EQUIVALENT DETERMINISTIC INPUTS FOR ATMOSPHERIC TURBULENCE

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Abstract

The equivalent deterministic input for a certain physical random process should satisfy some conditions and be of certain properties so that it can be realized easily and reasonably with physical equipments. These properties and conditions are proposed in this paper. With the techniques of separating partial block matrices, the given power spectrum matrix can be separated into the summation of several multiplications of column matrices and row matrices. Then based on Parseval's theorem and Fourier transformation, the equivalent deterministic inputs can be found. The presented method is more reasonable, and the processes obtained with the method can be simulated more easily. For example, the equivalent deterministic inputs for atmospheric turbulence are obtained with the new method.

I. Introduction

If there is a deterministic process for which the energy response of a dynamic system to the deterministic process equals the mean-square value response of the dynamic system to a given random process, the deterministic process is called the equivalent deterministic process (or input) of the system for the random process. The method to find the deterministic process is called equivalent deterministic technique (or EDT). So if the equivalent deterministic process for a random process is known, we can easily use it to obtain the mean-square value response of a dynamic system to the random process.

B. Etkin first introduced the concept of equivalent deterministic input[1] in 1961. The EDT that he proposed was very simple and restricted to one-dimensional problems. The equivalent deterministic input of a dynamic system can be obtained directly. In 1984, he expanded one-dimensional EDT into multi-dimensional EDT[2]. EDT was further developed by Xiao[3,4] and Zhu[5].

Besides EDT, several other methods, such as power spectrum response methods, system responses to random signals in a time domain, have also been presented elsewhere. There are two distinct advantages with EDT in solving the mean-square value response of a dynamic system to a random process. Because it is a deterministic process, the equivalent deterministic input (EDI) of a dynamic system for a random process can be simulated readily and exactly on physical equipments. Secondly, the precision to find the mean-square value response of the dynamic system to the random process can be controlled easily and effectively.

II. EDI Properties

It is necessary for us to propose some standards that EDI should satisfy. For controlling precision easily and realizing it exactly on physical equipments, EDI $u(t)$ should have properties as follows:

(1) $u(t)$ is a one-sided function. Or $u(t)=0$ for time $t \le 0$.
(2) $u(t)$ must be a real function.
(3) $u(t) \to 0$ when $t \to \infty$
(4) $u(t) \neq 0$ when $t > M$ (M is a large positive number)
(5) $u(t) \neq \pm\infty$ for $t \in [0, +\infty)$

In addition, because frequency properties of a dynamic system are based on a stable system, the dynamic system should be stable when EDT is used to solve the mean-square value response of the system to a random process.

III. New EDT

At first, we need to define a Hermit matrix. If a matrix $\bar{J}(\omega)$ can be expressed as

$$\bar{J}(\omega) = \begin{pmatrix} U_{11}^*(\omega)U_{11}(\omega) & U_{22}^*(\omega)U_{11}(\omega) & \cdots & U_{nn}^*(\omega)U_{11}(\omega) \\ U_{11}^*(\omega)U_{22}(\omega) & U_{22}^*(\omega)U_{22}(\omega) & \cdots & U_{nn}^*(\omega)U_{22}(\omega) \\ \vdots & \vdots & \vdots & \vdots \\ U_{11}^*(\omega)U_{nn}(\omega) & U_{22}^*(\omega)U_{nn}(\omega) & \cdots & U_{nn}^*(\omega)U_{nn}(\omega) \end{pmatrix}$$

it is called a Hermit matrix. Here superscripts - and * denote a matrix and the conjugate of a complex number, respectively.

The discussion on EDT in this paper is mainly restricted in that the power spectrum matrix $\bar{\Phi}(\omega)$ of random processes can be written as a diagonal block Hermit matrix, i.e.

$$\bar{\Phi}(\omega) = \begin{pmatrix} \bar{J}_1 & & & \\ & \bar{J}_2 & & \\ & & \ddots & \\ & & & \bar{J}_m \end{pmatrix} \quad (1)$$

where ω denotes frequency. Actually, a lot of power spectrum matrices of random processes such as atmospheric turbulence power spectrum matrices can be rearranged into the form of diagonal block Hermit matrices.

The main consideration about new EDT will be discussed with an example. Assuming that the transfer function matrix of a stable dynamic system is $\bar{G}(s)$, power spectrum matrix of input random processes is $\bar{\Phi}_x(\omega)$ and power spectrum response matrix of the system is $\bar{\Phi}_y(\omega)$. So we have

$$\bar{\Phi}_y(\omega) = \bar{G}^*(i\omega)\bar{\Phi}_x(\omega)\bar{G}^T(i\omega)$$

Transpose

$$\overline{\Phi}_y^T(\omega) = \overline{G}(i\omega)\overline{\Phi}_x^T(\omega)\overline{G}^H(i\omega)$$

Here superscripts T and H denote the transpose of a matrix and the transpose & conjugate of a matrix, respectively. Assuming $\overline{\Phi}_x^T(\omega)$ has the same form as Eq. (1), or

$$2\pi\overline{\Phi}_x^T(\omega) = \begin{pmatrix} U_{11}^*(\omega)U_{11}(\omega) & U_{22}^*(\omega)U_{11}(\omega) & 0 \\ U_{11}^*(\omega)U_{22}(\omega) & U_{22}^*(\omega)U_{22}(\omega) & 0 \\ 0 & 0 & U_{33}^*(\omega)U_{33}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} \overline{J}_1 & \\ & \overline{J}_2 \end{pmatrix}$$

we can write

$$\overline{\Phi}_y^T(\omega) = \frac{1}{2\pi}\overline{G}(i\omega)\begin{pmatrix} \overline{J}_1 & \\ & \overline{J}_2 \end{pmatrix}\overline{G}^H(i\omega)$$

$$= \frac{1}{2\pi}\overline{G}(i\omega)\begin{pmatrix} \overline{J}_1 & \\ & 0 \end{pmatrix}\overline{G}^H(i\omega) + \frac{1}{2\pi}\overline{G}(i\omega)\begin{pmatrix} 0 & \\ & \overline{J}_2 \end{pmatrix}\overline{G}^H(i\omega)$$

Further assuming

$$\overline{P}_1(\omega) = [U_{11}(\omega), U_{22}(\omega), 0]^T$$

$$\overline{P}_2(\omega) = [0, 0, U_{33}(\omega)]^T$$

we get

$$\overline{\Phi}_y^T(\omega) = \frac{1}{2\pi}\{[\overline{G}(i\omega)\overline{P}_1(\omega)]\,[\overline{G}(i\omega)\overline{P}_1(\omega)]^H$$

$$+ [\overline{G}(i\omega)\overline{P}_2(\omega)]\,[\overline{G}(i\omega)\overline{P}_2(\omega)]^H\}$$

In terms of the relationship between a relation function and a power spectrum function

$$\overline{R}(\tau) = [R_{ij}] = \int_{-\infty}^{+\infty}\overline{\Phi}(\omega)e^{i\omega\tau}d\omega$$

we have

$$R_{y_jy_j}(0) = \sigma_{y_j}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\overline{G}(i\omega)\overline{P}_1(\omega)]_j[\overline{G}(i\omega)\overline{P}_1(\omega)]_j^* d\omega$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\overline{G}(i\omega)\overline{P}_2(\omega)]_j[\overline{G}(i\omega)\overline{P}_2(\omega)]_j^* d\omega$$

From parseval's theorem

$$\sigma_{y_j}^2 = \int_{-\infty}^{+\infty}y_j^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}Y_j(\omega)Y_j^*(\omega)d\omega$$

new EDT can be derived. Here $Y(\omega)$ is the Fourier transform of $y(t)$. Thus if $\overline{P}_1(\omega)$ and $\overline{P}_2(\omega)$ are the fourier transforms of $\overline{u}_1(t)$ and $\overline{u}_2(t)$, we can derive the conclusion that the mean-square value response σ^2 of the stable dynamic system with a transfer function $\overline{G}$ to a random process $\overline{x}(t)$ with a power spectrum Φ_x equals the summation of the energy responses of the system to deterministic processes $\overline{u}_1(t)$ and $\overline{u}_2(t)$. Here $\overline{u}_1(t)$ and $\overline{u}_2(t)$ are called equivalent deterministic inputs of the system for the random process $\overline{x}(t)$. In general, the solution of EDI is not unique.

IV. EDI for Atmospheric Turbulence

As an example, using new EDT, we can get the equivalent deterministic inputs of an aircraft for atmospheric turbulence with given power spectrum matrices[4].

1. Longitudinal Motion of an Aircraft

The longitudinal dynamic equation of an aircraft is

$$\frac{d(\overline{y}_1)}{dt} = \overline{A}_1\overline{y}_1 + \overline{B}_1\overline{x}_1 \qquad (2)$$

Here $\overline{x}_1 = [W_x, W_y, W_{yx}]^T$ and $\overline{y}_1$ denotes the longitudinal motion parameters of the aircraft such as $[V, \alpha, \omega_z, \vartheta]^T$. W_x and W_y denote the velocity components of the atmospheric turbulence in the directions of x coordinate and y coordinate, respectively. W_{yx} denotes the gradient of W_y in the direction of x coordinate. The two-sided atmospheric turbulence power spectrum matrix is

$$\overline{\Phi}_{x_1}(\omega) = \begin{pmatrix} \Phi_{W_xW_x} & 0 & 0 \\ 0 & \Phi_{W_yW_y} & \Phi_{W_yW_{yx}} \\ 0 & \Phi_{W_{yx}W_y} & \Phi_{W_{yx}W_{yx}} \end{pmatrix}$$

where

$$\Phi_{W_xW_x} = \frac{L_{W_x}}{\pi V}\,\frac{1}{1 + (\frac{L_{W_x}}{V}\omega)^2}\,\sigma_{W_x}^2$$

$$\Phi_{W_yW_y} = \frac{L_{W_y}}{\pi V}\,\frac{1 + 12(\frac{L_{W_y}}{V}\omega)^2}{[1 + 4(\frac{L_{W_y}}{V}\omega)^2]^2}\,\sigma_{W_y}^2$$

$$\Phi_{W_{yx}W_{yx}} = \frac{(\omega/V)^2}{1 + (\frac{4L}{\pi V}\omega)^2}\,\Phi_{W_yW_y}$$

$$\Phi_{W_yW_{yx}} = \frac{(i\omega/V)}{1 + \frac{4L}{\pi V}i\omega}\,\Phi_{W_yW_y}$$

$$\Phi_{W_{yx}W_y} = \Phi_{W_yW_{yx}}^*$$

where V, L, Lw and σ_W denote the velocity of the aircraft, the wing span of the aircraft, the relation scale of atmospheric turbulence and the strength of atmospheric turbulence.

Using new EDT, we have

$$\overline{P}_1(\omega) = [U_{11}(\omega), 0, 0]^T$$

$$\overline{P}_2(\omega) = [0, U_{22}(\omega), U_{33}(\omega)]^T$$

where

$$U_{11}(\omega) = \sigma_{W_x}\sqrt{\frac{2L_{W_x}}{V}}\frac{1}{1+\frac{L_{W_x}}{V}i\omega}$$

$$U_{22}(\omega) = \sigma_{W_y}\sqrt{\frac{2L_{W_y}}{V}}\frac{1+2\sqrt{3}\frac{L_{W_y}}{V}i\omega}{[1+\frac{2L_{W_y}}{V}i\omega]^2}$$

$$U_{33}(\omega) = \frac{i\omega/V}{1+\frac{4L}{\pi V}i\omega}U_{22}(\omega)$$

Using inverse Fourier transformation, we can obtain the equivalent deterministic inputs (one-sided functions):

$$u_{11}(t) = \sigma_{W_x}\sqrt{\frac{2V}{L_{W_x}}}\exp(-\frac{V}{L_{W_x}}t)$$

$$u_{22}(t) = \sigma_{W_y}\sqrt{\frac{V}{2L_{W_y}}}[\sqrt{3}+(1-\sqrt{3})\frac{V}{2L_{W_y}}t]\exp(-\frac{V}{2L_{W_y}}t)$$

and for $\frac{4L}{\pi V} = \frac{2L_{W_y}}{V}$,

$$u_{33}(t) = \sigma_{W_y}\sqrt{\frac{2L_{W_y}}{V^3}}[\frac{\sqrt{3}V^2}{4L^2_{W_y}} + \frac{(1-2\sqrt{3})V^3}{(2L_{W_y})^3}t - \frac{(1-\sqrt{3})V^4}{2(2L_{W_y})^4}t^2]\exp(-\frac{V}{2L_{W_y}}t)$$

for $\frac{4L}{\pi V} \neq \frac{2L_{W_y}}{V}$,

$$u_{33}(t) = \sigma_{W_y}\sqrt{\frac{2L_{W_y}}{V^3}}.[\frac{\pi V}{4L}a_1\exp(-\frac{\pi V}{4L}t) + c_1(\frac{V}{2L_{W_y}})^2\exp(-\frac{V}{2L_{W_y}}t) + (b_1 - \frac{c_1V}{2L_{W_y}})(\frac{V}{2L_{W_y}})^2 t\exp(-\frac{V}{2L_{W_y}}t)]$$

where

$$a_1 = (2\sqrt{3}\frac{L_{W_y}}{V} - \frac{4L}{\pi V})/(\frac{2L_{W_y}}{V} - \frac{4L}{\pi V})^2$$

$$b_1 = -a_1$$

$$c_1 = 1 - (\frac{4L_{W_y}}{V} - \frac{4L}{\pi V})a_1$$

2. Lateral Motion of an Aircraft

The lateral dynamic equation of an aircraft is

$$\frac{d(\overline{y}_2)}{dt} = \overline{A}_2\overline{y}_2 + \overline{B}_2\overline{x}_2$$

Here $\overline{x}_2 = [W_z, W_{zx}, W_{yz}]^T$ and $\overline{y}_2$ denotes the lateral motion parameters of the aircraft such as $[\beta, \omega_x, \omega_y, \gamma]^T$. W_z denotes the velocity component of the atmospheric turbulence in the direction of z coordinate. W_{zx} and W_{yz} denote the gradient of W_z in the direction of x coordinate and the gradient of W_y in the direction of z coordinate, respectively. The two-sided atmospheric turbulence power spectrum matrix is

$$\overline{\Phi}_{x_2}(\omega) = \begin{pmatrix} \Phi_{W_zW_z} & \Phi_{W_zW_{zx}} & 0 \\ \Phi_{W_{zx}W_z} & \Phi_{W_{zx}W_{zx}} & 0 \\ 0 & 0 & \Phi_{W_{yz}W_{yz}} \end{pmatrix}$$

where

$$\Phi_{W_zW_z} = \frac{L_{W_z}}{\pi V}\frac{1+12(\frac{L_{W_z}}{V}\omega)^2}{[1+4(\frac{L_{W_z}}{V}\omega)^2]^2}\sigma^2_{W_z}$$

$$\Phi_{W_{zx}W_{zx}} = \frac{(\omega/V)^2}{1+(\frac{3L}{\pi V}\omega)^2}\Phi_{W_zW_z}$$

$$\Phi_{W_{yz}W_{yz}} = \frac{1}{2L_{W_y}}\frac{0.4(\frac{\pi L_{W_y}}{2L})^{1/3}}{1+(\frac{4L}{\pi V}\omega)^2}\sigma^2_{W_y}$$

$$\Phi_{W_zW_{zx}} = \frac{(i\omega/V)}{1+\frac{3L}{\pi V}i\omega}\Phi_{W_zW_z}$$

$$\Phi_{W_{zx}W_z} = \Phi^*_{W_zW_{zx}}$$

Using new EDT, we have

$$\overline{P}_1(\omega) = [U_{11}(\omega), U_{22}(\omega), 0]^T$$

$$\overline{P}_2(\omega) = [0, 0, U_{33}(\omega)]^T$$

where

$$U_{11}(\omega) = \sigma_{W_z}\sqrt{\frac{2L_{W_z}}{V}}\frac{1+2\sqrt{3}\frac{L_{W_z}}{V}i\omega}{[1+\frac{2L_{W_z}}{V}i\omega]^2}$$

$$U_{22}(\omega) = \frac{i\omega/V}{1+\frac{3L}{\pi V}i\omega}U_{11}(\omega)$$

$$U_{33}(\omega) = \sigma_{W_y}\sqrt{\frac{0.4\pi}{V L_{W_y}}}\sqrt{(\frac{\pi L_{W_y}}{2L})^{1/3}}\frac{1}{1+\frac{4L}{\pi V}i\omega}$$

Using inverse Fourier transformation, we can obtain the equivalent deterministic inputs (one-sided functions):

$$u_{11}(t) = \sigma_{w_z}\sqrt{\frac{V}{2L_{w_z}}}\left[\sqrt{3} + (1-\sqrt{3})\frac{V}{2L_{w_z}}t\right]\exp\left(-\frac{V}{2L_{w_z}}t\right)$$

for $\frac{3L}{\pi V} = \frac{2L_{w_z}}{V}$,

$$u_{22}(t) = \sigma_{w_z}\sqrt{\frac{2L_{w_z}}{V^3}}\left[\frac{\sqrt{3}\,V^2}{4L_{w_z}^2} + \frac{(1-2\sqrt{3})V^3}{(2L_{w_z})^3}t - \frac{(1-\sqrt{3})V^4}{2(2L_{w_z})^4}t^2\right]\exp\left(-\frac{V}{2L_{w_z}}t\right)$$

for $\frac{3L}{\pi V} \neq \frac{2L_{w_z}}{V}$,

$$u_{22}(t) = \sigma_{w_z}\sqrt{\frac{2L_{w_z}}{V^3}}\left[\frac{\pi V}{3L}a_2\exp\left(-\frac{\pi V}{3L}t\right) + c_2\left(\frac{V}{2L_{w_z}}\right)^2\exp\left(-\frac{V}{2L_{w_z}}t\right) + \left(b_2 - \frac{c_2 V}{2L_{w_z}}\right)\left(\frac{V}{2L_{w_z}}\right)^2 t\,\exp\left(-\frac{V}{2L_{w_z}}t\right)\right]$$

where

$$a_2 = \left(2\sqrt{3}\,\frac{L_{w_z}}{V} - \frac{3L}{\pi V}\right) / \left(\frac{2L_{w_z}}{V} - \frac{3L}{\pi V}\right)^2$$

$$b_2 = -a_2$$

$$c_2 = 1 - \left(\frac{4L_{w_z}}{V} - \frac{3L}{\pi V}\right)a_2$$

$$u_{33}(t) = \sigma_{w_z}\frac{\pi}{4L}\sqrt{\frac{0.4\pi V}{L_{w_y}}\left(\frac{\pi L_{w_y}}{2L}\right)^{1/3}}\exp\left(-\frac{\pi V}{4L}t\right)$$

3. Computational Results

Assume

$V = 130$ (m/s), $L = 38$ (m)
$\sigma_{w_x} = \sigma_{w_y} = \sigma_{w_z} = 1.766$ (m/s)
$L_{w_x} = 2L_{w_y} = 2L_{w_z} = 530$ (m)

$$\overline{A}_1 = \begin{pmatrix} -0.01978 & 7.302 & 0 & -9.807 \\ 0.001151 & -1.061 & 1 & 0 \\ 0.0005736 & -2.807 & -1.368 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\overline{B}_1 = \begin{pmatrix} 0.01978 & -0.01884 & 0 \\ 0.001151 & -0.008113 & 0 \\ -0.0005736 & -0.02161 & 0.7184 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{A}_2 = \begin{pmatrix} -0.1408 & 0.06495 & 1 & 0.07528 \\ -4.000 & -2.328 & -0.7038 & 0 \\ -1.000 & 0.1013 & -0.4000 & 0 \\ 0 & 1 & -0.06504 & 0 \end{pmatrix}$$

$$\overline{B}_2 = \begin{pmatrix} 0.001083 & 0 & 0 \\ 0.03077 & -0.7038 & -2.328 \\ 0.007692 & -0.4000 & 0.1013 \\ 0 & 0 & 0 \end{pmatrix}$$

Some computational results are given in Table 1.

V. Conclusion

New EDT presented in this paper is more effective and reasonable.

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References

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Table 1 Mean-Square Value Responses of an Aircraft to Atmospheric Turbulence

Longitudinal Direction	Mean-Square Value	σ_v (m/s)	σ_α (rad)	σ_{ω_z} (1/s)	σ_ϑ (rad)	σ_{n_y}
	New EDT	3.206	0.01357	0.006761	0.03190	0.1243
	Exact	3.160	0.01347	0.006668	0.03179	0.1244
Lateral Direction	Mean-Square Value	σ_β (rad)	σ_{ω_x} (1/s)	σ_{ω_y} (1/s)	σ_γ (rad)	σ_{n_z}
	New EDT	0.01832	0.02969	0.01558	0.09998	0.1157
	Exact	0.01793	0.02793	0.01481	0.09280	0.1103