F.E.M. ANALYSIS OF ACOUSTIC PROPAGATION IN DUCT WITH MEAN COMPRESSIBLE FLOW

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Abstract

This paper describes the development of a F.E.M. code capable of computing the propagation of acoustic waves in a ducts having a pointwise variable mean compressible flow.
The field equations are derived for both the mean flow and for the perturbation in terms of the relative velocity potentials by assuming: steadiness for the mean flow and harmonic time and azimuthal dependence for the acoustics.
A Galerkin procedure is used for the acoustic problem to get the weak residual formulation of the equation, in order to develop a FEM procedure based on quadrilateral elements of the "serendipity family".
Two codes are developed: one that solves the basic flow field, the second one that solves the acoustic field; both codes use the same geometry and the same discretization; they can interchange, with the same order of approximation, field data (geometries and flow parameter).
The codes are considered as a general purpose ones apt to have built in capabilities of accepting (in a friendly manner) boundary conditions of industrial interest.
This paper documents the preliminary validation phase and presents results relative to cases where analytical solutions are available.

1. INTRODUCTION

The limitations of the acoustic emission of flow apparatus is one of the main and more stringent demand of the modern market. To cope with such demand it is requested the development of methodologies (of analytical and numerical nature) to furnish the designer a detailed and realistic understanding of such complex phenomenologies in order to reduce the source of noise, to study its propagation, to realize efficient protections.
Examples of fields of interest include both aeronautical applications (intake and exhaust of turboengines, propellers, wind tunnels etc.) and terrestrial applications (internal combustion engines, turbomachinaries, air conditioning plants, etc). The acoustic pollution is of such current interest so that we shall not go into further detail.
This work is a result of a Research Contract between ALENIA and the University of Naples on "Determination of the Acoustic Field in axisymmetric Duct with Flow". The research was aimed toward the development of analytical modelling and numerical codes. This work, in particular, details the results relative to the second phase of the research that had the scope of developing a formulation having as primary variable the acoustic velocity potential valid for a pointwise mean compressible flow.
The work is articulated as follows: in Sect.2 the field equations are derived; in Sect.3 the problem is defined with the boundary condition of interest; in Sect.4 the resolution by Finite Element Method is detailed; in Sect.5 the main result of the first validation phase is presented.

2. DERIVATION OF THE FIELD EQUATIONS

Scope of the analysis is the derivation of the general equations for the mean flow and for the propagation of the acoustic waves in terms, respectively, of the velocity potential and of the acoustic velocity potential. The motivation of using scalar potential is based on the hope of obtaining models and a "numerically robust" algorithm.

2.1 Basic Mean Flow

Under the assumptions of:
i. non dissipative and irrotational motion
ii. absence of body forces
iii. perfect gas
iv. subsonic mean flow
the general equation of the velocity potential, relative to a compressible unsteady flow, writes as:

$$\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial}{\partial t}(\underline{V}\cdot\underline{V}) + \underline{V}\cdot\,\underline{\nabla}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) = c^2\,\nabla^2\Phi \qquad (1)$$

$$\frac{c^2}{c_r^{\,2}} = 1 - (\gamma\text{-}1)\left(\frac{\partial\Phi}{\partial t} + \frac{\underline{V}\cdot\underline{V}}{2}\right) / \; c_r^{\,2} \qquad (2)$$

where:
Φ is the velocity potential
$\underline{V}$ is the velocity
c is the sound speed
γ is the ratio of specific heats.

Equation (1), under the assumption of steady flow simplifies as:

$$\nabla^2\Phi = \frac{1}{c^2}\,\underline{V}\cdot\,\underline{\nabla}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) \qquad (3)$$

2.2 Acoustic perturbation

By perturbating in (1) and (2) the variables
($\underline{V}$, Φ, c) with the relative acoustic contributions ($\underline{v}$, φ, a), and by subtracting the terms relative to the base equations, one obtains:

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial}{\partial t}\left[\underline{V}_0\cdot\underline{\nabla}\varphi + \underline{\nabla}\varphi\cdot\underline{V}_0\right] + \frac{1}{2}\,\underline{V}_0\cdot\underline{\nabla}\left[\underline{V}_0\cdot\underline{\nabla}\varphi + \underline{\nabla}\varphi\cdot\underline{V}_0\right]$$

$$+\,\underline{\nabla}\varphi\cdot\,\underline{\nabla}\left(\frac{V_0^{\,2}}{2}\right) = c_0^{\,2}\,\nabla^2\varphi + a^2\,\nabla^2\varphi_0 \qquad (4)$$

$$a^2 = -\,(\gamma\text{-}1)\left[\frac{\partial\varphi}{\partial t} + \underline{V}_0\cdot\underline{\nabla}\varphi\right] \qquad (5)$$

By recalling:
i) the vectorial identity $\underline{\nabla}\,[\underline{a}\cdot\underline{b}] = (\underline{\nabla}\,\underline{a})\cdot\underline{b} + (\underline{\nabla}\,\underline{b})\cdot\underline{a}$,
ii) the irrotationality of the base flow,
and by duly combining the (4) and (5) it is possible to obtain the general equation for the acoustic velocity potential:

$$c_0^{\,2}\,\nabla^2\varphi - (\gamma\text{-}1)\,(\underline{\nabla}\cdot\underline{V}_0)\left[\frac{\partial\varphi}{\partial t} + \underline{V}_0\cdot\underline{\nabla}\varphi\right] - \frac{\partial^2\varphi}{\partial t^2} -$$

$$2\,\underline{V}_0\cdot\underline{\nabla}\left(\frac{\partial\varphi}{\partial t}\right) - 2\,\underline{V}_0\cdot\underline{\nabla}\,\underline{V}_0\cdot\underline{\nabla}\varphi - \underline{V}_0\cdot\underline{\nabla}\,\underline{\nabla}\varphi\cdot\underline{V}_0 = 0 \qquad (6)$$

3. PROBLEM FORMULATION

We suppose that the acoustic velocity potential has, in the plane case, a time dependence of the type:

$$\varphi(x,\, y,\, t) = \varphi'(x,\, y)\,\exp\,[-i\omega t] \qquad (7)$$

and, in the case of axi-symmetry, a time and azimuthal dependence of the type:

$$\varphi(r,\, \theta,\, z,\, t) = \varphi'(r,\, z)\,\exp\,[-i(\omega t - m\theta)] \qquad (8)$$

It is obvious that with such notation the acoustic velocity potential is a complex variable.
In order to have an unitary formulation for both cases (plane and axisymmetric) we shall denote with "x" the longitudinal axis (x=z) and with "y" the radial axis (y=r).
We use the following non-dimensionalization:

$$\tilde{x} = x/D \;\; ; \;\; \tilde{y} = y/D \;\; ; \;\; \tilde{\varphi} = \varphi'/(c_0 D)$$

$$U = \frac{u_0}{c_0} \;\; ; \;\; V = \frac{v_0}{c_0} \;\; ; \;\; \underline{M} = \frac{\underline{V}}{c_0} \;\; ; \;\; \tilde{\omega} = \omega\, D/c_0 \qquad (9)$$

where:
D = reference length (diameter/height)
c_0 = speed of sound (constant)

Under the above assumptions, equation (6) assumes the non-dimensional form:

$$\nabla\cdot\underline{\nabla}\tilde{\varphi} + B\,\frac{\partial^2\tilde{\varphi}}{\partial\tilde{x}^2} + C\,\left(\frac{\partial^2\tilde{\varphi}}{\partial\tilde{x}\partial\tilde{y}} + \frac{\partial^2\tilde{\varphi}}{\partial\tilde{y}\partial\tilde{x}}\right)$$

$$+\,E\,\frac{\partial^2\tilde{\varphi}}{\partial\tilde{y}^2} + F\,\frac{\partial\tilde{\varphi}}{\partial\tilde{x}} + G\,\frac{\partial\tilde{\varphi}}{\partial\tilde{y}} + H\,\tilde{\varphi} = 0 \qquad (10)$$

having denoted:

$$B = -\,U^2 \;\; ; \;\; C = -\,UV \;\; ; E = -\,V^2 \qquad (11a)$$

$$F = \left[-(\gamma\text{-}1)(\underline{\nabla}\cdot\underline{M}_0) + 2i\omega U - 2(UU_x + VU_y)\right] \qquad (11b)$$

$$G = \left[-(\gamma\text{-}1)(\underline{\nabla}\cdot\underline{M}_0) + 2i\omega V - 2(UV_x + VV_y)\right] \qquad (11c)$$

$$H = \left[\omega^2 - (m/r)^2 + (\gamma\text{-}1)i\omega(\underline{\nabla}\cdot\underline{M}_0)\right] \qquad (11d)$$

By looking to problems of practical interest, the boundaries conditions are determined as four kind of boundaries (Fig.1):

C_1: assigned acoustic potential; this represents a Dirichlet condition that is very easily implemented in a F.E.M. code;

C_2: rigid wall, i.e. zero normal acoustic velocity; it is a Neumann condition, naturally satisfied in a variational formulation;

C_3: assigned normal component of the acoustic velocity; it is a Neumann condition for the variational formulation;

C_4: assigned normal acoustic impedance.

Whereas the first three conditions are very easy to be implemented in a F.E.M. formulation, the fourth condition on C_4 deserves some comments.
The energy equation for the basic flow, under the assumptions made above can be written as:

$$\frac{\partial \Phi}{\partial t} + \left(\frac{\gamma}{\gamma-1}\right)\frac{P}{\rho} + \frac{V^2}{2} = \text{const} \tag{12}$$

where P is the pressure.
By making a perturbation of (12) due to sound, by subtracting the basic contribution, it is possible to derive the linearized equation:

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho_0} + \underline{V}_0 \cdot \underline{\nabla}\varphi = 0 \tag{13}$$

that can be used to compute the acoustic pressure (p) as :

$$p = -\rho_0 \left(\frac{\partial \varphi}{\partial t} - \underline{V}_0 \cdot \underline{\nabla}\varphi\right) \tag{14}$$

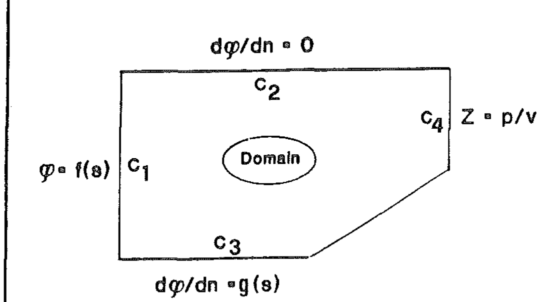

Fig.1 : Boundary Conditions

that in non-dimensional form looks as:

$$\tilde{p} = i\,\tilde{\omega}\,\tilde{\varphi} - U\frac{\partial \tilde{\varphi}}{\partial \tilde{x}} - V\frac{\partial \tilde{\varphi}}{\partial \tilde{y}} \tag{15}$$

so that the condition on C4 can be explicited as:

$$\frac{\partial \tilde{\varphi}}{\partial n} = \frac{\tilde{p}}{Z_n} = \frac{\tilde{1}}{Z_n}\left(i\,\tilde{\omega}\,\tilde{\varphi} - U\frac{\partial \tilde{\varphi}}{\partial \tilde{x}} - V\frac{\partial \tilde{\varphi}}{\partial \tilde{y}}\right) \tag{16}$$

where Z_n is the acoustic impedance.

4. F.E.M. SOLUTION

4.1 Basic Mean Flow

The equations that govern such field are:

$$\nabla^2 \Phi = \frac{1}{c^2}\,\underline{V}\cdot\underline{\nabla}\left(\frac{\underline{V}\cdot\underline{V}}{2}\right) \tag{17}$$

$$\frac{c^2}{c_r^{\,2}} = 1 - (\gamma-1)\,\frac{\underline{V}\cdot\underline{V}}{2\,c_r^2} \tag{18}$$

They describes, for subsonic flows, an elliptic problem composed by a non linear partial differential equation coupled with an algebraic equation.
By looking toward an iterative technique (at level n+1) based on FEM, equation (17) can be written as a Poisson problem of the type:

$$\nabla^2 \Phi^{n+1} = f^n \tag{19}$$

where the source term (f^n) :

$$f^n = \left[\underline{\nabla}\Phi^n \cdot \underline{\nabla}\left(\underline{\nabla}\Phi^n \cdot \underline{\nabla}\Phi^n\right)\right]/2(c^n)^2 \tag{20}$$

is computed at the previous iteration level.
A FEM technique for (19), even if very simple in principle, is limited to cases where second order elements are used to allow the computation of the second derivatives contained in the source term (20).
An alternative formulation, that does not have such inconvenience, is based on the continuity equation written in terms of the velocity potential:

$$\underline{\nabla}\cdot\left(\rho^n(\underline{\nabla}\Phi^{n+1})\right) \tag{21}$$

where the density (ρ^n) is computed at the previous iteration, by assuming the isentropicity:

$$\rho^n=\rho_r\left[1-(\gamma-1)(\underline{\nabla}\Phi^n\cdot\underline{\nabla}\Phi^n)/2c_r^2\right]^{1/\gamma-1} \quad (22)$$

Problem (21) and (22) admits the classical variational formulation, i.e. the solution field (Φ) must minimize the functional:

$$I(\Phi^n)=\frac{1}{2}\iint\rho^n(\underline{\nabla}\Phi^{n+1}\cdot\underline{\nabla}\Phi^{n+1})\,dA+\int_{c_1}\rho^n\,\Phi^{n+1}\,g\,dc \quad (23)$$

where Φ must belong to the space of functions that:

$$\{L^2(\Phi):\Phi=\text{constant on } C_3;\ \partial\Phi/\partial n=g \text{ on } C_1\}\ .$$

4.2 Acoustic Field

To derive a formulation of the problem defined by (10-11) with the boundary conditions C_1, C_2, C_3, C_4, apt to be utilized for a F.E.M., a Galerkin technique was used in order to obtain a weak variational formulation. In this regard the (10) is multiplied by a weight function $W(x,y) > 0$ and thereafter integrated on the domain.
By recalling the Gauss theorem and the generalized gradient theorem (projected on the "x" axis), by introducing the BC.s on C_3 and on C_4 one obtains the integral equation:

$$\iint\left\{(1+B)W_x\Phi_x+C(W_y\Phi_x+W_x\Phi_y)+(1+E)W_y\Phi_y\right\}dA$$

$$-\iint\left\{F\Phi_x+G\Phi_y+H\Phi\right\}W\,dA+\frac{1}{z_n}\int_{c_4}\left[-i\omega\Phi+U\Phi_x+V\Phi_y\right]W\,dc$$

$$-\int_{c}\left[B\Phi_xU_x+C(\Phi_xU_y+\Phi_yU_x)+F\Phi_yU_y\right]W\,dc=\int_{c_3}Wg\,dc \quad (24)$$

By following the usual procedure of the F.E.M. , the equation (24) is discretized by subdividing the field in a number (k) of elements. On each element the field variable $\tilde{\varphi}(x,y)$ is approximate as

$$\tilde{\varphi}=\sum_{j=1}^{n}N_j\,\tilde{\varphi}_j \quad (25)$$

where:

n is the number of nodal points of the element
N_j are the shape functions
$\tilde{\varphi}_j$ are the values of $\tilde{\varphi}$ in the (k) nodal points.
According to the Galerkin procedure the same discretization is used for the weight function W(x,y):

$$W=\sum_{i=1}^{n}N_iW_i \quad (26)$$

By substituting, and by integrating on each element and boundary, summing up on each element, one obtains a (discretized) functional that can be minimized by taking its derivative with respect to the nodal values of the weight function W_i (supposed all different from zero). It results a system of complex algebraic equations of the type:

$$K_{ij}\,\tilde{\varphi}_j=b_i \quad (27)$$

that is afterwards manipulated to impose the conditions on C_1.
It is worthy to note that:
- *the matrix K_{ij} is not symmetric (due to* terms deriving from the basic flow)
- the resulting system is defined in a complex space.

The novelty of the code lies in the fact that it makes use directly of a complex variable to represent the acoustic potential (instead of dealing with its real and imaginary parts). Despite the cost of developing new routines, the resulting code exhibits a very clear and cogent representation of the equations, of the various terms in the construction of the matrices needed to the implementation of the system of equations.
Once the solution for the acoustic velocity potential is computed, the other acoustic field variables are derived as:

$$u=\frac{\partial\tilde{\varphi}}{\partial\tilde{x}}\quad;\quad v=\frac{\partial\tilde{\varphi}}{\partial\tilde{y}}$$

$$\tilde{p}=i\,\tilde{\omega}\,\tilde{\varphi}-U\frac{\partial\tilde{\varphi}}{\partial\tilde{x}}-V\frac{\partial\tilde{\varphi}}{\partial\tilde{y}} \quad (28)$$

5. RESULTS

The results herein presented are relative to the first validation phase of the code: simple but significative cases are analyzed where analytical solutions are available.

5.1 Type of elements

The elements used in the code belong to the "serendipity family"[2]; in particular quadrilateral elements with four (linear) and eight nodes (bi-linear) were considered.
The first numerical experiments have been devoted to the capability of such elements to describe the sound propagation process.

It was noted that, whereas for planar problems both elements show good behaviour, in the case of axisymmetric problems the eight nodes elements are not able to successfully describe the physics when the symmetry line is part of the field or is near to the inner element (toroidal geometries). Such phenomenon is present in other structural codes.
For these reasons all the axisymmetric cases are solved with four nodes quadrilateral elements.

5.2 Cylindrical Duct

The geometry considered is a cylindrical cavity, having radius half of the length, that is discretized with a regular mesh of 50 quadrangular elements.
The acoustic boundary conditions at the inlet simulate fixed constant acoustic velocity potential; anecoical conditions at the exit are assumed (absence of acoustic reflections).
A constant mean flow is supposed to be present along the duct.

5.2.1 Discretization limitations

The criteria to determine the finiteness of the mesh apt to adequately solve the acoustic problem for various sound frequency was searched.
Let us define a "discretization parameter" (δ) :

$$\delta = \Delta / \left[(c(1-M^2)^{1/2})/(D\ f)\right] \qquad (29)$$

where:

c is the speed of sound
Δ is the minimum (dimensionless) distance between nodes of an element
D is the reference length
f is the acoustic frequency

(δ) represents the ratio between the minimum discretized dimension and the acoustic wave length; its inverse represents the number of nodes contained in one wave length.
The results of the experimentations show that for values of δ up to 0.09 (more than 10 nodes per wave length) satisfactory results are obtained.

5.2.2 Results for non-planar spinning waves

Non planar sound waves can exist in axisymmetric ducts if the sound frequency is larger than the first critical frequency [1,3].
Being "m" the azimuthal number and "s" the radial number, a waves with a (m,s) mode can exist only if the sound frequency (f) is larger than the critical one, $f_{m,s}$, defined as:

$$f_{m,s} = j_{m,s} \left[(c(1-M^2)^{1/2})\right]/(\pi\ D)$$

where $j_{m,s}$ is the first zero of the Bessel function $J'_{m,s}$.
Fig.2 shows the fields of the critical frequencies and the resultant acoustical modes.

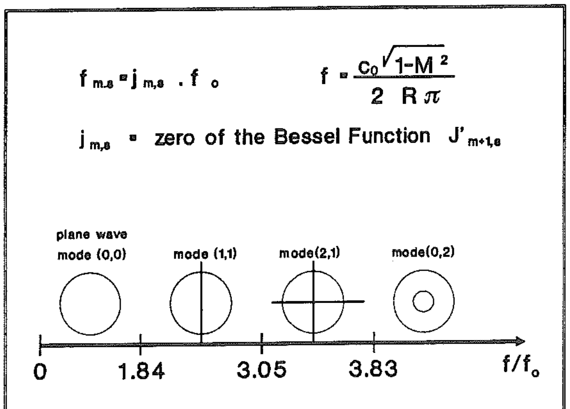

Fig.2 : Critical Frequency fields and modes

For a cylindrical square geometry (unitary diameter equal to length) we report experimental runs relative to two cases:

a. M=0, f=200 [Hz], mode (1,1)
b. M=0.4, f=168.75 [Hz], mode (1,1)

Fig.3 show the longitudinal profiles of the acoustical velocity potential at r=0.5 for the case (a), here the numerical values of the real and of the imaginary parts of the acoustical potential are compared with the corresponding values of the analytical solution. Fig.4 show, for case (a) the radial profiles of the acoustic potential (numerical and analytical) at the exit. In both cases the quality of the numerical data is satisfactory.

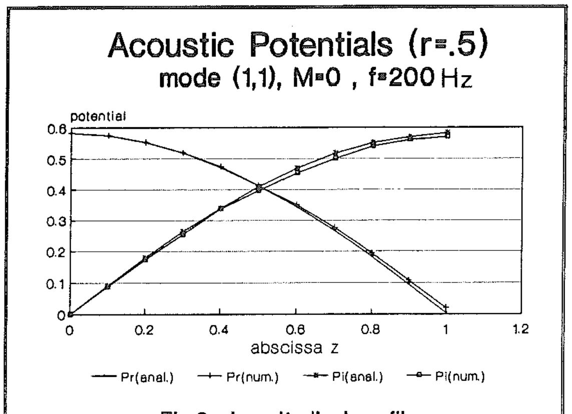

Fig.3 : Longitudinal profiles

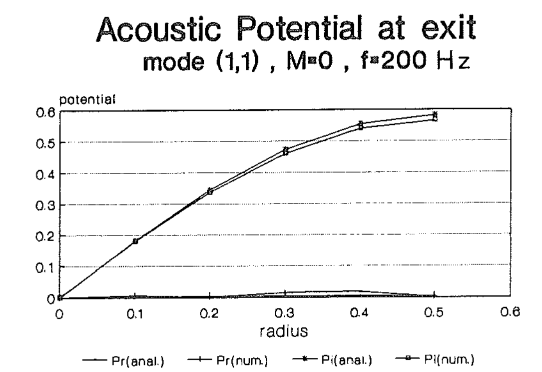

Fig.4 Radial profiles

Fig.5 and Fig.6 display for case (a), respectively the longitudinal and the radial profiles of the real and imaginary parts of acoustic potential, acoustic pressure, and acoustic velocity.

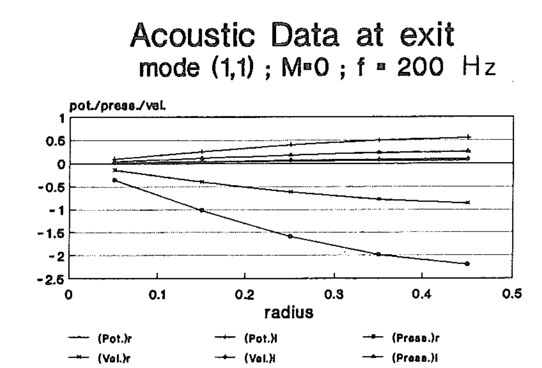

Fig.5 : Profiles at exit

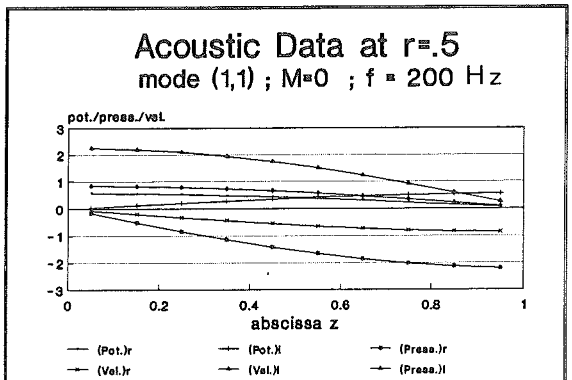

Fig.6 : Axial Profiles

Fig.7 and Fig.8 reports, for case (b), the longitudinal and radial profiles of the computed acoustic potential against the analytical ones. When applicable the agreement between analytical and numerical results is more than satisfactory.

5.3 Conical Horn

Such problem was identified to test the code for non uniform geometry.
As well known, small angles conical horns at low frequency admits as solution a spherical wave, emanating from an equivalent point source located at the apex of the cone, whose amplitude is inversely proportional to the radial distance from the point source.
Fig. 9 shows the geometry used and the BC.s : solid wall and anecoical termination are assumed for a horn having 21.5° semi-angle, starting at 1.36 from the source and having a length of 2.
Fig. 10 reports the acoustic potential profiles along the horn for the case of no mean flow (m=0) and a frequency of 200 [Hz]. Real and

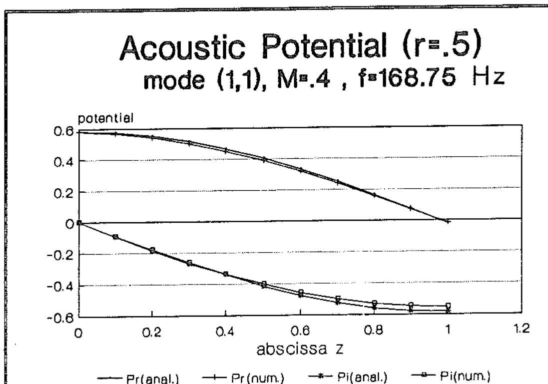

Fig.7 : Longitudinal Profile

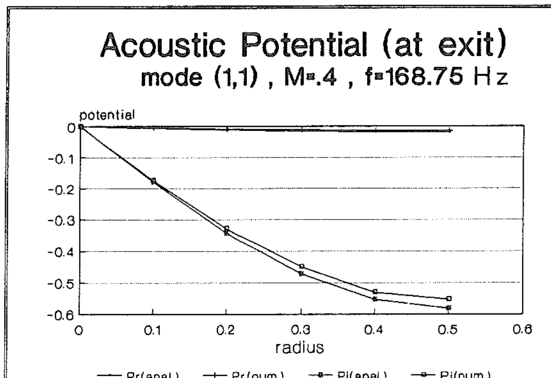

Fig.8 : Profiles at exit

Imaginary parts of the Acoustic Velocity Potential are plotted together with its modulus, the latter is compared with the theoretical one. Fig. 11 reports the acoustic velocity potential at the exit: the almost constancy of both real and imaginary parts reveals the performance of the code.
Finally, Fig. 12 reports the effect of the mean flow (m=0.5) on the potential profiles.

6. CONCLUSIONS

The capability of the modelling and the resultant code to describe the acoustical properties of ducts with mean flow, for simple but significative cases, has been demonstrated. The code is able to simulate, in a friendly manner, boundary conditions of industrial interest, and provides fruitful contributions to the design of flow systems.
The preliminary validation runs, herein reported, refer to cases were analytical solutions are available; in the following of the research, comparisons with experimental data are foreseen.

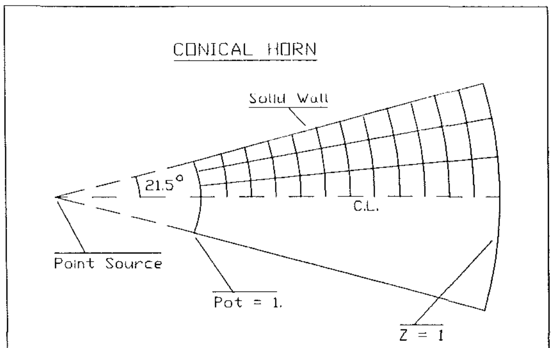

Fig. 9 Conical Horn

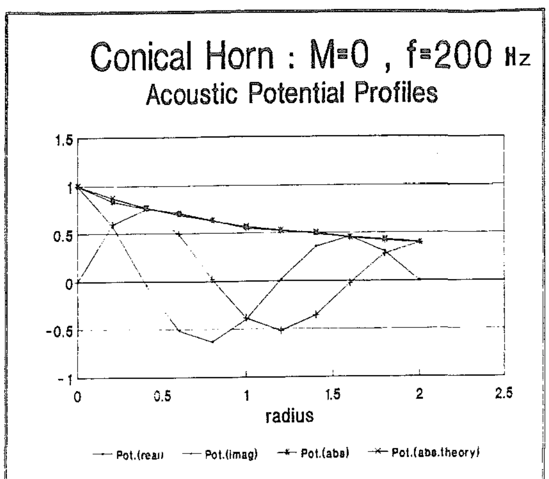

Fig. 10

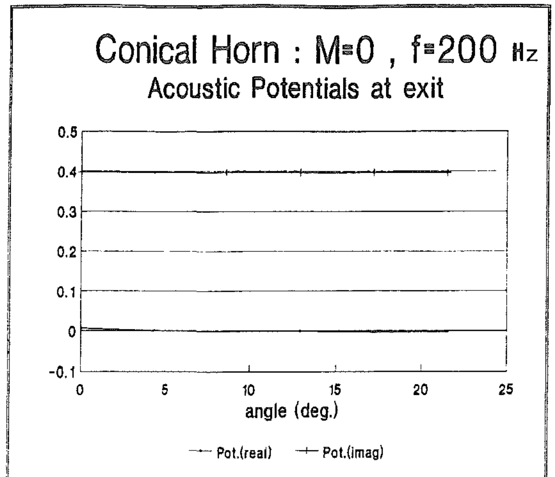

Fig. 11

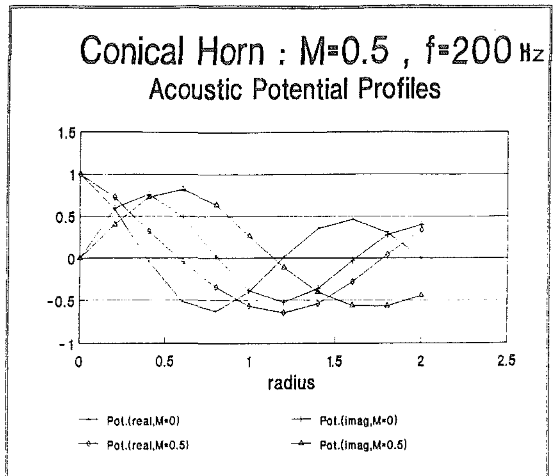

Fig. 12

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