream Mach cone with apex M) and zero elsewhere

$$(r^H)^2 = (x_M - x_P)^2 - (M_\infty^2 - 1)[(y_M - y_P)^2 + (z_M - z_P)^2] \qquad (2)$$

$$\mathrm{curl^H}\,\overline{X} = \nabla^H \times \overline{X} = \left(-\frac{\partial X_3}{\partial y} + \frac{\partial X_2}{\partial z}\right)\overline{i} - \left[(M_\infty^2-1)\frac{\partial X_3}{\partial x} + \frac{\partial X_1}{\partial z}\right]\overline{j} + \left[(M_\infty^2-1)\frac{\partial X_2}{\partial x} + \frac{\partial X_1}{\partial y}\right]\overline{k} \qquad (3)$$

$$\nabla^H = (M_\infty^2 - 1)\frac{\partial}{\partial x}\overline{i} - \frac{\partial}{\partial y}\overline{j} - \frac{\partial}{\partial z}\overline{k} \qquad (4)$$

## 3. The velocity field induced by a 2-D vortex distribution in supersonic regime

The configuration of the 2-D domain containing the vortex distribution is represented in Fig.1. The domain D enclosing the vortex distribution is assumed to be cylindrical. The intersection of D with the plane y=0 is the surface S, bounded by curve $C_0$.

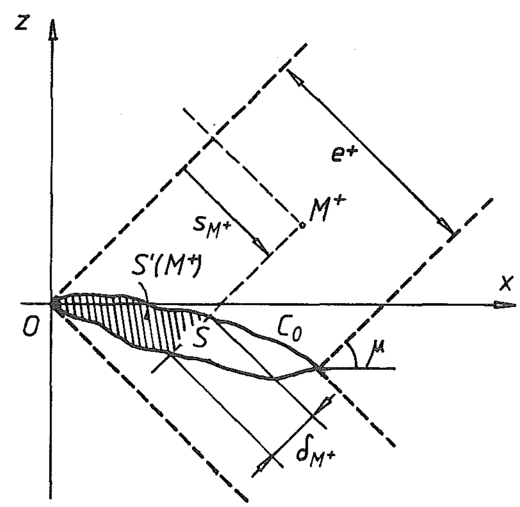


Fig.1

In order to comply with the small perturbations assumption, the surface S must be pointed at both leading and trailing edges, such as to be contained within the Mach cones issuing from the leading edge and the Mach forecones from the trailing edge.

Although the principal characteristics in Fig.1 depend on the position of point M above or below surface S (e.g. $e^+$), in the following argument the absence of the superscript will denote that this position is arbitrary and does not affect the reasoning.

Let us denote by $g(x_p, z_p)$ the local value of the vorticity density , so that

$$\overline{\Omega}(P) = g(x_p, z_p)\,\overline{J} \tag{5}$$

By means of (5) (1) becomes

$$\overline{V}(M) = -\frac{1}{2\pi}\,\mathrm{curlh}\int_{S'(M)} g(x_p, z_p)\,\overline{J}\,d\sigma_p\int_{-y_0}^{y_0}\frac{dy_p}{r^H} \tag{6}$$

where S'(M) is that part of S situated in the interior of the Mach forecone with vertex M, and thus influences M.

In the following we shall denote by

$$A^2 = (x_M - x_p)^2 - B^2(z_M - z_p)^2 \tag{7}$$

where

$$B^2 = M_\infty^2 - 1 \tag{8}$$

Since the integration in (6) is limited to S'(M), $A^2>0$ .

Introducing (7) and (8) in (2) it follows that

$$(r^H)^2 = A^2 - B^2\,y_p^2 \tag{9}$$

The ordinate $y_0$ in (6) indicates the point where $r^H$ becomes zero, and thus

$$y_0 = \frac{A}{B} \tag{10}$$

By means of (9) and (10), the induced velocity (6) can be expressed as

$$\overline{V}(M) = -\frac{1}{2\pi}\,\mathrm{curlh}\int_{S'(M)} g(x_p, z_p)\,\overline{J}\,d\sigma_p\int_{-\frac{A}{B}}^{\frac{A}{B}}\frac{dy_p}{\sqrt{A^2 - B^2 y_p^2}} \tag{11}$$

Integrating with respect to $y_p$ in (11) we obtain

$$\overline{V}(M) = -\frac{1}{2B}\,\mathrm{curlh}\,\overline{J}\int_{S'(M)} g(x_p, z_p)\,d\sigma_p \tag{12}$$

The integral in (12) represents the sum of vortices in S'(M)

$$\Gamma(s_M) = \int_{S'(M)} g(x_p, z_p)\,d\sigma_p \tag{13}$$

The coordinate $s_M$ gives the position of M with respect to the Mach front wave, measured normally to the wave, see Fig.1.

Observing that

$$\sin\mu = \frac{1}{M_\infty} \tag{14}$$

$$\cos\mu = \frac{B}{M_\infty} \tag{15}$$

$s_M$ can be expressed as

$$s_M = x_M\frac{1}{M_\infty} - |z_M|\frac{B}{M_\infty} \tag{16}$$

We shall notice that $s_M$ and hence $\Gamma(s_M)$ display different properties above ($s_M+$) and below ($s_M-$) S.

From (12) and (13), by taking into account (3), it follows that

$$\overline{V}(M) = -\frac{1}{2B}\left[\frac{\partial}{\partial z}\Gamma(s_M)\,\overline{I} + B^2\frac{\partial}{\partial x}\Gamma(s_M)\,\overline{k}\right] \tag{17}$$

Introducing (16) in (17) it results

$$\overline{V}(M) = \frac{1}{2}\frac{\partial\Gamma(s_M)}{\partial s_M}\,\overline{m} \tag{18}$$

where $\overline{m}$ is the unit vector normal to the wave front giving the orientation of the induced velocity

$$\overline{m} = \mathrm{sgn}(z_M)\frac{1}{M_\infty}\overline{I} - \frac{B}{M_\infty}\overline{k} \tag{19}$$

The flowfield due to the 2-D vortex distribution is depicted in Fig.2. Contrary to the source case[4] the induced velocities have different signs with respect to the Mach waves above and below surface S.

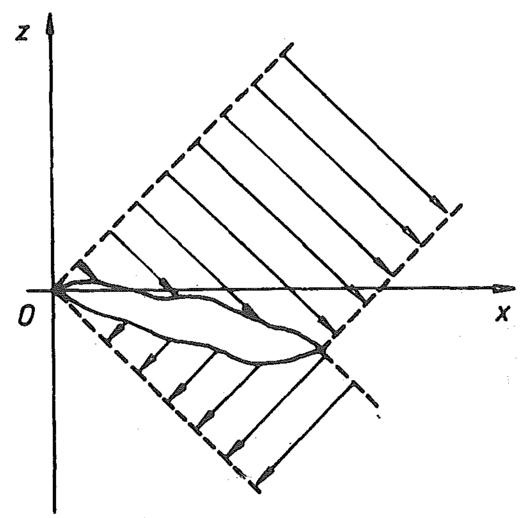


Fig.2

It will be observed that for $s_M>e$, it results $\Gamma(s_M) = \Gamma(e) = \Gamma_t$ i.e. a constant circulation, so that (18) implies $\overline{V}(M)=0$. Thus there are no perturbations behind the Mach cones issuing from the trailing edge of a supersonic airfoil. The same result is obtained in Ackeret's theory of 2-D supersonic small perturbations flow.

Let us denote by $\delta$ the "thickness" of S along the line s=constant, see Fig.1. Then, by applying the mean value theorem for the integration along the line $s_p$=const., (13) yields

$$\Gamma(s_M) = \int_0^{s_M} g(s_p)\,\delta(s_p)\,ds_p \tag{20}$$

where,

$$g(s_p)\,\delta(s_p) = \int_0^{\delta(s_p)} g(x_p, z_p)\,dt$$

Finally, the sum of vortices in S'(M) takes the form

$$\Gamma(s_M) = \int_0^{s_M} \gamma(P)\, ds_P \qquad (21)$$

where

$$\gamma(P) = g(x_P, z_P)\ \delta(P) \qquad (22)$$

By means of (22), (18) becomes

$$\overline{V}(M) = \frac{1}{2}\ \gamma(M)\ \overline{m} \qquad (23)$$

Let us calculate now the mean value of the induced velocity

$$\overline{V}_{mean} = \frac{1}{e} \int_0^e \overline{V}(P)\ ds_P \qquad (24)$$

Taking into account (18) the integration of (24) yields

$$\overline{V}_{mean} = \frac{\overline{m}}{2}\ \frac{\Gamma_t}{e} \qquad (25)$$

where $\Gamma_t$ is the total circulation :

$$\Gamma_t = \Gamma(e)$$

Similarly, one can obtain from (23) and (24)

$$\overline{V}_{mean} = \frac{\overline{m}}{2}\ \gamma_{mean} \qquad (26)$$

In order to obtain the result concerning vortex lines we must take the limit of S for $e \to 0$. In this situation (25) becomes

$$\overline{V}_{mean} = \frac{\overline{m}}{2}\ \lim_{e\to 0} \frac{\Gamma_t}{e}$$

For the case of a vortex line of finite strength ($\Gamma_t \neq 0$) it results an infinite mean induced velocity $V_{mean}$. This kinematic solution would contradict the perfect fluid assumption, since the pressure equation can be expressed as[5]

$$\frac{1}{2} V^2 + \frac{a^2}{\varkappa - 1} = \text{constant} \qquad (27)$$

relation that cannot account for infinite velocities .

Therefore we shall consider only vortex lines of infinitely small strength (e.g. $d\Gamma = \gamma dx$) so that, according to (25) and (26)

$$\gamma_{mean} = \lim_{e\to 0} \frac{\Gamma_t}{e}$$

From the small perturbations assumption it follows that $\gamma_{mean} \ll U_\infty$.

In addition it will be supposed that $\gamma$ satisfies the necessary continuity and boundedness conditions that enable the extension of all the properties of $\gamma_{mean}$ and $\overline{V}_{mean}$ discussed above to $\gamma$ and $\overline{V}(M)$.

## 4. The calculation of the force acting on a 2-D vortex distribution in supersonic regime

In order to determine the force acting on a vortex line under the assumptions considered, we shall apply the momentum theorem to the configuration depicted in Fig.3. As opposed to the arrangement used in[3], the leading and trailing edges of S lie no longer on the same axis Ox.

Let L and D be the components of the force with which the flow acts upon the vortex distribution along the axes Oz and Ox, respectively. Then, we can write[5]

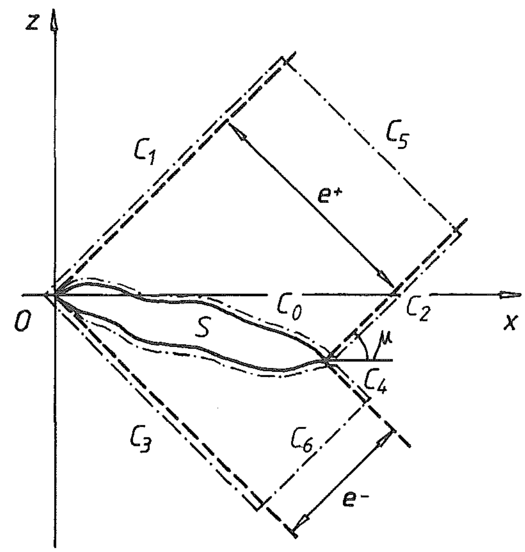


Fig.3

$$D\overline{i} + L\overline{k} = \int_{\sum_1^6 C} (p - p_\infty)\ \overline{n}\ ds + \int_{\sum_1^6 C} \rho\, \overline{V}_t\ (\overline{V}_t \overline{n})\ ds \qquad (28)$$

where $\overline{V}_t$ is the total velocity :

$$\overline{V}_t = U_\infty \overline{i} + \overline{V} \qquad (29)$$

and $\overline{V}$ is the induced velocity.

Next we shall observe that on $C_1$, $C_2$, $C_3$ and $C_4$ the induced velocities are zero, which leads to

$$D\overline{i} + L\overline{k} = \begin{cases} \displaystyle\int_{C_1+C_3} \rho_\infty (U_\infty \overline{i})(U_\infty \sin\mu)\, ds - \int_{C_2+C_4} \rho_\infty (U_\infty \overline{i})(U_\infty \sin\mu)\, ds \\ \\ \displaystyle - \int_{C_5+C_6} [(U_\infty + u)\overline{i} + w\overline{k}](U_\infty \cos\mu)\, ds + \int_{C_5+C_6} (p - p_\infty)\overline{n}\ ds \end{cases} \qquad (30)$$

Since

$$\int_{\sum_1^6 C} \rho_\infty (U_\infty \overline{i})([(U_\infty + u)\overline{i} + w\overline{k}]\overline{n})\ ds = 0$$

the drag D component along the Ox axis becomes

$$D = \begin{cases} \displaystyle - \int_{C_5+C_6} (p - p_\infty)\frac{B}{M_\infty}\, ds - \int_{C_5+C_6} (\rho - \rho_\infty)\ U_\infty^2 \frac{B}{M_\infty}\, ds \\ \\ \displaystyle - \int_{C_5+C_6} \rho\ u\ U_\infty \frac{B}{M_\infty}\, ds \end{cases} \qquad (31)$$

In the small perturbations approximation the following formulas apply[5]

$$p - p_\infty = - \rho_\infty U_\infty u \qquad (32)$$

$$\rho - \rho_\infty = -\rho_\infty \frac{u}{U_\infty} M_\infty^2 \tag{33}$$

By means of (32) and (33) the drag D can be expressed as

$$D = \int_{C_5+C_6} \rho_\infty u^2 M_\infty B \, ds \tag{34}$$

Taking into account (19) and (23), (34) becomes

$$D = \frac{\rho_\infty}{4} \frac{B}{M_\infty} \int_{C_5+C_6} [\gamma(P)]^2 \, ds_P \tag{35}$$

Integrating (35) one obtains

$$D = \frac{\rho_\infty}{2} \frac{B}{M_\infty} \gamma_1 \Gamma_t \tag{36}$$

where

$$\gamma_1 = \frac{\int_{C_5+C_6} [\gamma(P)]^2 \, ds_P}{\int_{C_5+C_6} \gamma(P) \, ds_P} \tag{37}$$

Finally, the drag takes the form

$$D = -\rho_\infty w_1 \Gamma_t \tag{38}$$

where $w_1$ represents a mean downwash associated to $\gamma_1$ (37) by means of (19) and (23) as follows

$$w_1 = -\frac{1}{2} \frac{B}{M_\infty} \gamma_1 \tag{39}$$

Equation (22) yields by projection on Oz axis

$$L = \left\{ \begin{array}{l} -\int_{C_5} (p-p_\infty) \frac{1}{M_\infty} ds + \int_{C_6} (p-p_\infty) \frac{1}{M_\infty} ds \\ \\ + \int_{C_5+C_6} \rho w \, U_\infty \frac{B}{M_\infty} ds \end{array} \right. \tag{40}$$

From (40) we obtain

$$L = \left\{ \begin{array}{l} \int_{C_5+C_6} \rho_\infty U_\infty \frac{1}{2M_\infty} \gamma(P) \, ds_P \; + \\ \\ + \int_{C_5+C_6} \rho_\infty (1 - \frac{u}{U_\infty} M_\infty^2) \; U_\infty \frac{B}{2M_\infty} \gamma(P) \, ds_P \end{array} \right. \tag{41}$$

Since as discussed previously $u \ll U_\infty$, (41) yields

$$L = \rho_\infty U_\infty \Gamma_t \tag{42}$$

The same result has been obtained by Linkowski[1] for airfoils in linearized supersonic flow using a different approach, mentioned in the introduction.

From (36) and (42) it can be inferred that

$$\frac{D}{L} = \frac{B}{M_\infty} \frac{\gamma_1}{2U_\infty} \tag{43}$$

The same ratio of drag to lift can be expressed by taking into account (38) and (42) as

$$\frac{D}{L} = -\frac{w_1}{U_\infty} \tag{44}$$

## 5. Conclusions

It must be observed that the force exerted by a supersonic uniform flow under the small perturbations assumption on a 2-D vortex distribution has a wave drag component directed along the freestream velocity, unlike in the incompressible case. From (43) and (8) it results that the wave drag tends to zero for $M_\infty \to 1$. This shows that for sonic speeds the drag obtained for the supersonic regime is in good correlation with the known result of zero drag (D'Alembert's paradox) characteristic for the subsonic regime.

We shall notice that according to formula (35) the drag is positive for any non-zero distribution of vorticity.

In the derivation of the above mentioned formulas it was assumed that the freestream velocity and the vorticity are perpendicular. If the angle $\vartheta = \angle(\bar{i}, \bar{\gamma}) \neq 90°$, the freestream velocity can be decomposed in a velocity normal to the vortex line, $U_\infty \sin\vartheta$, and another one along it. Obviously only the velocity normal to the vortex line interacts with it, so that the lift and drag become, respectively

$$L = \rho_\infty U_\infty \Gamma_t \sin\vartheta$$

and

$$D = -\rho_\infty w_1 \Gamma_t \sin\vartheta$$

Thus, it is possible to express the force on the vortex line similarly to the subsonic Kutta-Joukowski theorem

$$\bar{F} = \rho_\infty (U_\infty \bar{i} + w_1 \bar{k}) \times \bar{\Gamma}_t \tag{45}$$

In the small perturbations approximation $u \ll U_\infty$ and therefore the force exerted on a 2-D vortex distribution takes the form

$$\bar{F} = \rho_\infty \bar{V}_t \times \bar{\Gamma}_t \tag{46}$$

where $\bar{V}_t$ is given by (29).

In order to obtain the force on a vortex line, equation (46) will be considered in the limit $e \to 0$. In this case the strength of the vortex line becomes infinitesimal in order to preserve the basic assumptions of small perturbations and perfect fluid, as shown previously. The force generated by a supersonic flow on the vortex line results as

$$d\bar{F} = \rho_\infty \bar{V}_t \times d\bar{\Gamma}$$

This formula is somewhat similar to the subsonic one (the Kutta-Joukowski theorem). However, two differences must be pointed out :

1) only infinitely small strength vortex lines can be accounted for within the small perturbations assumption, and

2) the total velocity, including the one induced by the vortex under observation, has to be considered.

The latter characteristic might lead us to the conclusion that for supersonic flows the total velocity plays the same role as the freestream velocity for subsonic flows.

Moreover, the force is perpendicular to the total velocity which implies no force ("drag") along it[3].

For example, in the case of a 2-D flat plate modelled by means of a constant vortex distribution $\gamma$, the force can be written as

$$\bar{F} = \rho_\infty (U_\infty \bar{i} + \bar{V}) \times \gamma c \bar{j}$$

where c is the chord of the plate.

Since $U_\infty \bar{i} + \bar{V}$ is oriented along the flat plate $\bar{F}$ acts perpendicular to the plate, which implies that the lift is $L \doteq F$ while the drag is $D \doteq \alpha F$, as in Ackeret's theory.

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