APPROXIMATE CHATTERING ARC FOR PRACTICAL MANEUVERS

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Abstract

The purpose of this paper is to develop an approximate chattering arc for practical maneuver of the shuttle-type space vehicle. Theoretically, in chattering arc of the first kind, the control chatters between its positive and negative maximum values at an infinite rate. Thus for shuttle-type vehicle at constant altitude flight, we can use either bank control or lift control to obtain theoretical chattering arc with the drag being maximized. This chattering arc could be useful for minimum-time aerobraking maneuver. In practice, switching of the control between its positive and negative maximum values at an infinite rate is not possible. The approximate chattering arc developed in this paper has finite rate of control switching. It has the potential for practical maneuvers in actual flight. As compared with the theoretical chattering arc, the effect of the finite rate approximation on the final conditions is discussed in detail. Particular attention is devoted to the effect on the flight range.

Nomenclature

C_D	=drag coefficient
C_{D0}	=zero-lift drag coefficient
C_L	=lift coefficient
C_L^*	=C_L for maximum lift-to-drag ratio
E^*	=maximum lift-to-drag ratio
g	=gravitational acceleration
H	=Hamiltonian
$\bar{H}$	=reduced Hamiltonian
K	=induced drag factor
l	=length of approximate chattering arc
m	=vehicle mass
N	=number of control switching
$p_v, p_\theta, p_\phi, p_\psi$	=adjoint variables
(P_1, P_2)	$=(p_\psi \frac{(1-v)}{\sqrt{v}}, -p_v \frac{\Omega(1-v)^2}{E^*\sqrt{v}})$
(Q_1, Q_2)	$=(\tan\sigma, \tan^2\sigma)$
r	=vehicle distance from center of earth
s	=dimensionless time
$s_1, s_2, s_3, \ldots$	=instants at which control switching occurs
S	=reference area
t	=time
v	=dimensionless speed
V	=vehicle speed

Subscripts

i	=initial value
f	=final value
max	=maximum value

I. Introduction

The frequent flight of the space shuttle and the developments of the national aerospace plane (NASP) and personnel launch system (PLS) indicate that the atmospheric maneuvers in the moderate dense layer of the atmosphere are becoming more and more important. The evidence of the requirement for practical application such as, for example, the maneuvers for atmospheric rendezvous, is more obvious than before. Two eminent books[1,2] and much literature in the area of trajectory optimization have discussed the maneuvers for optimal trajectories. In particular, the characteristics and applications of the chattering arc have been investigated in some published papers.[3-5] Theoretically, in chattering arc of the first kind, the control chatters between its positive and negative maximum values at an infinite rate.[3] In practice, switching of the control between its maximum and minimum values at an infinite rate is not possible. The purpose of this paper is to develop an approximate chattering arc in which the control chatters at a finite rate. In other words, the approximate chattering arc we defined in this paper is resulted from the finite frequency of control switching between its positive and negative maximum values. It is expected that the developed approximate chattering method will be more proper for practical maneuvers.

In Ref. 5, the minimum-time aerobraking maneuver at constant altitude was investigated. The final conditions specified consist of the final position and final velocity vector (both magnitude and direction) of the vehicle. This is required for the purpose of atmospheric rendezvous or ground landing. Obviously, it is a set of very strong final conditions. The chattering arc, when exists, is an arc along a great circle at the specified altitude. Now, assume the maximum bank angle σ_{max} is used at first. The vehicle turns to the left at constant altitude. Then at a certain instant the control is switched to its minimum value (or negative maximum value, $-\sigma_{max}$). The switching time is selected such that the vehicle will return to its original latitude at the same final time as the theoretical chattering arc. Thus the control "chatters" only once in this case, and the specified final conditions may not be completely satisfied. For the second case, we allow the control to "chatter" two times during the flight. The flight path will have one fluctuation and two of the four final conditions can be satisfied at the same final time. As the number of chattering (or switching) of the control variable is increased to three, the only final condition which remains unsatisfied is the final longitudinal range. The final longitudinal range will be shorter than the theoretical chattering value, but the difference will be reduced when the number of control switching is increased. In order to compensate the shortage of final longitudinal range due to the finite frequency of control switching, we can allow the vehicle to coast a proper distance from the initial state. The approximate chattering arc then follows. With this strategy there will be some penalty on the minimum flight time.

II. Equations of Motion

The geometry of constant altitude flight is shown in Figure 1, where θ is the longitude and ϕ is the latitude, and $\theta\phi$-surface represents the constant altitude spherical surface. For a vehicle with lift capability and zero thrust, the flight in the $\theta\phi$-surface is governed by the equations:[2]

$$\frac{dV}{dt} = -\frac{\rho S C_D V^2}{2m} \tag{1a}$$

$$\frac{d\theta}{dt} = \frac{V\cos\psi}{r\cos\phi} \tag{1b}$$

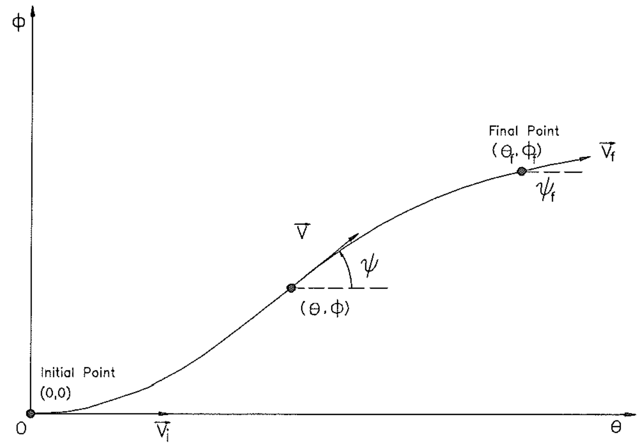

Figure 1. Geometry of Constant Altitude Flight.

$$\frac{d\phi}{dt} = \frac{V \sin\psi}{r} \tag{1c}$$

$$V\frac{d\psi}{dt} = \frac{\rho S C_L V^2}{2m}\sin\sigma - \frac{V^2}{r}\cos\psi\tan\phi \tag{1d}$$

where V is the speed, ρ is the atmospheric density, S is the reference area, C_D is the drag coefficient, m is the mass, θ is the longitude, ϕ is the latitude, ψ is the heading, r is the distance of the vehicle from the planet center, C_L is the lift coefficient, σ is the bank angle, and t denotes the independent variable, time. Using a parabolic drag polar of the form:

$$C_D = C_{D0} + KC_L^2 \tag{2}$$

we define the normalized lift coefficient λ such that

$$C_L = \lambda C_L^* \tag{3}$$

where C_L^* is the lift coefficient corresponding to the maximum lift-to-drag ratio E^*. With given values of C_{D0} and K assumed constant at hypersonic speed, we have

$$E^* = \frac{1}{2\sqrt{C_{D0}K}} \tag{4}$$

By introducing the dimensionless altitude Ω, dimensionless kinetic energy v, and dimensionless time s, defined as

$$\Omega = \frac{2m}{\rho S C_L^* r}, \quad v = \frac{V^2}{gr}, \quad s = \int_0^t \sqrt{\frac{g}{r}}dt \tag{5}$$

where g is the gravitational acceleration and is constant for constant altitude flight, we have the dimensionless equations of motion:

$$\frac{dv}{ds} = -\frac{\sqrt{v^3}}{E^*\Omega}[1 + \frac{\Omega^2(1-v)^2}{v^2}(1+\tan^2\sigma)] \tag{6a}$$

$$\frac{d\theta}{ds} = \frac{\sqrt{v}\cos\psi}{\cos\phi} \tag{6b}$$

$$\frac{d\phi}{ds} = \sqrt{v}\sin\psi \tag{6c}$$

$$\frac{d\psi}{ds} = (\frac{1-v}{\sqrt{v}})\tan\sigma - \sqrt{v}\cos\psi\tan\phi \tag{6d}$$

The constraining relation for constant altitude flight is

$$L\cos\sigma = m(g - \frac{V^2}{r}) \tag{7}$$

where L is the lift. Or, in dimensionless form,

$$\lambda\cos\sigma = \Omega(\frac{1-v}{v}) \tag{8}$$

This relation has been used in deriving Eqs. (6).

The above equations are for the general two-dimensional flight. For one-dimensional flight along the θ-axis, we have $\phi = \psi = \sigma = 0$ and Eqs. (6) become the reduced form:

$$\frac{dv}{ds} = -\frac{\sqrt{v^3}}{E^*\Omega}[1 + \Omega^2(\frac{1-v}{v})^2] = -\frac{\sqrt{v^3}}{E^*\Omega}(1+\lambda^2) \tag{9a}$$

$$\frac{d\theta}{ds} = \sqrt{v} \tag{9b}$$

and Eq. (8) becomes

$$\lambda = \Omega(\frac{1-v}{v}) \tag{10}$$

III. Theoretical Chattering Arc

In Eqs. (6) there are four state variables, θ, ϕ, v and ψ, and we use the bank angle σ as the sole control variable. The Hamiltonian can be written as

$$H = -p_v\frac{\sqrt{v^3}}{E^*\Omega}[1 + \frac{\Omega^2(1-v)^2}{v^2}(1+\tan^2\sigma)] + p_\theta\frac{\sqrt{v}\cos\psi}{\cos\phi} + p_\phi\sqrt{v}\sin\psi + p_\psi[(\frac{1-v}{\sqrt{v}})\tan\sigma - \sqrt{v}\cos\psi\tan\phi] \tag{11}$$

Regarding the optimal bank angle, it suffices to consider the part of the Hamiltonian containing σ:

$$\bar{H} = -p_v\frac{\Omega(1-v)^2}{E^*\sqrt{v}}\tan^2\sigma + p_\psi\frac{(1-v)}{\sqrt{v}}\tan\sigma \tag{12}$$

This reduced Hamiltonian can be considered as the dot product of the two vectors (P_1, P_2) and (Q_1, Q_2) such that[2]

$$P_1 \equiv p_\psi\frac{(1-v)}{\sqrt{v}}, \quad P_2 \equiv -p_v\frac{\Omega(1-v)^2}{E^*\sqrt{v}} \tag{13a}$$

$$Q_1 \equiv \tan\sigma, \quad Q_2 \equiv \tan^2\sigma \tag{13b}$$

and

$$\bar{H} \equiv P_1Q_1 + P_2Q_2 \tag{14}$$

When σ varies, the vector $\vec{Q} = (Q_1, Q_2)$ describes the domain of maneuverability which is the parabola as shown in Figure 2,

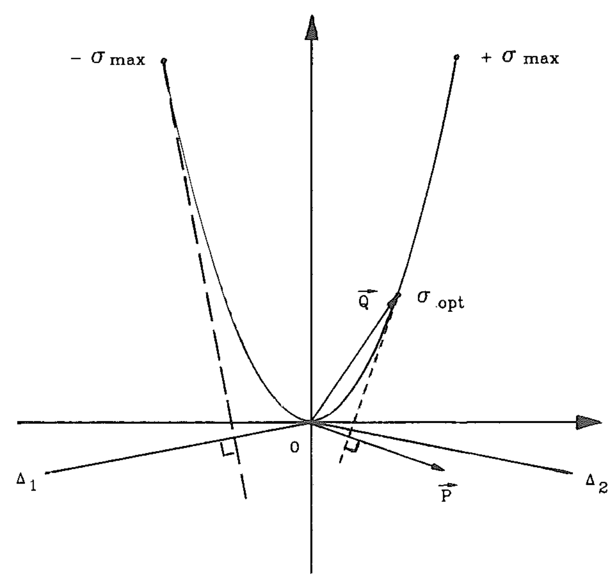

Figure 2. Domain of Maneuverability for the Bank Angle.

$$Q_2 = Q_1^2 \tag{15}$$

To maximize $\bar{H}$, if the vector $\overrightarrow{P} = (P_1, P_2)$ is inside the angle $\Delta_1 O \Delta_2$, the optimal bank angle used is an interior bank angle such that the tangent to the parabola is perpendicular to $\overrightarrow{P}$. This is expressed by

$$\frac{\partial Q_2}{\partial Q_1} = 2Q_1 = -\frac{P_1}{P_2}$$

or

$$\tan\sigma = \frac{E^* p_\psi}{2\Omega(1-v)p_v} \tag{16}$$

Thus it is necessary that $P_2 < 0$, or, in other words, $p_v > 0$. When the vector $\overrightarrow{P}$ is outside the angle $\Delta_1 O \Delta_2$, the bank angle used is $\sigma = \sigma_{max}$ when $p_\psi > 0$ and $\sigma = -\sigma_{max}$ when $p_\psi < 0$. In the case when $p_v < 0$ and $p_\psi = 0$ for a finite time interval, there may exist a chattering control in which the bank angle switches rapidly between $+\sigma_{max}$ and $-\sigma_{max}$ at an infinite rate. The resulted flight path is called the chattering arc. Theoretically, the rapid switching of bank angle between $+\sigma_{max}$ and $-\sigma_{max}$ at an infinite rate keeps the heading at constant and at the same time maximizes the drag. Therefore, the chattering arc is along a great circle with maximum deceleration for aerobraking maneuver. Its application on time-minimization flight at constant altitude has been presented in Ref. 5.

From the discussion above, it can be concluded that the chattering arc is equivalent to one-dimensional flight with maximum drag. Thus the equations of motion for the chattering arc are the same as those in Eqs. (9) except that λ is replaced by its maximum value λ_{max}. That is, we have

$$\frac{dv}{ds} = -\frac{\sqrt{v^3}}{E^*\Omega}(1 + \lambda_{max}^2) \tag{17a}$$

$$\frac{d\theta}{ds} = \sqrt{v} \tag{17b}$$

for the chattering arc. The analytic solutions of Eqs. (17) are

$$s_f = \frac{2E^*\Omega}{1+\lambda_{max}^2}\left(\frac{1}{\sqrt{v_f}} - \frac{1}{\sqrt{v_i}}\right) \tag{18a}$$

$$\theta_f = \frac{E^*\Omega}{1+\lambda_{max}^2}\ln\left(\frac{v_i}{v_f}\right) \tag{18b}$$

where the conditions $\theta_i = s_i = 0$ have been used. We see that s_f is identically determined when v_i and v_f are prescribed. For numerical computation, we assume the maximum normalized lift coefficient λ_{max}=2.5, the maximum lift-to-drag ratio $E^* = 2$, the dimensionless altitude $\Omega = 0.2$, the initial speed $v_i = 0.95$, and the final speed $v_f = 0.0741$. The initial speed is selected to be a little smaller than the orbital speed and the final speed is equal to the stall speed calculated from Eq. (10). The length of the chattering arc θ_f and the dimensionless time of chattering flight s_f for $E^* = 2$ are $\theta_f = 0.1408$ and $s_f = 0.2922$, respectively.

In summary, for constant altitude flight, the theoretical chattering arc is along a great circle. For $\lambda_{max} = 2.5$, $E^* = 2$ and $\Omega = 0.2$, and with the specified initial states $s_i = 0$, $v_i = 0.95$, $\theta_i = \phi_i = \psi_i = 0$, the chattering arc is along the θ-axis and the final states are $s_f = 0.2922$, $v_f = 0.0741$(specified), $\theta_f = 0.1408$ and $\phi_f = \psi_f = 0$. In the following section, the approximate chattering arc which has finite frequency of control switching will be developed. Its characteristics will be investigated as compared with the theoretical chattering arc.

IV. Approximate Chattering Arc

Theoretically, in chattering arc of the first kind, the control chatters between its positive and negative maximum values at an infinite rate. In practice, switching of the control between its maximum and minimum values at an infinite rate is not possible. The main purpose of this section is to develop an approximate chattering arc which will be more proper for practical maneuver.

The approximate chattering arc we defined in this paper is resulted from the finite frequency of control switching between its positive and negative maximum values. Let N be the number of control switching. As the first approximate chattering arc, we allow the vehicle to make only one switch during the flight and N=1. In other words, the vehicle makes a positive maximum bank (to the left) from the initial point. Then, at a certain instant s_1, it switches to the negative maximum bank and keeps this attitude till the final instant s_f. From Eq. (8), the maximum bank angle is a function of the vehicle speed:

$$\cos(\sigma_{max}) = \frac{\Omega}{\lambda_{max}}\left(\frac{1-v}{v}\right) \tag{19}$$

Thus the value of σ_{max} is decreasing when the vehicle speed is becoming smaller due to the aerodynamic drag. There is only one parameter which can be adjusted, it is the instant of switching s_1. Therefore, only one final condition can be satisfied. To obtain the flight trajectory, we integrate Eqs. (6) from the initial states with the σ_{max} calculated from Eq. (19). Then at s_1, the bank angle switches to $-\sigma_{max}$ till s_f. From Eq. (6a) and Eq. (19), we can prove that Eq. (18a) is also valid for two-dimensional turning with maximum bank angle. This means that with specified v_i and v_f, the final flight time s_f can be calculated from Eq. (18a) and is exactly the same as the theoretical chattering flight. The parameter s_1 is selected such that the vehicle returns to the θ-axis and $\phi_f = 0$. The flight trajectory for $E^* = 2$ is shown in Figure 3. It is clear that two final conditions remain unsatisfied: $\psi_f = -0.5827 \neq 0$, $\theta_f = 0.1333 \neq 0.1408$.

Now, we shall consider the second approximate chattering arc in which the vehicle makes two switches during the flight and $N = 2$. From the initial instant s_i=0, the vehicle banks to the left by using the positive maximum bank angle σ_{max}. Then at a certain instant s_1, the vehicle banks to the right with the control switches from σ_{max} to $-\sigma_{max}$. And then, at another certain instant s_2, the vehicle banks to the left again and the control switches from $-\sigma_{max}$ to σ_{max}. In this case, besides the final condition $v_f = 0.0741$ which can be satisfied when s_f is reached, two more final conditions can be satisfied since there are two parameters s_1 and s_2 which can be adjusted. We select s_1 and s_2 such that the two final conditions $\phi_f = 0$ and $\psi_f = 0$ are satisfied. The remaining final condition which is still unsatisfied is $\theta_f = 0.1373 \neq 0.1408$. Actually, this final condition can never be satisfied by using approximate chattering arc. What we shall

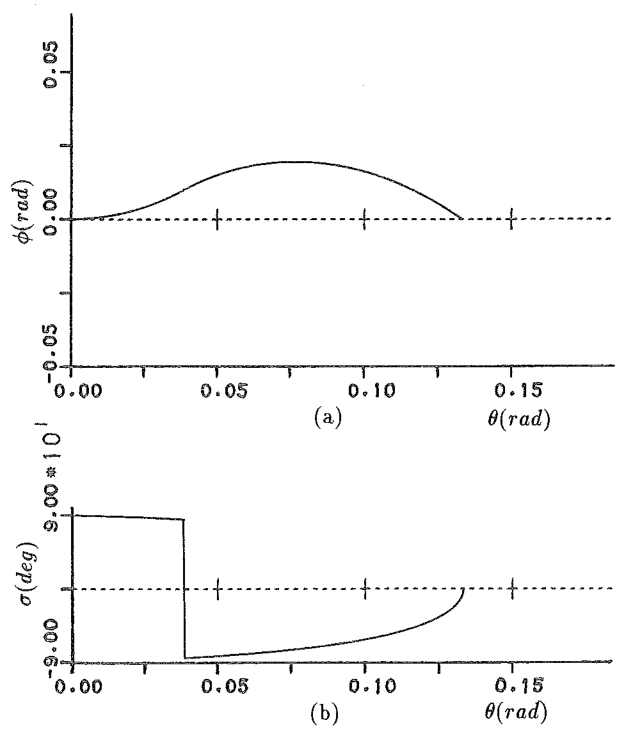

Figure 3. One - Switching Approximate Chattering Arc.

do is to develop the approximate chattering arc with some more switchings such that the final position on the θ-axis can approach $\theta_f = 0.1408$ as close as possible. The flight trajectory for the second approximate chattering arc is shown in Figure 4.

For further investigation, we consider the third approximate chattering arc which has three switching in the bank angle and $N = 3$. As compared with the second chattering arc, we now have three parameters s_1, s_2 and s_3 which represent the three consecutive instants at which the control switching happens. Therefore, it becomes a parameter optimization problem with the three parameters to be selected such that $\theta_f = \psi_f = 0$ and θ_f is maximized. The optimal trajectory obtained is shown in Figure 5. The final range obtained is $\theta_f = 0.1390$.

For N=4 and 5, the optimal trajectories obtained are plotted in Figures 6 and 7, respectively. The growth of maximum θ_f along with the increase of the number of control switching N is presented in Figure 8.

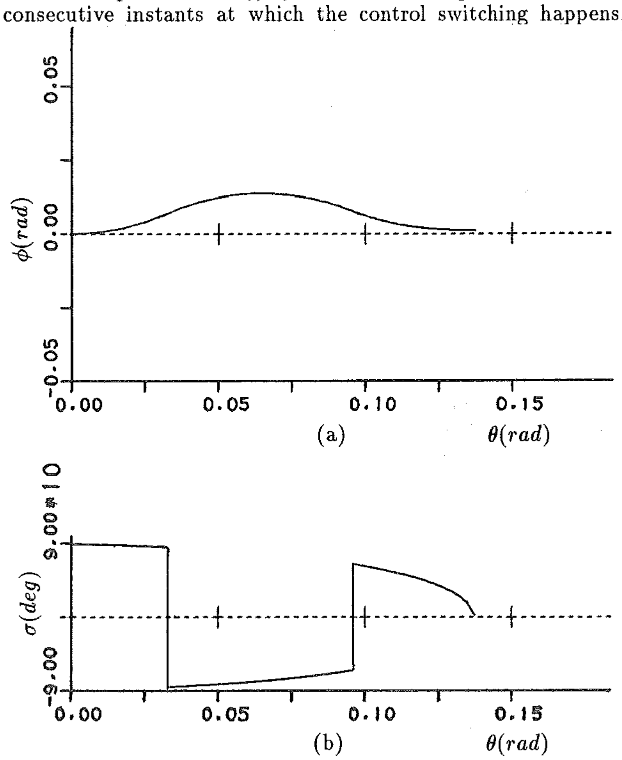

Figure 4. Two - Switching Approximate Chattering Arc.

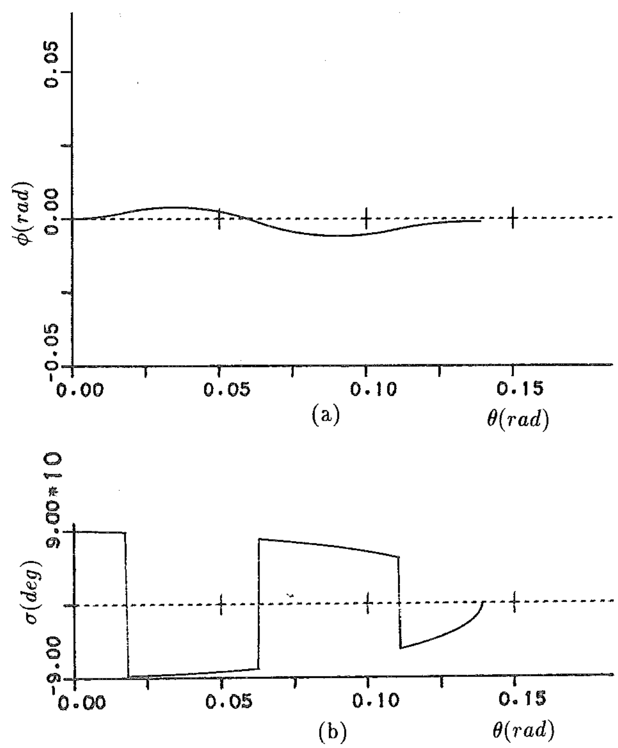

Figure 5. Three - Switching Approximate Chattering Arc.

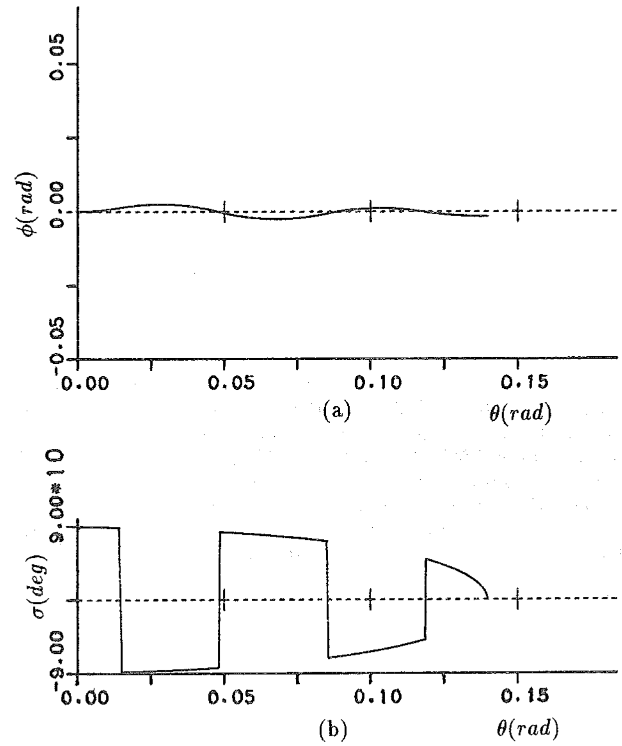

Figure 6. Four - Switching Approximate Chattering Arc.

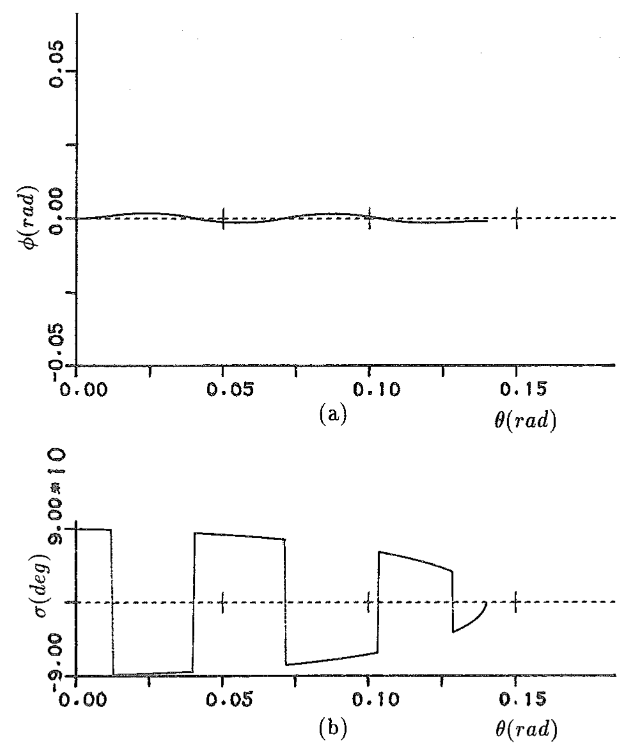

Figure 7. Five - Switching Approximate Chattering Arc.

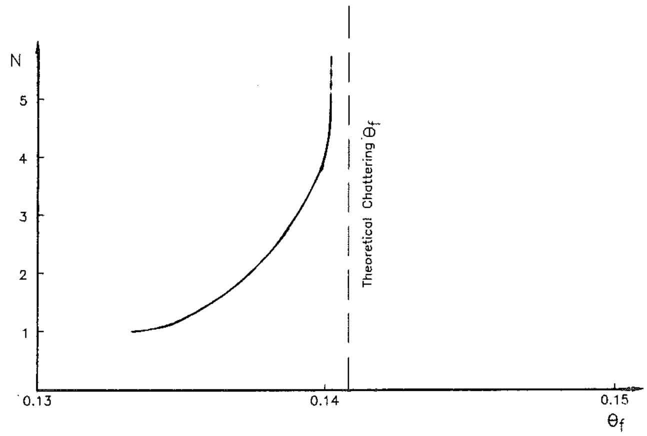

Figure 8. Maximum θ_f as a Function of N.

V. Discussion

From Figure 8, it is seen that the increase in maximum θ_f which can be obtained through the increase of N is limited beyond N=5. For N=5, the penalty on θ_f due to the approximation is only 0.5%. Generally speaking, this penalty is small enough. In both theoretical and approximate chattering arcs, the flight time is the same and is uniquely determined with prescribed v_i and v_f values. Also, both arcs have the same length which can be proved below. Let l denotes the length of the approximate chattering arc, we have

$$l = \int_0^t V dt$$

or, in dimensionless form,

$$\frac{l}{r} = \int_0^{s_f} \sqrt{v} ds \tag{20}$$

This is exactly the same as the integration of Eq. (17b). Physically it is understandable since in both arcs the vehicle flies at maximum drag. The time history, $v(s)$ or $V(t)$, is the same in any case.

There is a strategy by which we can obtain $\theta_f = 0.1408$ at the end of approximate chattering arc with some sacrifice on the flight time. The technique is to allow the vehicle to coast along the θ-axis for a short distance from the initial states. Then the approximate chattering arc follows. The length of the coasting arc can be calculated through numerical iteration since the final longitudinal range (coasting range plus approximate chattering range) must be 0.1408. There will be a penalty on the flight time due to the reason that the vehicle does not use the maximum drag for deceleration during the coasting phase.

VI. Conclusion

The approximate chattering arc has been introduced in which the frequency of control switching between its maximum and minimum values is finite. As compared with the theoretical chattering arc in which the frequency of control switching is infinite, the approximate chattering arc at constant altitude is a two-dimensional flight. It has the same final states as the theoretical chattering arc when the number of control switching is equal to or greater than two except that the final longitudinal range is shorter. The final longitudinal range can be maximized when the number of control switching is equal to or greater than three. The shortage in the final longitudinal range is 0.5% when the number of control switching is five, and can not be improved much even with more switchings. It can be completely eliminated by adding a short coasting arc and with some sacrifice on the flight time.

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