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THEORY AND EXPERIMENTS ON OPERATING PRINCIPLE OF HEMISPHERICAL RESONATOR GYRO†

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Abstract

On the basis of the thin–wall shell's theory and practical construction feature of hemispherical shell resonator(HRG), this paper establishes the HRG's math model and derives the relation between scale factor and construction parameters. Experiments were carried out with laser holographic interferometry which quite agrees with theoretical model. The results provide a reliable theoretical basis for developing the new gyro, especially for designing the hemispherical shell resonator.

I .Introduction

As compared with the conventional gyro, hemispherical resonator gyro without spinning–wheel and bearing takes advantage of mechanical resonating technique and Coriolis effect occured as the hemispherical shell resonator rotated. It has many unique advantages, such as small size, short reaction time, long time constant, allowing operation over a wide temperature range with no temperature control and warm–up delay ect. It appears that HRG is one of the best components in the new generation of strap–down inertial navigation system. USA firstly reported that Delco System Operation had developed an inertial grade HRG performance in 1982(1), but no theoretical derivation therein. Soon after that, former USSR also reported some results studied on HRG (2). But there some considerable difference between the math model and reality, up to now.

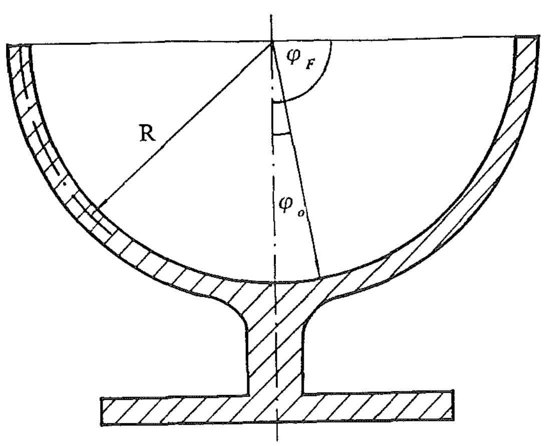

FIGURE 1. Hemispherical Shell Construction

A thin–wall hemispherical shell in resonant vibrating state, with the top free and the bottom clamped (Fig.1), is the sensing component of the HRG. When the shell rotates angle ψ_1 about its central axis, the vibrating shape in circumferential direction of the hemispherical shell will move or precess an angle ψ relative to the shell reverse movenent(Fig.2). It's so surprised that the ratio of ψ / ψ_1 is a constant, and independent of the rotation angular velocity of the above shell. So, the angle ψ can be calulated by measuring the precession of the vibrating shape in circumferential direction. This is the operating principle of HRG.

In fact, the above precession phenomenon for the hemispherical shell, was pointed out in 1890 by G.H.bryan, a British Scientist (3). In this paper, the precession of the vibrating shape in circumferential direction for the practical hemispherical shell is analyzed, based on the thin–wall shell's theory. And some valuable results for designing and developing HRG are proposed through the theoritical and experimental researches.

II. Vibrating Shape

Fig.3 is the sketch of the hemispherical shell, $\vec{X}$ is the central axis, the mean radius of the shell is R, the material curved surface density is $D(\varphi,\theta)$, the bottom and top angles are φ_o and φ_F respectively. φ and θ are the longitudinal and circumferential coordinates respectively.

The displacement of the arbitrary point B is

$$\vec{V} = u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 \quad (1)$$

Where u, v, w are respectively the longitudinal, circumferential and radial displacements. $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are respectively the relevent unit moving vectors.

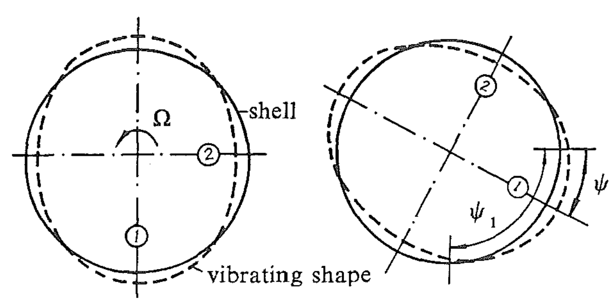

FIGURE 2. Precession Scheme of Shell

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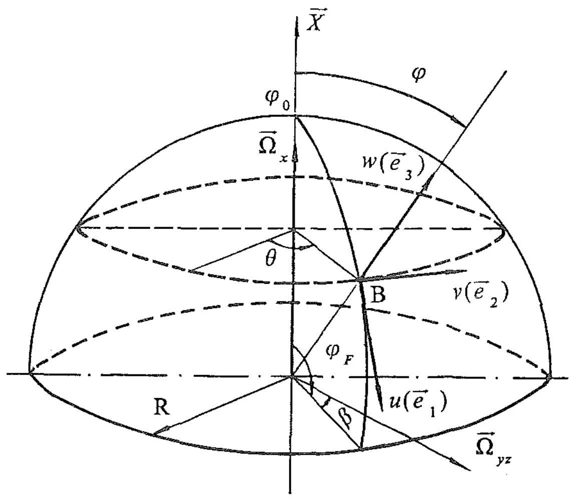

FIGURE 3. Sketch of Hemispherical Shell

When the hemispherical shell does not rotates, its nth axisymmetric vibrating shape can be written as

$$\left.\begin{aligned} u(\varphi,\theta,t) &= u(\varphi)cosn\theta cos\omega t \\ v(\varphi,\theta,t) &= v(\varphi)sinn\theta cos\omega t \\ w(\varphi,\theta,t) &= w(\varphi)cosn\theta cos\omega t \end{aligned}\right\} \tag{2}$$

Where u(s), v(s), w(s) are respectively vibrating shape in longitudinal direction and ω is the relative natural frequency of the hemispherical shell.

When the shell rotates at $\vec{\Omega}=\vec{\Omega}_x+\vec{\Omega}_{yz}$ in the inertial space (Fig.3). And in the rotation space, the corresponding nth axisymmetric vibrating shape can be written as

$$\left.\begin{aligned} u(\varphi,\theta,t) &= u(\varphi)cos[n(\theta+\psi)]cos\omega t \\ v(\varphi,\theta,t) &= v(\varphi)sin[n(\theta+\psi)]cos\omega t \\ w(\varphi,\theta,t) &= w(\varphi)cos[n(\theta+\psi)]cos\omega t \end{aligned}\right\} \tag{3}$$

$$\psi = \int_{t_0}^{t} Pdt$$

Where P is the precession rate of the vibrating shape related to the shell in circumferential direction. As compared with equations (2) and (3), when the shell rotates at $\vec{\Omega}$ about its central axis $\vec{X}$ and moves angle $\psi_1 = \int_{t_0}^{t}\Omega_x dt$, the vibrating shape retroacts at P in circumferential direction, and moves angle $\psi = \int_{t_0}^{t} Pdt$. From the point of view of the vibrating shape, the equation (3) includes two principal vibrating shape. One is the vibration of the shell, another is the rotation of the vibrating shape of the shell. Both were represented respectively by $cos\omega t$ and $cos[n(\theta+\psi)]$ or $sin[n(\theta+\psi)]$.

The actual hemispherical shell is free at the top (φ_F) and clamped at the bottom (φ_0). So, Lord Rayleigh's condition of inextensibility are satisfied. The vibrating shape in longitudinal direction should satisfy(4).

$$\begin{cases} u(\varphi) = v(\varphi) \\ w(\varphi) = -\dfrac{du(\varphi)}{d\varphi} \end{cases} \tag{4}$$

And when the bottom angle φ_0 is much smaller, the vibrating shape approximately is

$$\left.\begin{aligned} u(\varphi) = v(\varphi) &= Csin\varphi tg^n\frac{\varphi}{2} \\ w(\varphi) &= -C(n+cos\varphi)tg^n\frac{\varphi}{2} \end{aligned}\right\} \tag{5}$$

C is the constant which is dependent on the vibration energy of the shell.

III. Virtual Work

When the shell rotates at $\vec{\Omega}$, the inertial force in point B is

$$\vec{F} = -\vec{a}D(\varphi,\theta)R^2sin\varphi d\varphi d\theta \tag{6}$$

$$\vec{a} = \vec{a}_0 + \vec{a}(\Omega_x) + \vec{a}(\Omega_{yz}) \tag{7}$$

$$\vec{a}_0 = \frac{\partial^2 u}{\partial t^2}\vec{e}_1 + \frac{\partial^2 v}{\partial t^2}\vec{e}_2 + \frac{\partial^2 w}{\partial t^2}\vec{e}_3 \tag{8}$$

$a(\Omega_x)$, $a(\Omega_{yz})$ are the accelerations caused by the angular velocity Ω_x, Ω_{yz} respectively. So, the virtual work done by inertial $\vec{F}$ is

$$\delta T = \int_S \vec{F}\cdot\delta\vec{V} \tag{9}$$

S is the integrated curved surface domain. Using the relevent relationships in Reference (4), we have

$$\delta T = \delta T_0 + \delta T(\Omega_x) + \delta T(\Omega_{yz}) + \delta T(\Omega) - \delta W(\Omega) \tag{10}$$

$$\delta T_0 = \omega^2R^2cos^2\omega t\int_{\varphi_0}^{\varphi_F}\int_0^{2\pi}\Big\{\Big[u(\varphi)\delta u + w(\varphi)\delta w\Big]cos^2n(\theta+\psi) + v(\varphi)\delta v sin^2n(\theta+\psi)\Big\}D(\varphi,\theta)sin\varphi d\theta d\varphi \tag{11}$$

$$\begin{aligned}\delta T(\Omega_x) = R^2cos^2\omega t\Big(n^2P^2\int_{\varphi_0}^{\varphi_F}\int_0^{2\pi}&\Big\{\Big[u(\varphi)\delta u + w(\varphi)\delta w\Big]cos^2n(\theta+\psi) + v(\varphi)\delta v sin^2n(\theta+\psi)\Big\}D(\varphi,\theta)sin\varphi d\theta d\varphi \\ &+ 2nP\Omega_x\int_{\varphi_0}^{\varphi_F}\int_0^{2\pi}\Big\{cos\varphi\Big[v(\varphi)\delta u cos^2n(\theta+\psi) + u(\varphi)\delta v sin^2n(\theta+\psi)\Big] + sin\varphi\Big[v(\varphi)\delta w cos^2n(\theta+\psi) \\ &+ w(\varphi)\delta v sin^2n(\theta+\psi)\Big]\Big\}\ D(\varphi,\theta)sin\varphi d\theta d\varphi\Big)\end{aligned} \tag{12}$$

$$\delta T(\Omega_{yz}) = 2nP\Omega_{yz}R^2cos^2\omega t\int_{\varphi_0}^{\varphi_F}\int_0^{2\pi}\Big\{\Big[w(\varphi)sin\beta sinn(\theta+\psi) + v(\varphi)sin\varphi cos\beta cosn(\theta+\psi)\Big]cosn(\theta+\psi)\delta u + \Big[u(\varphi)sin\varphi$$

$$- w(\varphi)cos\varphi \Big] cos\beta sin^2 n(\theta+\psi)\delta v - [u(\varphi)sin\beta sinn(\theta+\psi)$$

$$+ v(\varphi)cos\beta cos\varphi cosn(\theta+\psi) \Big] cosn(\theta$$

$$+ \psi)\delta w \Big\} \cdot D(\varphi\ ,\theta)sin\varphi d\theta d\varphi \quad (13)$$

The definition of angle β is shown in Fig.3, $\delta W(\Omega)$ is the virtual work done by the initial elastic force which is caused by $\vec{\Omega}$, $\delta T(\Omega)$ is the virtual work, which is independent of the rotation rate P, done by the ″grave inertial″ force caused by $\vec{\Omega}$. Using the virtual work principle, we have

$$\delta T + \delta W_0 = \delta T_0 + \delta T(\Omega_x) + \delta T(\Omega_{yz}) + \delta T(\Omega) - \delta W(\Omega) + \delta W_0 = 0 \quad (14)$$

δW_0 is the virtual work done by the elastic force as the shell dose not rotate.

In the equation (14), each term represents the different physical meaning.

For a certain vibrating mode [ω,(u,v,w)](see equation (3)), $\delta T(\Omega) - \delta W(\Omega) + \delta W_0$ is the virtual work done by elastic force on the above mode, for the rotating shell, corresponding to the elastic potential energy; and δT_0 is the virtual work done by inertial force on the above mode, corresponding to the vibration kinetic energy. So, the vibration characteristics, vibrating frequency and vibrating shape in longitudinal direction, are dependent on $\delta T(\Omega) - \delta W(\Omega) + \delta W_o$ and δT_0. On the other hand, the rotation of the shell at Ω (or Ω_x) represents the rigid movement, which also is a principal vibration. According to the law of conservation of energy for the principal vibration, we have the following relationships by using equation (14)

$$\delta T_0 + \delta T(\Omega) - \delta W(\Omega) + \delta W_0 = 0 \quad (15)$$

$$\delta T(\Omega_x) + \delta T(\Omega_{yz}) = 0 \quad (16)$$

Where equation (15) can be used to calculate the natural frequency as the shell rotates at $\vec{\Omega}$(4).

And the equation (16) can be used to analyze and calculate precession of the vibrating shape in circumferential direction.

IV. Precession

Using the equation (4) and the boundary condition of the shell, we can conclude

$$2G_x\Omega_x + 2G_{yz}\Omega_{yz} + nP\Omega_p = 0 \quad (17)$$

$$G_p = \int_0^{2\pi} \Big[\int_{\varphi_0}^{\varphi_F} u(\varphi)D(\varphi,\theta)sin\varphi d\varphi + w(\varphi_F)D(\varphi_F,\theta)sin\varphi_F cos^2 n(\theta+\psi) \Big] d\theta \quad (18)$$

$$G_x = \int_0^{2\pi} \Big\{ \int_{\varphi_0}^{\varphi_F} \Big[\frac{1}{2}u(\varphi)sin2\varphi + w(\varphi)sin^2\varphi sin^2 n(\theta + \psi)\Big] D(\varphi,\theta)d\varphi + u(\varphi_F)D(\varphi_F,\theta)sin^2\varphi_F \Big\} d\theta \quad (19)$$

$$G_{yz} = \int_0^{2\pi} \Big\{ \int_{\varphi_0}^{\varphi_F} \Big[\frac{1}{2}w(\varphi)sin\beta sin2n(\theta+\psi) - w(\varphi)cos\varphi cos\beta sin^2 n(\theta+\psi) + u(\varphi)sin\varphi cos\beta\Big] D(\varphi,\theta)sin\varphi d\varphi - \Big[\frac{1}{2}sin\beta sin2n(\theta+\psi) + cos\varphi_F cos\beta cos^2 n(\theta+\psi)\Big] u(\varphi_F)D(\varphi_F,\theta)sin\varphi_F \Big\} d\theta \quad (20)$$

G_x, G_{yz} represent the Coriolis effects caused by $\vec{\Omega}_x$ and $\vec{\Omega}_{yz}$ respectively, and G_p represents the inertial effect for the shell.

As the curved surface density $D(\varphi,\theta)$ is constant in the circumferential direction,

$$D(\varphi,\theta) = D(\varphi) \quad (21)$$

Then the $G_{yz} = 0$, which indicates that the precession of the vibrating shape in circumferential direction is only caused by the angular velocity Ω_x, and independent of the Ω_{yz}. Therefore, only one directional angular message can be sensed, and there is no error of the cross influency in principle for HRG.

Then, by using the above relevent equations, we can conclude the precession factor

$$K = \frac{P}{\Omega_x} = \frac{2[\int_{\varphi_o}^{\varphi_F} (n - cos\varphi)sin^2\varphi tg^n\frac{\varphi}{2} D(\varphi)d\varphi - sin^3\varphi_F tg^n\frac{\varphi_F}{2} D(\varphi_F)]}{n[\int_{\varphi_0}^{\varphi_F} 2sin^2\varphi tg^n\frac{\varphi}{2} D(\varphi)d\varphi - sin\varphi_F(n + cos\varphi_F)tg^n\frac{\varphi_F}{2} D(\varphi_F)]} \quad (22)$$

Table 1 gives the results calculated by equation (22) as the curved surface density is uniform $D(\varphi) = D_o$.

It is shown that the n = 2 is the best vibrating mode for HRG from the point of the view of measurement, therein $K \approx 0.3$, moreover, whose relevent vibrating frequency is the lowest, thus easy making the shell in resonant vibrating state.

V. Error Model

For a actual hemispherical shell resonator, its curved surface density $D(\varphi,\theta)$ is a variable instead of a constant in circumferential direction. In this paper, by the way, we describe the deficiency of the hemispherical shell in circumferential direction. So, G_x, G_{yz}, G_p are all variables versus the angle ψ, which is shown as follows.

1. When $\Omega_{yz} = 0$, the precession rate P caused by Ω_x is related to the angle ψ or the distribution of the vibrating shape in circumferential direction. The precession factor K is a variable, which causes the measurement

n	φ_o	φ_F			
		72°	78°	84°	90°
2	0°	0.288288	0.305815	0.310811	0.297824
	5°	0.298485	0.305960	0.310942	0.297938
3	0°	0.088732	0.086866	0.079549	0.065987
	5°	0.088735	0.086867	0.079550	0.065988
4	0°	0.038224	0.035743	0.030764	0.023188
	5°	0.038224	0.035743	0.030764	0.023188

TABLE 1 The Precession Factor

relative error for HRG. We define the precession factor K_x which is only caused by the Ω_x.

$$K_x = \frac{P}{\Omega_x} = \frac{-2G_x}{nG_p} \tag{23}$$

In order to analyze the relative error, define

$$\sigma_x = \frac{K_{xmax} - K_{xmin}}{K_{xo}} \tag{24}$$

Where K_{xmax}, K_{xmin}, are respectively the maximum and minimum values of K_x, and K_{xo} is the ideal value of K_x that is the $D(\varphi,\theta)=D(\varphi)$.

2. Since $G_{yz} \neq 0$, the Ω_{yz} may also cause the precession in circumferential diredtion, which is not the same as the above ideal condition that the $D(\varphi,\theta)=D(\varphi)$, which causes the measurement absolute error for HRG, called as the error of cross influence.

We define the precession factor K_{yz}, which is only caused by Ω_{yz}.

$$K_{yz} = \frac{P}{\Omega_{yz}} = \frac{-2G_{yz}}{nG_p} \tag{25}$$

Therefore, using the equations (24) and (25), we can analyze the σ_x and $(K_{yz})_{max}$ in order to analyze and determine the influence of the deficiency on the $D(\varphi,\theta)$ for the precession of the actual hemispherical shell resonator in circumferential direction.

Ⅵ. Experimental Results

In this paper, only the experimental results are provided. We take advantage of the double pulse laser to carry out the holographic interferometric experiment, whose advantage is that the instantaneous vibration condition of the hemispherical shell resonator can be recorded accuretely, and the distribution of the vibrating shape in circumferential direction can be obtained. When the distributions of the vibrating shape at the two different times are obtained, the angle ψ, the precession angle of the vibrating shape in circumferential direction may be calculated. And the angle ψ_1, the angular displacement during the above two times is set in advance. The precession factor can be calculated by $K=\psi/\psi_1$.

According to the above principle, we carry out repetitively the experiment on a test sample made of Ni–Span–C, whose parameters are $\varphi_o=6°$, $\varphi_F=90°$, R=24.7mm. The average value of the precession factor is about 0.27, which agrees well with the above theory.

Ⅶ.Conclusions

1. The uniformity in circumferential direction, especially in the neighborhood of the top, is the important principle for designing and selecting the hemispherical shell resonator, hereupon, the precession only caused by the Ω_x rotated about the central axis of the shell. There is no error of cross influence for HRG in principle.

2. In this paper, based on the thin–wall shell's theory, orthogonality of the principal vibration mode and practical construction feature of the hemispherical shell resonator, we present the following formulae:

Equation (22), for finding the precession factor K versus φ_o, φ_F, $D(\varphi)$, reflected the precession of the vibrating shape in circumferential direction, caused by Ω_x.

Equation (24), for finding the maximum relative error in principle.

Equation (25), for finding the maximum absolute error in principle.

In a sense, equations (24), (25) may be determined the deficient limit in circumferential direction, or the permission variation limit of $D(\varphi,\theta)$ versus the circumferential coordinate θ, in principle. They may be referenced for selecting the parameters for the hemispherical shell resonator.

3. In this paper, a novel experimental project that takes advantage of the laser holographic interferometry was proposed. And the final experimental results obtained here, may provide the reliable theoretical basis for developing and designing HRG, especially for selecting the hemispherical shell resonator.

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