ACCELERATED METHOD OF THE EULER EQUATION SOLUTION IN TRANSONIC AIRFOIL FLOW PROBLEM

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Abstract

An efficient approximate method of the Euler equation solution is developed. The present method makes it possible to accelerate the airfoil flow calculation 10 to 20 times as compared to the direct Euler equation solution and its speed is practically the same as the speed of the potential equation solution methods. The method needs only minor modifications of the existent numerical methods for the solution of the potential equation. The comparison of the results with the direct Euler equation solution shows a satisfactory agreement both for the distributed and total aerodynamic characteristics up to the formation of strong shock waves when the potential flow model gives essentially wrong results.

I. Introduction

Ideal gas flows are governed by the Euler equations which include the equations of mass, momentum and energy conservation. In the absence of shock waves in case of a uniform freestream flow over a body it is potential and governed by a single equation for the velocity potential. The gasdynamic variables in this case are determined from algebraic expressions for isentropic flows based on known velocity. The use of the potential model is a significant simplification and the numerical solution of the potential equation needs considerably less time than the solution of the Euler equations.

In the presence of shock waves the flow behind a shock becomes neither irrotational nor potential. The appearance of vorticity is due to an increase in entropy at the shocks in accordance with the Crocco equation. However, because the entropy variation is of the third order of magnitude relative to the shock strength, the vorticity in case of weak shocks may be neglected and the flow may be considered to be potential everywhere. The resultant equation for the potential function can be effectively solved with the aid of the widely used relaxation methods.

However, as the shock strength grows the assumption of the flow potentiality becomes increasingly less justified and the results obtained with the potential model agree increasingly worse with those obtained by the Euler equation solution. In airfoil flow problems it is especially obvious in the presence of nonzero lift: the lift coefficient may differ for these cases by a factor of three or more, the pressure distribution and the wave drag are absolutely different. Usually for the potential model the shock wave on the upper surface is located more downstream and is stronger than for the Euler solution. Different modifications of the potential model were proposed in order to take into account of the entropy jump across the shock waves[1-4]. These approaches improve agreement with the Euler equation solution but they are approximate and the Euler equations are essentially not solved.

The Clebsch transformation in[5] yields the system of equations equivalent to the Euler equations that includes the continuity and energy equations and two additional equations of transfer. It was proposed to solve this system numerically or to use an approximate analytical solution of one of the equations of transfer. The accelerated Euler equation solution method was developed on the basis of this approach that retains all the positive features of the potential model.

The present paper discusses some shortcomings of the method developed in[5]. A new method of the approximate Euler equation solution is proposed based on the Clebsch transformation. In essence, the present approach requires the solution of the potential equation with modified density. The entropy field behind shocks is determined by the calculation of streamline forms and from the condition of the entropy conservation along the streamlines. The conditions at normal shock waves are accurately satisfied in the present method.

II. Governing equations and numerical method

Consider a transonic isoenergetic ideal gas flow and let the velocity vector be written in the following form

$$\vec{V} = \nabla\Phi + \lambda\nabla\mu \qquad (1)$$

that is as a sum of the potential part $\nabla\Phi$ and the part $\lambda\nabla\mu$ that provides for nonzero vorticity. The functions λ and μ are selected so that the Euler equations should be satisfied. In case of the potential flow, $\lambda\nabla\mu$ must be equal to zero. This velocity vector presentation is reffered to as the Clebsch transformation. The Euler equations describing this flow may be written in the following form:

– the continuity equation

$$\nabla(\rho\vec{V}) = 0 \qquad (2)$$

– the Crocco equation

$$\vec{V} \times \vec{\omega} = -\frac{a^2}{\varkappa}\nabla S \qquad (3)$$

– the Bernoulli equation

$$a^2 = M_\infty^{-2} + \frac{\varkappa-1}{2}(1 - V^2) = p\,(M_\infty^2 \rho)^{-1} \qquad (4)$$

Here ρ is the density, $\vec{\omega}$ = rot $\vec{V}$ the vorticity vector, a the speed of sound, $\varkappa$ the specific heat ratio, $S = \frac{1}{\varkappa-1}\ln(p\,\rho^{-\varkappa})$ the entropy, p the pressure, M_∞ the free stream Mach number. All the quantities are nondimensionalized relative to the free-stream velocity, pressure and density (so the entropy in the free-stream flow and in the entire flow field except for the flow behind shocks is equal to zero). The equation (3) results from the Bernoulli and momentum equations and replaces the latter in the Euler equations. In accordance with (1)

$$\vec{\omega} = \nabla\lambda \times \nabla\mu$$

and (3) takes the form

$$\vec{V} \times \nabla\lambda \times \nabla\mu = \nabla\lambda\,(\vec{V}\,\nabla\mu) - \nabla\mu\,(\vec{V}\,\nabla\lambda) = -\frac{a^2}{\varkappa}\nabla S$$

Let as yet undetermined function λ be λ = S then we shall obtain

$$\nabla S\,(\vec{V}\,\nabla\mu) - \nabla\mu\,(\vec{V}\,\nabla S) = -\frac{a^2}{\varkappa}\nabla S$$

Since the entropy is conserved along the streamlines then

$$\vec{V}\,\nabla S = 0 \qquad (5)$$

and hence

$$\vec{V}\,\nabla\mu = -\frac{a^2}{\varkappa} \qquad (6)$$

Substitution of (1) with λ = S into the continuity equation (2) yields

$$\nabla(\rho\,\nabla\Phi) = -\nabla(\rho\,S\,\nabla\mu) \qquad (7)$$

In accordance with the entropy definition the density is equal to

$$\rho = e^{-S}\,(M_\infty^2\,a^2)^{\frac{1}{\varkappa-1}}$$

and the speed of sound a is related to the flow velocity by the Bernoulli equation (4).

The equation (7) contains a source term in the right-hand side. This may explain the fact that the nonconservative schemes for the numerical potential equation solution, when the shocks are sources of mass, in some cases gives a better agreement with the Euler solutions than the conservative schemes where the mass is conserved.

Integration of the equations of transfer (5) and (6) requires the specification of initial conditions at the shock wave. Since the shock is close to the normal shock in a transonic flow about an airfoil, the initial values of the entropy may be obtained from the relation at the normal shock, namely

$$S = \frac{1}{\varkappa-1}\left[\ln\left(\frac{2\varkappa}{\varkappa-1}M_{sh}^2 - \frac{\varkappa-1}{\varkappa+1}\right) + \varkappa\ln\left(\frac{\varkappa-1}{\varkappa+1} + \frac{2}{\varkappa+1}M_{sh}^{-2}\right)\right]$$

where M_{sh} is the Mach number immediately ahead of the shock. If the function Φ is continuous across the shock, as in the potential flow, the condition μ = const must be satisfied along the shock, which provides the conservation of the velocity component tangential to the shock.

The above approach was developed in[5] and applied to the construction of a fast Euler equation solution method. The algorithm of the method is as follows. 1) The entropy is assumed to be zero in the entire flow field; 2) the equation (7) is solved by using the conservative finite-difference scheme; 3) the shock locations and the entropy jumps at them are determined; 4) the equations (5) and (6) are solved and the functions S and μ are determined; 5) the steps 2-4 are repeated until the convergence is achieved. It was proposed to obtain the solution of the equation (6) in[5] either by a direct numerical integration or approximately, by assuming that $\frac{\partial\mu}{\partial x} = -(\varkappa M_\infty^2)^{-1}$; $\frac{\partial\mu}{\partial y} = 0$.

The numerical integration of (6) and the subsequent differentiation of the function μ leads to some difficulties. Let us introduce a local orthogonal curvilinear coordinate system (s,n) fixed to the streamlines, where s is the arc length along a streamline and n is the distance normal to it. The metric tensor components are equal to $g_{ss} = (1 + Kn)^2$, $g_{nn} = 1$, where K is the streamline curvature. In this coordinate system eq. (6) may be written as

$$\mu_s = \frac{\partial\mu}{\partial s} = -(\varkappa M^2)^{-1}\,V \qquad (10)$$

where M is the local Mach number. Hence the μ value unrestictly grows along the streamlines because the right-hand side of (10) tends far downstream to a finite nonzero value (being different for various streamlines). Let us show that $\mu_n = \frac{\partial\mu}{\partial n}$ also tends to infinity far downstream along the streamlines past the shock. In accordance with (4) an equality may be written

$$\frac{\partial}{\partial s}\mu_n = \frac{\partial}{\partial n}\mu_s = -\frac{\partial}{\partial n}[(\varkappa M^2)^{-1}\,V] = \frac{M^{-2}+(\varkappa-1)}{\varkappa}\frac{\partial V}{\partial n}$$

The vorticity in coordinates (s,n) may be written as

$$\omega = -\frac{\partial V}{\partial n} - KV \qquad (11)$$

and

$$\frac{\partial\mu_n}{\partial s} = -\frac{M^{-2}+(\varkappa-1)}{\varkappa}(\omega + KV) \qquad (12)$$

Far downstream the right-hand side of (12) tends to a finite nonzero value different for various streamlines past the shock, which results in an unrestricted growth of μ_n along these streamlines.

Thus the function μ and its derivative normal to the streamline unrestrictly grows along the streamlines past the shock, and the character of this growth is different for various streamlines. This fact makes the use of this solution extremely difficult in the velocity approximation in the form of (1) as was proposed in[5].

In the present paper it is proposed to use the vector $-\frac{\vec{V}}{\varkappa M^2}$ instead of $\nabla\mu$ in (1), where λ = S in accordance with (10), e.g.

$$\vec{V} = \nabla\Phi - S\ (\varkappa M^2)^{-1}\ \vec{V} \qquad (13)$$

An estimation of the errors of such velocity presentation is given below. It follows from (13) that

$$\vec{\omega} = (\varkappa M^2)^{-1}[\nabla S \times \vec{V}] - S\ (\varkappa M^2)^{-1}\ \vec{\omega} - \frac{S}{\varkappa}\ [\nabla M^{-2} \times \vec{V}]$$

The substitution of this expression into (3) yields

$$\vec{V} \times \vec{\omega} = -\frac{a^2}{\varkappa}\nabla S - S\ (\varkappa M^2)^{-1}\ [\vec{V} \times \vec{\omega}] - \frac{S}{\varkappa}\ [\vec{V} \times \nabla M^{-2} \times \vec{V}] \qquad (14)$$

Comparing (3) and (14) we obtain that the sum of the second and the third terms in the right-hand side of (14) is an error of the Crocco equation due to the velocity representation (13). As S is of the third order of magnitude with respect to the shock strength, the second term has higher order of magnitude than the leading term having the order of magnitude of ω and the second term may be neglected. Let us assess the third term:

$$\nabla M^{-2} = -\ (M_\infty^{-2} + \frac{\varkappa-1}{2})\ \frac{\nabla V^2}{V^4} = -\ A\ \frac{\nabla V^2}{V^4}$$

In the (s,n) coordinate system we have

$$\nabla M^{-2} \times \vec{V} = -\ \vec{k}\ 2AV^{-2}\ \frac{\partial V}{\partial n}$$

and

$$\vec{V} \times \nabla M^{-2} \times \vec{V} = -\ [\vec{V} \times \vec{k}]\ 2AV^{-2}\ \frac{\partial V}{\partial n}$$

where $\vec{k}$ is the unit vector in the z-direction normal to the flow plane. Using (11) we obtain an assessment of the absolute value of the third term in (14) in the form

$$\frac{S}{\omega}\ |\vec{V} \times \nabla M^{-2} \times \vec{V}| = \frac{2AS}{\varkappa V}\ (\omega + KV)$$

Again, the product of $S\omega$ may be neglected. The term with the curvature K is formally of the leading order of magnitude, but, as a rule, the streamline curvature downstream the shock is small enough and this term is also small.

The flow velocity representation (13) allows us to simply modify the known transonic potential methods in order to account for the vorticity downstream the shock. It follows from (13) that

$$\vec{V} = \frac{\nabla\Phi}{1+S(\varkappa M^2)^{-1}} \qquad (15)$$

It is obvious that this representation is erroneous in the vicinity of the points, where V = M = 0, at the trailing edge, for example. But the numerical results given below showed that these errors doesn't have a global effect on the results.

The Bernoulli equation may be written in the form

$$V^2(1-\frac{\varkappa-1}{2\varkappa}S) - V|\nabla\Phi| + \frac{S}{\varkappa}(M_\infty^{-2}+\frac{\varkappa-1}{2}) = 0 \qquad (16)$$

Substituting (8) and (15) into (2) we eventually obtain

$$\nabla\left[\rho_i\ e^{-S}\ \frac{1}{1+S(\varkappa M^2)^{-1}}\ \nabla\Phi\right] = 0 \qquad (17)$$

where $\rho_i = (M_\infty^2\ a^2)^{\frac{1}{\varkappa-1}}$ is the density calculated by the isentropic relation. Thus, the approximate solution of the Euler equations is reduced to the solution of (17) that has the structure of the potential equation $\nabla(\hat{\rho}\ \nabla\Phi)$ with the modified density

$$\hat{\rho} = \rho_i\ e^{-S}\ [1 + S\ (\varkappa M^2)^{-1}]^{-1} \qquad (18)$$

and may be effectively solved by the relaxation schemes. Note that the density modifications are required only behind shocks. The entropy values are determined by the calculation of streamlines and by the use of the condition of the entropy conservation along the streamlines. The velocity value is calculated from (16) using known values of S and $|\nabla\Phi|$ at any particular point. This approximate method of the Euler equation calculation is practically as fast as the potential methods and enables a better approximation to the Euler equations than the potential model, because it takes into account the nonzero vorticity downstream the shocks. Moreover, this method accurately satisfies the Rankin-Hugoniot relations at normal shocks. Actually the mass and total enthalpy are conserved across the shocks and the entropy jump is directly calculated from the Rankin-Hugoniot conditions for normal shocks. The method is approximate because the Crocco equation contains some additional terms which may be neglected, as mentioned above.

The pressure (referred to p_∞) is obtained from

$$p = e^{-S}(M_\infty^2\ a^2)^{\frac{\varkappa}{\varkappa-1}} \qquad (19)$$

The determination of the function Φ from (17) requires the statement of the boundary conditions. For the isentropic flow those conditions are zero normal velocity condition at the airfoil boundary, Kutta-Joukovsky condition for the circulation determination, fixed asymptotic behavior of the potential at infinity and the condition of the flow continuity at the body streamline behind the airfoil. In the nonisentropic flow the pressure at infinity is constant. In this case, the velocity at infinity is various for different streamlines passed across the shock. At the body streamline behind the airfoil the pressure is continuous and the velocity is discontinuous. That is why the boundary conditions for the equation (17) require an additional investigation. The function Φ in accordance with (15) must satisfy the usual condition $\frac{\partial\Phi}{\partial n} = 0$.

Let us find out the boundary conditions for Φ at infinity and at the body streamline behind the airfoil. Further should be assessed the difference of the pressure from the value calculated by isentropic relations when $S = 0$. Regarding S as a small parameter and retaining only the first order terms we can obtain from (19) the following expression:

$$p - p_i = - p_i \left(S + \frac{\varkappa M_i^2}{V_i} \Delta V\right) \qquad (20)$$

where $\Delta V = V - V_i = O(S)$, and subscript i corresponds to the isentropic values for $S = 0$. Let us show that $|\nabla\Phi| \to 1 + O(S^2)$ for $r = \sqrt{x^2+y^2} \to \infty$ along any direction. Actually, in accordance with (15) we have

$$|\nabla\Phi| - |\nabla\Phi_i| = \Delta V + V_i\, S\, (\varkappa M_i^2)^{-1} + O(S^2)$$

If we require the pressure to be homogeneous at infinity, i.e. $p = p_i$ for $r \to \infty$, from (20) we shall obtain

$$|\nabla\Phi| = |\nabla\Phi_i| + O(S^2)$$

for $r \to \infty$ and the value $|\nabla\Phi|$ is constant at infinity in the first order with respect to S as for the isentropic case.

The representation (15) also provides the continuity of the vector $\nabla\Phi$ in the first order with respect to S along the body streamline behind the airfoil. Generally speaking, velocity, density and entropy are discontinuous and pressure is continuous along this streamline. It follows from (19) and the pressure continuity condition that

$$\delta p = - p_m \left(\delta S + \frac{\varkappa M_m^2}{V_m} \delta V\right) + O(\delta S^2)$$

where δp, δV and δS are the small jumps of corresponding values at the body streamline behind the airfoil, and the subscript m marks the values by respect to which an expansion is performed (mean values at the body streamlinemay be used for these parameters, for example). From (15) we can obtain

$$\delta|\nabla\Phi| = \delta V + \frac{V_m^2}{\varkappa M_m^2} \delta S + O(\delta S^2) = $$

$$= \frac{\varkappa M_m}{V_m} \delta p + O(\delta S^2) = O(\delta S^2)$$

Hence, the pressure continuity enables the $|\nabla\Phi|$ continuity in the first order (and, consequently, the continuity of the vector $\nabla\Phi$ in the same order, because the velocity vector is tangential to the streamline).

So, the boundary conditions for the boundary-value problem for the function Φ are formulated similarly to the conditions in isentropic flows. It was also assumed that the asymptotic expression for Φ includes a term proportional to the circulation as in case of the isentropic model, and the Kutta-Joukovsky condition was used to determine the circulation. Hence, the problem formulated coincides with the potential calculation problem. Only two modifications are needed, namely: 1)the entropy determination behind shocks and 2)the density modification in this region. The numerical method must be conservative, because the additional sources of mass in nonconservative schemes may significantly change the solution.

On the basis of the present method, a numerical code of the transonic airfoil calculation was developed. The algorithm was the following: 1)the entropy is set to zero in the entire flow field; 2)a number of iterations of the numerical solution of (17) are performed by using the finite difference scheme from[6]; 3)shock locations and strengths are determined and entropy jumps across the shocks are calculated; 4)streamline forms are calculated; 5)entropy values at mesh nodes are determined by linear interpolation; 6)modified density values at the mesh cell centers are determined by (18) and velocity values are calculated by (16); 7)the items 2-6 are repeated until the convergence is reached.

The mesh was constructed by a conformal mapping of the flow field onto the unit circle. The pressure was determined by (19). The aerodynamic force and pitching moment coefficients were obtained by pressure integration over the airfoil contour.

III Results

The results obtained with the present method (grid 80×16) were compared with the AGARD test cases given in[7] (method No.9 from[7], grid 320×64). Figs. 1 to 3 show some of these comparisons. In Fig.1 a result obtained with the potential model is also shown. This result was obtained with the same code, when the entropy was set to zero in the entire flow field. For other test cases it turned out to be impossible to carry out the potential calculations because of the divergence problems: the shocks were too strong and too close to the trailing edge. Fig.1 presents the results for the NACA0012 airfoil at $M_\infty = 0.8$, $\alpha = 1.25^\circ$. The result obtained by the present method is in good agreement with the Euler equation solution both for the pressure distribution and for the total characteristics. In this case, the Mach number ahead of the shock at its foot is equal to $M_{sh} = 1.38$. The result obtained by the potential model is absolutely wrong.

The results for the same airfoil at $M_\infty = 0.85$, $\alpha = 1^\circ$ are shown in Fig.2. The Mach number ahead of the shock at its foot is equal to $M_{sh} = 1.45$. Finally, Fig.3 shows the results for the RAE2822 airfoil at $M_\infty=0.75$, $\alpha = 3^\circ$. In this case the Mach number ahead of the shock is equal to $M_{sh} = 1.5$. In both last cases, the agreement of the results is satisfactory.

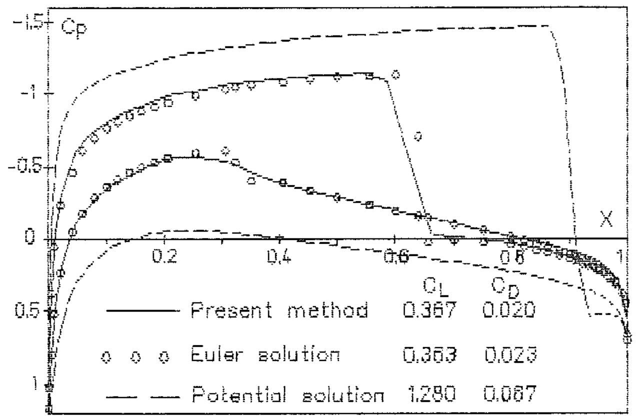

Figure 1 - Pressure distribution for NACA0012 airfoil. $M_\infty = 0.8$, $\alpha = 1.25^\circ$.

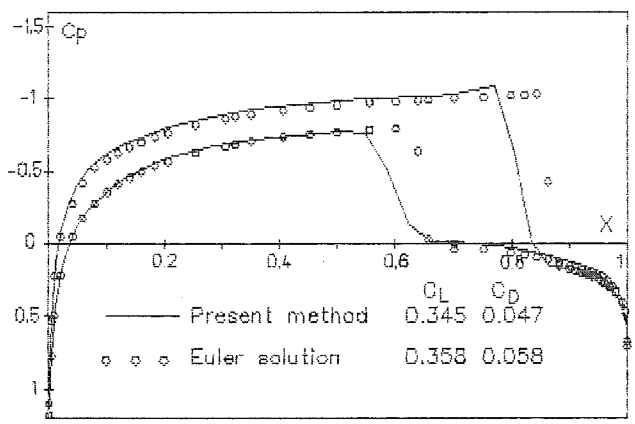

Figure 2 - Pressure distribution for NACA0012 airfoil. $M_\infty = 0.85$, $\alpha = 1^\circ$.

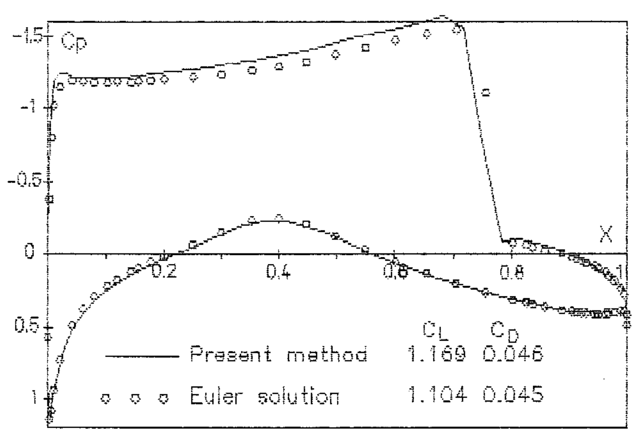

Figure 3 - Pressure distribution for RAE2822 airfoil. $M = 0.75$, $\alpha = 3^\circ$.

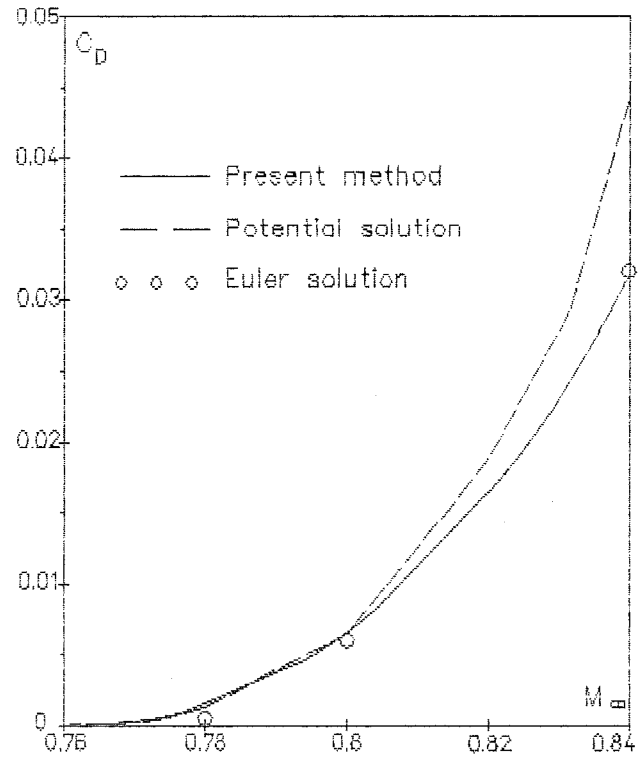

Figure 4 - Wave drag vs Mach number for NACA0012 airfoil at $\alpha = 0$.

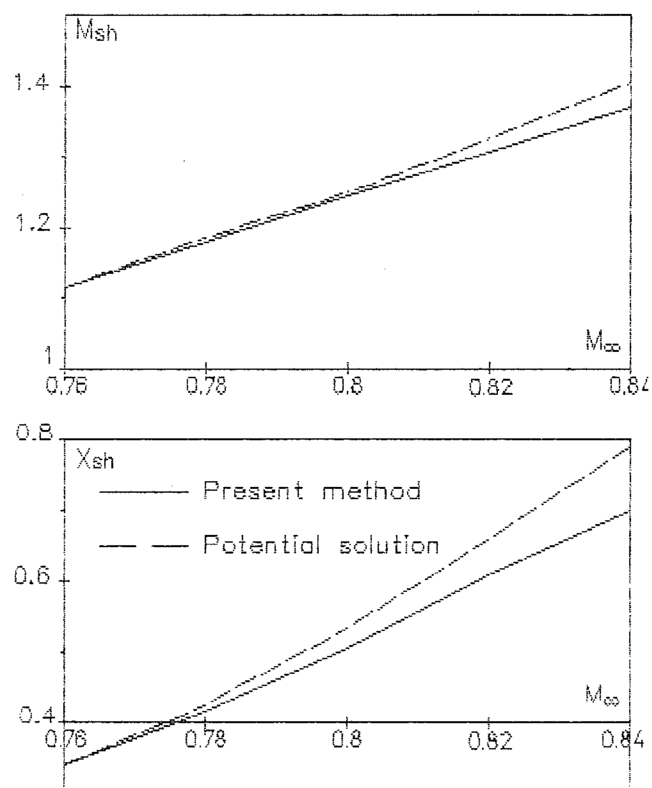

Figure 5 - Shock location and Mach number before shock for NACA0012 airfoil at α=0.

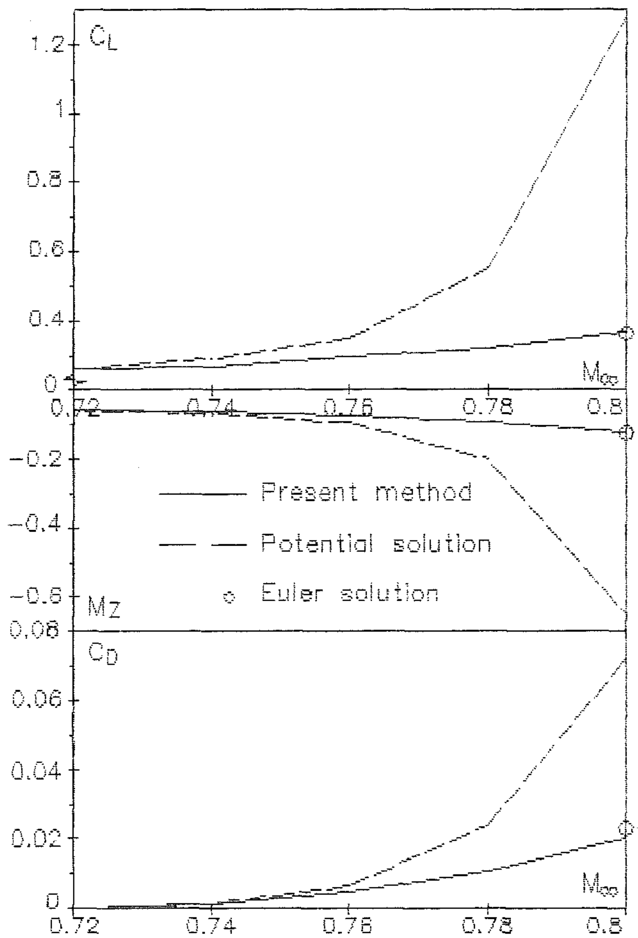

Figure 6 - Total aerodynamic characteristics vs Mach number for NACA0012 airfoil at α = 1.25°.

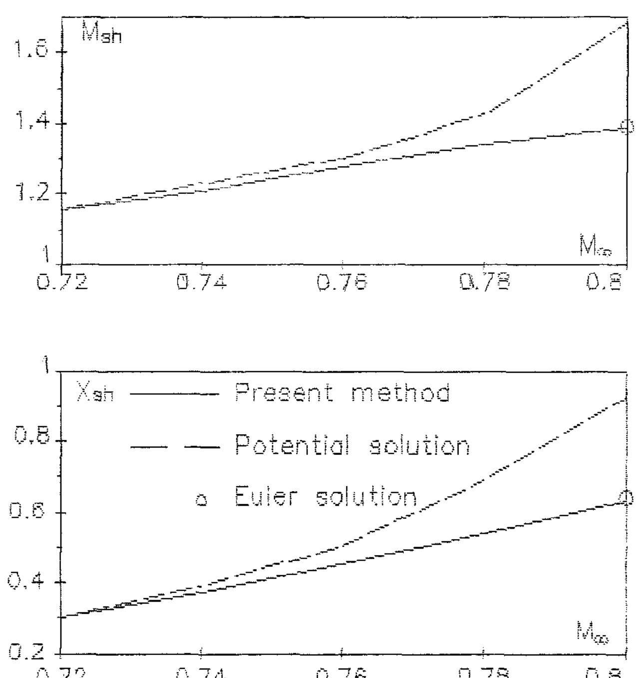

Figure 7 - Shock location and Mach number before shock for NACA0012 airfoil at α = 1.25°.

Figs.4 and 5 show the results of the NACA0012 calculations for α = 0 and various free stream Mach numbers. The wave drag coefficient C_D, Mach number ahead of the shock at its foot M_{sh} and shock location x_{sh} are presented. It can be seen that the wave drag coefficient values become different starting from M_∞ = 0.8 which corresponds to the Mach number ahead of the shock at its foot M_{sh} = 1.25 (Fig.5). The inclusion of the entropy variation in the present method results in a decrease in the wave drag, the Mach number ahead of the shock M_{sh}, and in an upstream movement of the shock. The wave drag values agree well with the results of the Euler solutions[8].

Figs.6 and 7 present the similar results for the NACA0012 airfoil at α = 1.25° and various free-stream Mach numbers. In this case, a noticeable difference between the present method results and the potential model begins at $M_\infty \approx$ 0.76 (Fig.6) which corresponds to the Mach number ahead of the shock M_{sh}= 1.3 (Fig.7). The discrepancies in the lift coefficient C_L are observed even someone earlier, and at M_∞= 0.8 their ratio amounts to 3.6. The total aerodynamic characteristics obtained with the present method agree well with the Euler equation solution at M_∞= 0.8. When calculated by using the present method the shock becomes weaker and moves upstream as compared to the potential equation solutions.

The results given above show that up to the values of the Mach number ahead of the shock of $M_{sh} \approx$ 1.25 to 1.3 the results obtained with the potential model are close to those obtained with the present method. As the shock strength grows further, the potential model gives absolutely wrong results, while the results obtained with the present method agree well with the Euler equation solutions up to the formation of rather strong shocks.

Conclusions

The minor modifications of the potential transonic flow model made it possible to develop the fast effective method of the approximate Euler equation numerical solution. The modifications require a change of the flow density formulation downstream the shocks, and this allows one to take into account the nonzero vorticity behind the shocks as compared to the potential model, where the vorticity is equal to zero. The method is approximate, because the equations solved contain some additional terms which are shown to be small and may be neglected. The computational time is practically the same as for the potential model and 10 to 20 times less than that of the direct Euler equation solution. The comparisons of the results of transonic airfoil flow calculations obtained by using the present method with the Euler equation solutions showed good agreement up to the formation of rather strong shocks, when the potential model gives essentially wrong results.

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