CONVERGENCE ACCELERATION AND WAVE DRAG DETERMINATION IN TRANSONIC AIRFOIL CALCULATIONS

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Abstract

It is shown that one of the reasons for a relatively slow iteration process convergence during transonic potential flow calculations by relaxation methods is the calculation in the vicinity of the infinity point. The exclusion of this domain from the calculation region and using of the Dirichlet type condition on its boundary leads to an appreciable convergence acceleration and computational time reduction. The analogous method can be utilized for the calculations of axisymmetrical bodies and wings.

The second question involved deals with the determination of the wave drag in the potential airfoil flow calculations. The drag values were corrected for the nonconservativity of the finite-difference scheme and potential model errors and the result agrees well with the Euler equation solutions.

I. INTRODUCTION

The ideal gas flows are governed by the Euler equations. A numerical solution of these equations requires a great deal of compututer time. Thus in practice the potential flow model is often used which gives satisfactory results in case of weak shocks. The potential flows are successfully calculated by relaxation methods.

The present work deals with two problems connected with potential airfoil flows as an example. In the first part of the work the method of the convergence rate acceleration in the solution of the potential equation by the relaxation technique is considered. It is known that the relaxation techniques are characterized by a rather slow convergence. When solving such problems as the aerodynamic form optimization, taking account of the viscosity effects by the boundary layer approach, repeated calculations of the flow about different configurations are required. This stimulates the computational time reduction escpecially in connection with the application of the mini- and personal computers.

Lately various methods of transonic problems convergence acceleration are explored. To do it extrapolation methods[1], direct Poisson solvers[2], ADI- and AF-methods[3-4], multigrid methods[5], schemes for the vectorized computers[6] and some others are used. In several cases such methods makes it possible to reduce the computer time by an order of magnitude as compared with the successive line overrelaxation (SLOR) methods. However, as a rule, such methods are considerably more complex from the programming point of view than the SLOR methods, and the last of the above pointed methods requires special vectored processor computers.

In the present paper it is shown that the slow convergence of the SLOR-methods is to a significant extent connected, with the need of calculation in the vicinity of the infinity point. The exclusion of the far field domain from the calculation region and using the Dirichlet type condition for the potential on its boundary taken from the crude mesh calculations gives an appreciable convergence acceleration. This method can be used with other acceleration methods for a further computer time reduction. It can be also utilized for three-dimensional calculations.

Another problem involved is connected with the wave drag calculation in the potential airfoil flow. Various finite-difference schemes are developed for the solution of the potential equation. They may be divided into conservative and nonconservative ones. In the conservative schemes, the mass conservation on shocks is maintained. In the nonconservative schemes, shocks are sources of mass. For weak shocks the nonconservative schemes give a good agreement with the Euler equation solutions and experimental results, and this agreement is often better than for the conservative schemes. The examples illustrating this fact may be found in[7-9]. An apparent reason is that the shock mass sources model the entropy changes on shocks, which also gives rise to mass sources in the potential flow model[10].

One of the drawbacks of the nonconservative schemes is that shock wave mass sources give additional drag. This fact was pointed out in[8].

The second reason for the divergence

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between the wave drag values calculated by the solution of the potential equation and the Euler equations is that the conditions on shocks and the sources of the wave drag rise in these cases are different[11]. This remark is common for both the conservative and the nonconservative schemes.

In[8] it is suggested to take into account the effect of the scheme nonconservativity on wave drag by subtraction of the drag due to mass sources. In[11] some corrections for the potential flow wave drag values are suggested with the aim of approaching to the Euler equation results. In the present paper both of these corrections are used in the airfoil calculations and it is shown that taking account of these corrections gives good agreement with the Euler equation solutions for wave drag values in weak shock cases.

II. CONVERGENCE ACCELERATION OF RELAXATION METHODS

CONVERGENCE ANALISYS IN THE VICINITY OF THE INFINITY POINT

In airfoil calculations a conformal mapping of the flow region onto a circle as a method of grid generation is often used. The infinity point in such cases corresponds to the circle centre. The velocity potential equation in the polar coordinates (r,θ) in this circle takes the form

$$\frac{\partial}{\partial\theta}\left[\rho\Phi_\theta\right] + r\frac{\partial}{\partial r}\left[r\rho\Phi_r\right] = 0 \qquad (1)$$

where Φ is the potential, ρ - the density. Let us consider an incompressible fluid flow for simplicity. The equation (1) becomes the Laplace equation

$$\Phi_{\theta\theta} + r\frac{\partial}{\partial r}\left[r\,\Phi_r\right] = 0 \qquad (2)$$

or

$$\Phi_{\theta\theta} + r^2\Phi_{rr} + r\,\Phi_r = 0 \qquad (3)$$

Consider the finite-difference approximation of this equation by the use of difference scheme from[7] where the derivatives in (3) are approximated as follows

$$(\Delta\theta)^2\Phi_{\theta\theta} = \Phi_{k+1,j} - \frac{2}{\omega}\Phi^+_{k,j} - \left[2 - \frac{2}{\omega}\right]\Phi_{k,j} + \Phi^+_{k-1,j}$$

$$(\Delta r)^2\Phi_{rr} = \Phi^+_{k,j+1} - 2\Phi^+_{k,j} + \Phi^+_{k,j-1} \qquad (4)$$

$$2\Delta r\Phi_r = \Phi_{k,j+1} - \Phi_{k,j-1}$$

where the plus sign means a potential value from the current iteration, otherwise it is taken from the previous iteration, ω - relaxation parameter (which is equal to 1.4 in[7]).

According to the von Neumann difference scheme stability method let us consider the behaviour of the Fourier harmonics

$$\Phi_{k,j} = e^{iA\theta}e^{iBr}$$
$$\Phi^+_{k,j} = \lambda(r)\Phi_{k,j} \qquad (5)$$

The necessary scheme stability condition is $|\lambda| \le 1$. The substitution of (5) in (4) and then in (3) and neglecting the terms of the order of $O(\Delta r^2)$ gives

$$e^{i\alpha} - \frac{2}{\omega}\lambda - \left[2 - \frac{2}{\omega}\right] + e^{-i\alpha}\lambda + r^2\gamma^2\left[2\,i\sin\beta\,\Delta r\,\lambda' - 4\sin^2\frac{\beta}{2}\lambda\right] + \qquad (6)$$
$$+ i\,r\,\gamma\,\Delta\theta\sin\beta = 0$$

where $\alpha = A\,\Delta\theta$, $\beta = B\,\Delta r$, $\gamma = \Delta\theta\,/\,\Delta r$.

If $\sin\beta \ne 0$ then at $r \to 0$ which corresponds to the infinity point, one can obtain the following differential equation for the determination of λ

$$r^2\lambda' + G\lambda + H = 0$$

where $G = \dfrac{e^{-i\alpha} - \frac{2}{\omega}}{2\,i\,\Delta r\,\gamma^2\sin\beta}$,

$H = \dfrac{e^{i\alpha} - \left[2 - \frac{2}{\omega}\right]}{2\,i\,\Delta r\,\gamma^2\sin\beta}$. Its solution is

$$\lambda = Ce^{G/r} - \frac{H}{G}, \quad \mathrm{Re}\,G = -\frac{\sin\alpha}{2\,\Delta r\,\gamma^2\sin\beta} < 0$$

for all harmonics $0 < \alpha < \pi$, $0 < \beta < \pi$ and consequently

$$\lambda \xrightarrow[r\to 0]{} -\frac{H}{G} = \frac{1 - \frac{\omega}{2}\left[2 - e^{i\alpha}\right]}{1 - \frac{\omega}{2}e^{-i\alpha}}$$

It may be shown that the maximum value of $-\frac{H}{G} = 1$ is attained at $\alpha = 0$. If $\alpha = 0$, $\beta = \pi$, which corresponds to high frequency oscilations along r coordinate, the expression for λ in accordance with (6) takes the form

$$\lambda = \frac{1}{1 + \dfrac{4\,r^2\,\gamma^2}{\frac{2}{\omega} - 1}} \qquad (7)$$

which at $r \to 0$ $(0 < \omega < 2)$ tends to unity and, hence, high frequency oscillations along r coordinate decay most slowly in the vicinity of the infinty point.

Thus the presense of the infinity point in the calculation region is the reason for a slow convergence of relaxation methods. Such a conclusion may also be drawn for other types of difference schemes, for example for the conservative scheme of the form

$$(\Delta\theta)^2\ \Phi_{\theta\theta} = \Phi_{k+1,j} - 2\ \Phi^+_{k,j} + \Phi^+_{k-1,j}$$

$$(\Delta r)^2\ \frac{\partial}{\partial r}\left[r\ \Phi_r\right] = r_{j+1/2}\left[\Phi^+_{k,j+1} - \Phi^+_{k,j}\right] - r_{j-1/2}\left[\Phi^+_{k,j} - \Phi^+_{k,j-1}\right] \qquad (8)$$

In this case the expression for λ at $\alpha = 0$, $\beta = \pi$ coincides with (7) for $\omega = 1$.

The solution in the vicinity of the infinity point may be evaluated from the well-known asymthotic behaviour of the potential or from the potential solution obtained on a crude grid. The potential "freezing" in the vicinity of the point $r = 0$ (at $r < r_0$) and the application of the Dirichlet type condition on its boundary, that is fixing the potential values at $r=r_0$, substantially increase the value of residual convergence, as this show the numerical results given below.

NUMERICAL RESULTS

The calculations were carried out for NACA0012 airfoil for subcritical (M_∞=0.6, α = 2°) and supercritical (M_∞=0.8, α = 1°) cases. The polar grid was used with the number of nodes 80*16 (80 points on the airfoil). The preliminary calculations were carried out on a crude grid with the number of nodes 40*8. Then the potential values were "freezed" in the circle of radius r_0 and at the boundary of this circle the Dirichlet type condition were imposed. This "freezing" did not touch the circulation term, that has been recalculated after each iteration.

Fig.1 shows the convergence of the maximum residual logarithm for the NACA0012 airfoil calculation at M_∞=0.6, α=2°. The calculations were carried out with the aid of the code[7] (nonconservative scheme (4)). The numbers on the curves mean the amount of "freezed" grid layers along r-coordinate. A noticeable convergence acceleration is observed which *increases with the "freesing" region* growth. To illustrate the fact that this acceleration is not connected with the decrease in the node number along r-coordinate, the results of the calculations on the 80*8 grid are presented in the same figure. In this case no convergence acceleration is observed.

In Fig.2 analogous results are presented which were obtained using conservative scheme (8). The convergence rate is practically the same as in the previous case. Similar variations of the convergence rate are observed in the supercritical flow case (Fig.3).

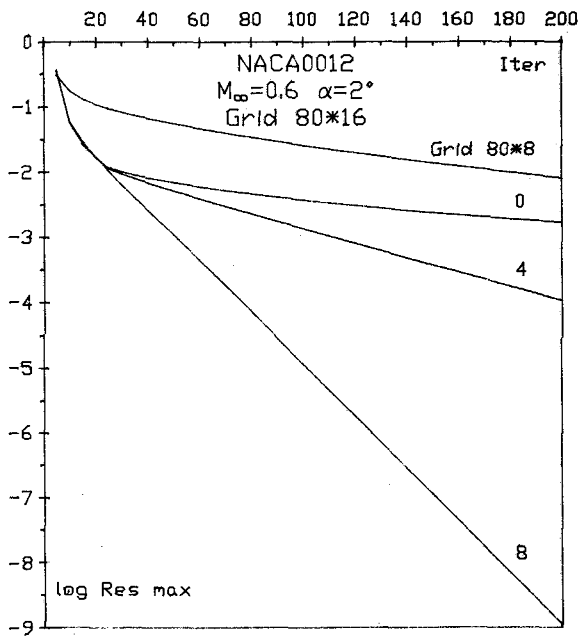

Fig.1 Convergence history. Nonconservative scheme.

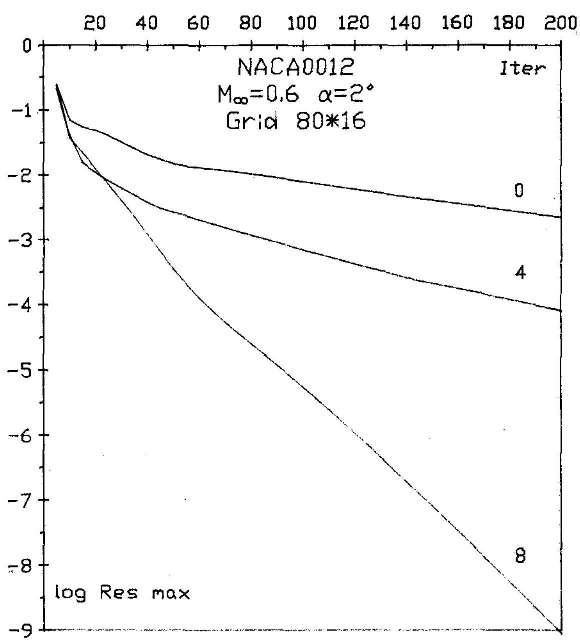

Fig.2 Convergence history. Conservative scheme.

The potential "freezing" even in a large region does not lead to a substantial accuracy deterioration in the calculations of distributed and total aerodynamic characteristics. The changes in lift Cy and wave drag Cx coefficients (determined from the pressure integration) with the number of "freezed" grid layers N are presented in Fig.4. The variation of N om 0 to 8 leads to a maximum variation Cy and Cx of about 2%.

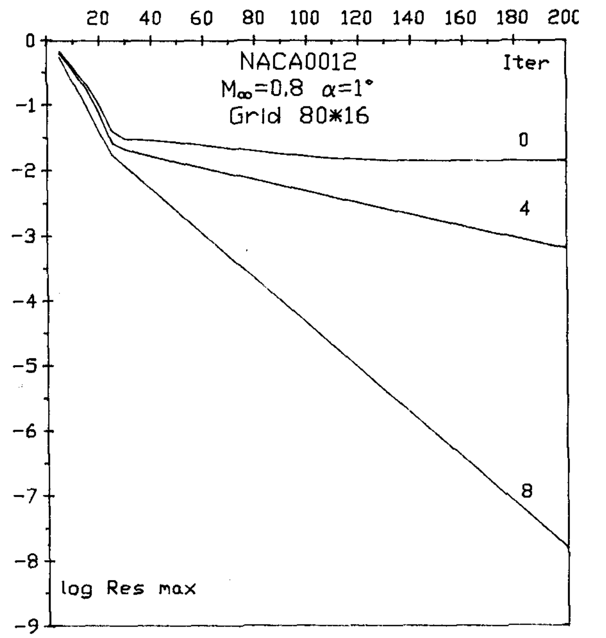

Fig.3 Convergence history. Nonconservative scheme.

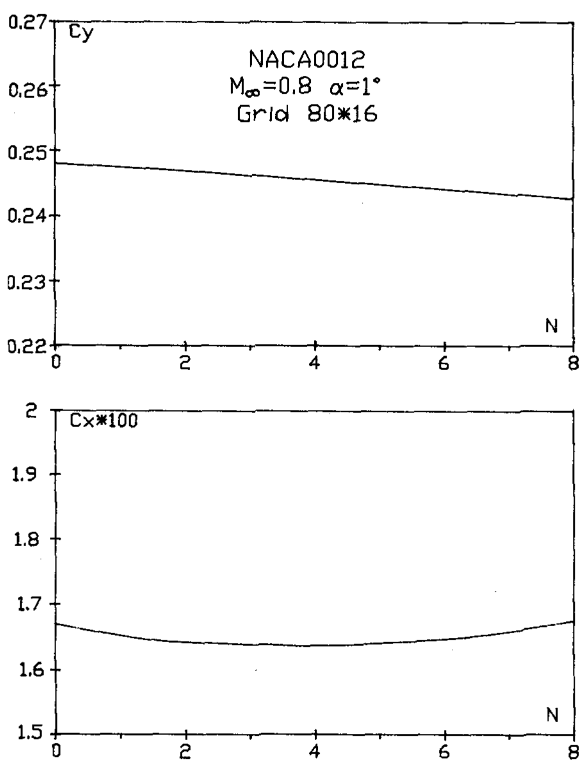

Fig.4 The dependence of Cx and Cy on the "freezed" region.

Thus the exclusion of the vicinity of the infinity point from the calculation region substantially accelerates the convergence of the iteration process for transonic calculations by relaxation methods both for the nonconservative and for the conservative schemes without any visible deterioration of the calculations accuracy. This method was successfully used for calculations of axisymmetrical and three-dimensional flows[1,2].

III WAVE DRAG CALCULATION

Let us consider an isoenergetic and isentropic ideal gas flow. This flow is potential in accordance with the Crocco equation. In this case the changes of the momentum $I = p + \rho V^2$ and mass flow $q=\rho V$ on the normal shock take the form

$$[I] = V_* [q] - \frac{1}{3} \rho_* \frac{\partial M^2}{\partial V}\Big|_* [(\Delta V)^3] + O((\Delta V)^4) \qquad (9)$$

$$[q] = -\frac{1}{2} \rho_* \frac{\partial M^2}{\partial V}\Big|_* [(\Delta V)^2] + \frac{1}{6} \rho_* \left(\frac{2}{V_*}\frac{\partial M^2}{\partial V}\Big|_* - \frac{\partial^2 M^2}{\partial V^2}\Big|_*\right) [(\Delta V)^3] + O((\Delta V)^4)$$

where p is the pressure, subscript * denotes critical values, square brackets denote value jump on the shock, $\Delta V=V-V_*$. If mass conserves on the shock then $[q]=0$ and momentum change has the third order of magnitude (relative to ΔV, or $M_{sh}-1$ where M_{sh} is the local Mach number before the shock on the airfoil surface). Using the momentum conservation law, drag may be expressed as follows

$$X = \int_{sh} [I] \cos \psi \, dl - Q$$

where Q is the total source intensity in the flow field. Because the main mass source in the nonconservative schemes is the shock, $Q = \int_{sh} [q] \, dl$ and, hence

$$X = \int_{sh} \Big\{ [I] \cos \psi - [q] \Big\} dl$$

Since for the airfoil flow the shock is nearly normal and ψ is small enough it is possible taking (9) into account, to write the drag in the following form

$$X = (V_* - 1) Q - \frac{1}{3} \int_{sh} \Big\{ \rho_* \frac{\partial M^2}{\partial V}\Big|_* [(\Delta V)^3] + O((\Delta V)^4) \Big\} dl \qquad (10)$$

For the conservative scheme $[q] = Q = 0$ and only the third order value is integrated. When the Hugoniot conditions for the Euler equations are used then the wave drag can also be expressed as an integral along the shock from the function proportional to the entropy jump across the shock which has the third order of magnitude[13]. Hence the conservative scheme drag has the right order. It can be shown that the length of the shock is proportional to ΔV i.e. it is the value of the first order of magnitude. In this connection the wave drag calculated with the aid of the conservative scheme, as by the Euler equations solution must be the forth order of magnitude. As for the nonconservative schemes in accordance with the last line of (9) the [q] value has second order and the wave drag itself the third order just like the Q value. To obtain the wave drag with a right order it was suggested in[8] to subtract the value $(V_{*}-1)Q$ from the drag obtained by the pressure integration. The Q value was obtained by integrating the mass flow along the circle of a large radius. Such a correction to wave drag values obtained with the nonconservative scheme gives hope to get a value close to that obtained with the conservative scheme only if the position and the strength of the shocks are close in case of both schemes.

Below are given the results of the investigation of the applicability of this correction for the scheme nonconservativity for symmetrical flow about NACA0012 airfoil as an example. The calculations are carried out with the aid of the nonconservative[7] and conservative schemes. Fig.5 shows the results of the wave drag coefficient calculations depending on $M_{sh} - 1$. The results are given in logarithmic scales for the nonconservative and conservative schemes. The computed data were approximated by straight lines using the least square method. The slope of such line for the nonconservative scheme is equal to 3.0 which is consistent with the theoretical considerations concerning a cubical dependence of the wave drag on the shock strength given above. So the drag in this case is mainly due to a growth of the source intensity Q with the growth of the shock strength. The slope of log Q (log $(M_{sh}-1)$) dependence equals to 2.8 which is close to the theoretical value 3.0. The Q value was calculated by integrating the mass flow along the circle of a large radius and was made dimensionless with respect to freestream parameters and airfoil chord. For the conservative scheme, a slope of line approximating log Cx $(\log(M_{sh}-1))$ dependence is equal to 4.0 what is also consistent with the theoretical considerations given above. Comparison of wave drag values for the nonconservative and conservative schemes is given in Fig.6. It is seen that the presence of the shock mass source in the nonconservative scheme substantially overpredicts wave drag. Supposing closeness of the shock wave to the normal one and taking (10) into account one can insert the following correction for the mass source[8]

$$X = \int p\, dy - (V_{*} - 1)\, Q \qquad (11)$$

Here integral is taken along the airfoil contour. As a result drag will be of the fourth order with regard to the shock strength. The results with this correction are also shown in Fig.6. Up to $M_{\infty}=0.81$ the results of the wave drag calculations by the conservative and corrected nonconservative schemes agree well. At greater M_{∞} numbers there is a divergence which can be explained by the fact that different schemes give different results for pressure distribution, in particular, for shock position and strength. An example is given in Fig.7 where the dependences of the Mach number upstream of the shock on the airfoil surface M_{sh} and shock position X_{sh} (sonic point) on the M_{∞} number are shown. Up to $M_{\infty}= 0.8$ both values are close for different schemes but at $M_{\infty}> 0.8$ the conservative scheme gives a more aft shock position, which explains a more rapid drag rise.

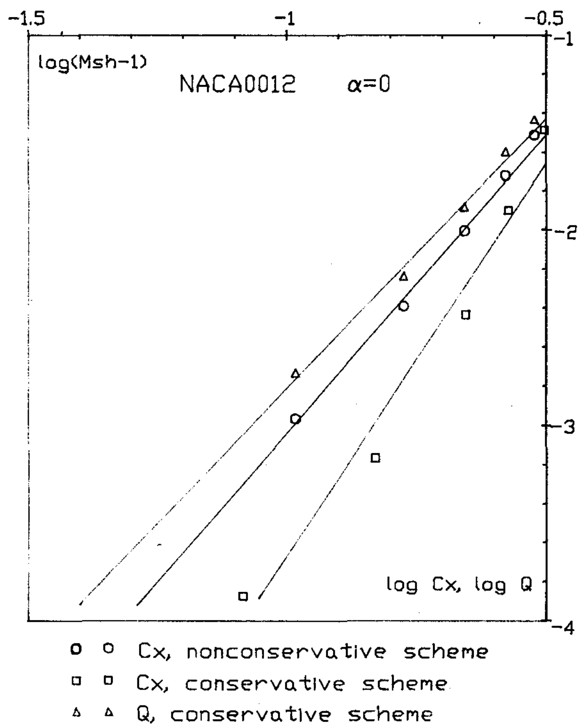

Fig.5 Dependence of Cx and source intensity on shock strength.

When the Euler equations with the shock wave Hugoniot conditions are considered, the drag is the result of an entropy jump on shocks[13]. This entropy jump like the momentum jump across an isentropis shock is of the third order of magnitude. Hence, the calculation of

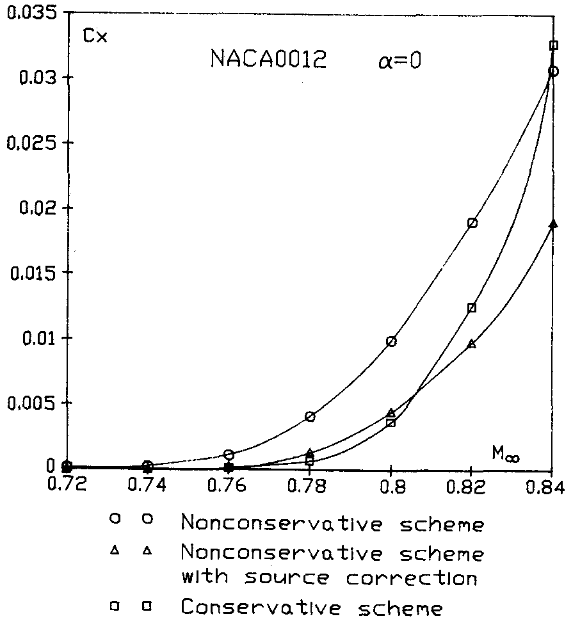

Fig.6 Dependence of Cx on M_∞.

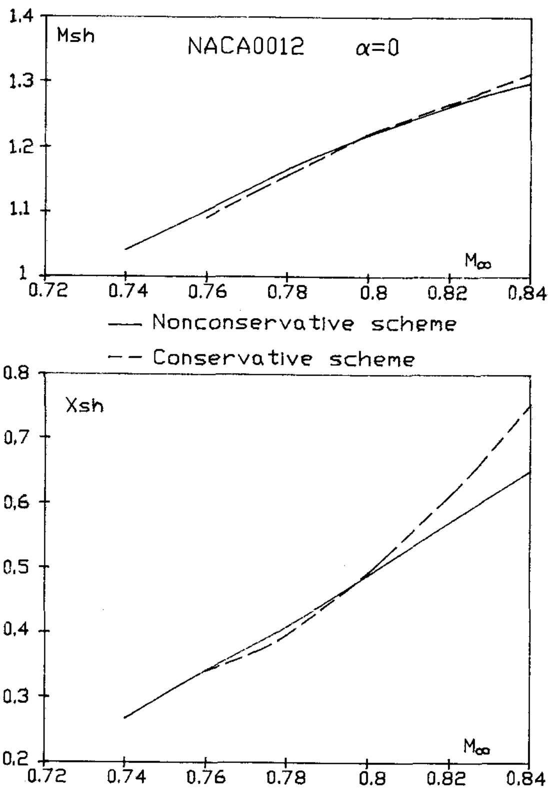

Fig.7 Dependence of shock strength and position on M_∞.

isentropic flows with the conservative schemes or nonconservative ones with the correction for the mass source presence gives the wave drag with right order of magnitude. But the ratio of coefficients at leading terms of the least order in these both cases differs from unity and is equal to

$$\frac{Xe}{Xi} = \frac{1}{M_\infty \left[1 + \frac{\varkappa-1}{\varkappa+1} (M_\infty^2 - 1) \right]^{1/2}}$$

where Xe is the Euler flow drag and Xi - the isentropic flow drag. This formula was obtained in[11] in the assumptions that the shock is normal for both flow models, the Mach number values upstream of the shock and its position are identical. This equation allows us to correct the potential flow results for the lack of the Hugoniot shock conditions. So by the solution of the potential equation with the nonconservative scheme and following taking into account the corrections for the mass source and shock conditions allows us to approach to the wave drag values obtained by the Euler equation solutions. Here the shock waves must be weak, and their position and strength must be close for both solutions. The results of the intoduction of these corrections are shown in Fig.8. Also shown are the results of the Euler equation calculations[14] and the results of calculating of the wave drag by the following formula

$$Cx = \frac{0.243}{k} \left(\frac{1+0.2\,M_\infty^2}{M_\infty} \right)^3 \frac{(M_{sh} - 1)^4 (2 - M_{sh})}{M_{sh} (1 + 0.2\,M_{sh}^2)} \qquad (12)$$

from[15], where k is the airfoil surface curvature at the foot of the shock. This formula was obtained for normal shocks assuming a linear velocity variation along shocks and the fulfilment of the Hugoniot conditions on them. It may be seen from Fig.8 that at moderate supercritical Mach number values the wave drag is overpredicted by the nonconservative scheme and the corrections made provide a good agreement with the Euler equation solutions and the results obtained from (12) up to M_∞=0.8. A later disagreement may be explained by a violation of the assumptions of shock wave position and strength closeness and by the influence of higher order of magnitude terms in wave drag expansions. It should be noted that in the design of airfoils for civil and transport aircraft wings it is the region of the wave drag origin that represents the most interest because when strong shocks occur aerodynamic characteristics are quite unsatisfactory. Thus using the above corrections to the wave drag values obtained by the solutions of the potential equation with the aid of the nonconservative scheme allows us to obtain wave drag values which are close to the Euler equation calculation results for weak shocks.

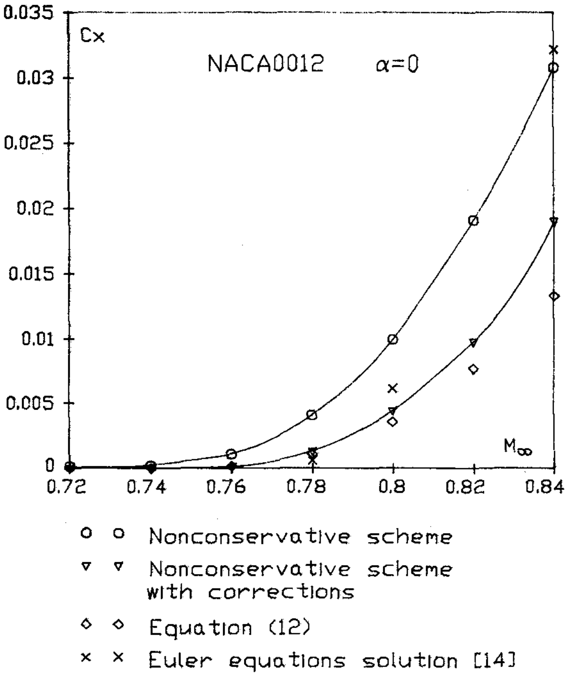

Fig.8 Dependence of Cx on M_∞.

IV CONCLUSION

An analysis of the difference scheme stability for the potential equation solution in subsonic and transonic airfoil flow calculations shows that high frequency harmonics in the solution decay most slowly in the vicinity of the infinity point. This fact is one of the reasons for a relatively slow convergence of the relaxation schemes. The exception of this vicinity from the calculation region and imposition of the Dirichlet type conditions on its boundary leads to a substantial convergence acceleration and considerable time reduction without any essential accuracy deterioration. The potential values at this boundary may be taken for example from a preliminary calculation on a crude grid. An analogous method may be used for axisymmetrical and three-dimensional calculations of wings and bodies.

The wave drag calculated in the potential flow model by the nonconservative schemes contains two sources of errors. The first one is connected with the fact that the shock waves are sources of additional mass and drag that is of the third order of magnitude with respect to the shock strength. The subtraction of this additional drag allows to obtain a drag with the right (fourth) order of magnitude. The second source of errors is connected with different conditions on shocks in the potential and the Euler flow models. The ratio of leading terms in the expansions of the wave drag in the shock strength depends solely on the free stream Mach number. The introduction of the corrections to the results of wave drag calculations from the potential flow model by the nonconservative scheme allows one to approach to the results obtained from the Euler flow model when the shock position and strength are close in both models and shocks themselves are weak.

REFERENCES

1. Yu N.J.,Rubbert P.E. Acceleration schemes for transonic potential flow calculations.- AIAA Paper N80-0338,1980.
2. Jameson A. Transonic potential flow calculations using conservation form .- AIAA Comp. Fluid Dyn. Conf., Hartford. CT, June, 1975.
3. Ballhaus W.F. ,Jameson A. ,Albert J. Implicit approximate factorization schemes for the efficient solution of steady transonic flow problems. Proc. AIAA 3rd Comp. Fluid Dyn. Conf., 1977.
4. Holst T.L. A fast conservative algorithm for solving the transonic full potential equation.- AIAA Paper N79-1456, 1979.
5. Jameson A. Acceleration of transonic potential flow calculations on arbitrary meshes by the multiple grid method.- AIAA Paper N79-1458, 1979.
6. Hafez M.,Lovell D. Improved relaxation schemes for transonic potential calculations.- Int. Journal for Num. Meth. in Fluids., vol.8,1-16,1988.
7. Bauer F.,Garabedian P.,Korn D.,Jameson A. Supercritical wing sections II.- Lect. Notes in Econ. and Math. Syst.,108,1975.
8. Bauer F.,Garabedian P.,Korn D. Supercritical wing sections III.- Lect. Notes in Econ. and Math. Syst.,150,1977.
9. Hinson B.L.,Burges K.P. An evaluation of three-dimensional transonic codes using new correlation-tailored test data.- AIAA Paper N80-0003,1980.
10. Dang T.Q.,Chen L-T. An Euler correction method for two- and three-dimensional transonic flows.- AIAA Paper N87-0522, 1987.
11. Steger L.M.,Baldwin D. Shock waves and drag in the numerical calculations of isentropic transonic flow.- NASA TM D-6997, 1972.
12. Vladimirova N.A., Vyshinskii V.V., Lyapunow S.V., Serebriiskii Ja.M. Convergence acceleration of computational methods for two- and three-dimensional transonic unconstrained flows about bodies.- Uchenye zapiski TSAGI, v.XVI, N4, 1985.
13. Serebriiskii Ja.M., Christianowich S.A. On wave drag.- Trudy TSAGI,1944.
14. Salas M.D.,Gumbert C.R.,Turkel E. Nonunique solutions to the transonic potential flow equation. - AIAA Journal, vol. 22,N 1, 1984.
15. Lock R.C. The prediction of the drag of aerofoils and wings at high subsonic speeds. - Aeronautical Journal June/July, 1986.