VISCOUS SUPERSONIC FLOW PAST A WEDGE-SHAPED BODY

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Abstract

Steady-state viscous plane supersonic flow past a wedge-shaped body is examined by means of the two-dimensional Navier--Stokes equations in the nonstationary conservation-low form. An implicit factored finite-difference scheme is developed together with the method of fractional steps and a time-dependent iteration procedure. The steady field of supersonic flow can be found as a limit of unsteady flows determined in the course of iterations.

Numerical analysis of the flow is carried out for moderate values of Mach number and Reynolds number as well as several values of wedge angle.

Nomenclature

c_p, c_v	specific heats
e	internal energy
E	total energy
F, G	flux vectors, Eq. /3/
f, f_m	vectors, Eq. (25)
F_p	vector, Eq. (24)
G_1, G_2, G_3, G_4	boundary of computation region, Fig. 1
k	Poisson adiabatic exponent
M, M_∞	Mach number
n	direction normal to the surface, number of time layer
Pr	Prandtl number
q_1, q_2	transformed variables
R	gas constant
Re	Reynolds number
s	direction tangent to the velocity vector
S	vector, Eq. (7)
S_x, S_y, S_E	components of the vector S, Eq. (8)
t	time
T	temperature
u	velocity in the x direction
U	vector of conserved quantities, Eq. (2)
v	velocity in the y direction
w	velocity of gas
x, y	rectangular coordinates, Fig. 1
x_1, y_1	dimensionless rectangular coordinates, Eq. (22)
$\varkappa$	coefficient of heat conduction
μ	coefficient of dynamic viscosity
ω	constant, Eq. (10)
ρ	density
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$	components of the viscous stress tensor, Eq. (9)
τ	time step

Subscript

∞	undisturbed flow condition

1. Introduction

Numerical computations based on the Navier-Stokes equations developed during the last two decades.[1-6] The method of finite elements was used in the case of incompressible flows.[2-3] As regards the supersonic flow range, various algorithms were applied based on methods of finite differences.[4-7]

The problem to be studied in the present paper is that of a wedge-shaped body flown-past by viscous, heat conducting gas, the complete Navier-Stokes equations being used. In that case a boundary layer and an oblique shock wave is formed simultaneously in the same region of flow, thus enabling us to determine the influence of the dissipative properties of the gas on the course of those phenomena.

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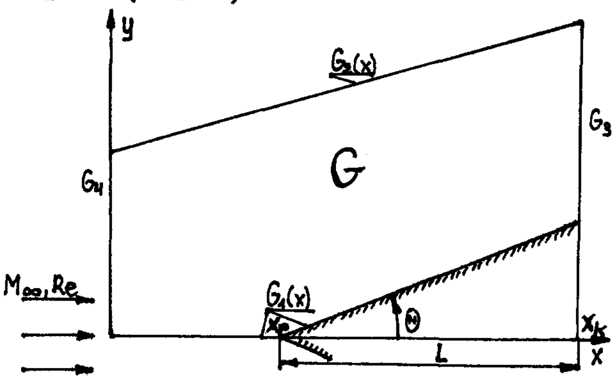
The present paper is a development of[7] and contains some results discussed in the reports.[8] The problem has been solved numerically by the method of decomposition of the set of equations, fractional steps and an iterative procedure with respect to time.[9,10] This set of equations has been approximated by an implicit double-layer difference scheme of the Cranck--Nicholson type.[10]

2. Formulation of the Problem

Let us consider plane supersonic flow of gas past a body having the form of a wedge with a sharp edge, an angle 2Θ and length L (Fig. 1).



It is assumed that the stream of gas is homogeneous and parallel to the symmetry plane of the flown-past body.

The gas is treated as a viscous and heat conducting medium to be perfect in the thermodynamic sense, the coefficient of dynamic viscosity μ and that of heat conduction $\varkappa$ being known functions of the temperature T, while the Prandtl number Pr and the Poisson adiabatic exponent k being constants.

The two-dimensional equations of the problem can be expressed in the rectangular coordinates x, y in the form of the conservation low[1]

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S \tag{1}$$

where U is the vector of conserved quantities

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{Bmatrix} \tag{2}$$

and F , G are flux vectors

$$F = \begin{Bmatrix} \rho u \\ p+\rho u^2 \\ \rho u v \\ u(\rho E+p) \end{Bmatrix}$$
$$G = \begin{Bmatrix} \rho v \\ \rho u v \\ p+\rho v^2 \\ v(\rho E+p) \end{Bmatrix} \tag{3}$$

The total energy of gas is

$$E = e + \frac{1}{2}(u^2 + v^2) \tag{4}$$

and the internal energy

$$e = C_v T\ , \quad C_v = const \tag{5}$$

For a perfect gas,the equation of state is

$$p = \rho R T\ , \quad R = C_p - C_v = const \tag{6}$$

The vector S in the right-hand member of Eq.(1) has the form

$$S = \begin{Bmatrix} 0 \\ S_x \\ S_y \\ S_E \end{Bmatrix} \tag{7}$$

where

$$S_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}$$

$$S_y = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \qquad (8)$$

$$S_E = \frac{\partial}{\partial x}\left(\varkappa \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varkappa \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\sigma_{xx} + v\sigma_{xy}\right) + \frac{\partial}{\partial y}\left(u\sigma_{xy} + v\sigma_{yy}\right)$$

The components of the viscous stress tensor are assumed taking into consideration the condition that the volume viscosity of the gas is disregarded[1]

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (9)$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The coefficient of dynamic viscosity can be found from the equation[1]

$$\mu = \left(\frac{T}{T_\infty}\right)^{\omega} \mu_\infty, \quad \omega = \text{const} \qquad (10)$$

The coefficient of heat conduction $\varkappa$ is assumed to vary proportionally to the coefficient μ and, therefore, the Prandtl number is

$$P_r = \frac{\mu c_p}{\varkappa} = \text{const} \qquad (11)$$

The boundary conditions of the problem can be assumed to be (Fig. 1)

$$u = v = 0$$
$$\frac{\partial T}{\partial n} = 0 \qquad (12)$$

at the surface of the body and

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial \rho}{\partial y} = v = 0 \qquad (13)$$

at the symmetry axis of the flow, for $y = 0$, and

$$u = u_\infty, \quad v = v_\infty = 0$$
$$T = T_\infty, \quad \rho = \rho_\infty \qquad (14)$$

in the region of undisturbed flow.

3. Transformation of the Equations and the Computation Region

The equations of the problem will be considered in a dimensionless form. In this connection the following quantities are introduced

$$\bar{t} = \frac{t u_\infty}{L}, \quad \bar{x} = \frac{x}{L}, \quad y = \frac{y}{L}$$

$$\bar{u} = \frac{u}{u_\infty}, \quad \bar{v} = \frac{v}{u_\infty}, \quad \bar{\rho} = \frac{\rho}{\rho_\infty}$$

$$\bar{T} = \frac{T c_p}{u_\infty^2}, \quad \bar{p} = \frac{p}{\rho_\infty u_\infty^2} \qquad (15)$$

$$\bar{\mu} = \frac{\mu}{\mu_\infty}, \quad \bar{\varkappa} = \frac{\varkappa}{\varkappa_\infty}$$

$$\bar{E} = \frac{E}{u_\infty^2}$$

Equations (1) to (9) can, then, be presented in the dimensionless form, the bars over particular symbols being rejected and the constant parameters involved in those equations are referred to the undisturbed flow parameters

$$R_e = \frac{\rho_\infty u_\infty L}{\mu_\infty}; \quad P_r = \frac{c_p \mu_\infty}{\varkappa_\infty} \qquad (16)$$

The computation region G of the flow is assumed in the form shown in Fig. 1. The lower boundary of the region is the Ox-axis /the symmetry axis of the flow/

and the surface of the body. The front and upper boundaries are assumed to lie outside the perturbation region of flow.

The boundary conditions assumed for computation within the region G are:

for $G_2(x)$ $(0 \leq x \leq x_k;\ y = G_2(x))$

and $G_4\,(x=0;\ 0 \leq y \leq G_2(0))$

$$u = 1;\ v = 0;\ \rho = 1 \qquad T = T_\infty = \frac{1}{(k-1)M_\infty^2}, \tag{17}$$

for $G_1/x/$, at the symmetry axis of the flow, $(0 \leq x \leq x_p;\ y = 0)$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial \rho}{\partial y} = v = 0 \tag{18}$$

and on the surface of the body $(x_p \leq x \leq x_k;\ y = G_1(x))$

$$u = v = 0;\ \frac{\partial T}{\partial n} = 0 \tag{19}$$

At the rear boundary of the computation region, for $G_3\,(x = x_k;\ G_1(x_k) \leq y \leq G_2(x_k))$, we assume approximate boundary conditions, which are necessary to close the boundary-value problem. It can be taken, for instance, that the gradients in the flow direction can be set to zero

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s} = \frac{\partial T}{\partial s} = 0 \tag{20}$$

The problem of supersonic viscous flow past a wedge is treated as an initial-boundary-value problem and the steady-state field of flow will be determined by an iteration process as a limit of unsteady fields.

The initial conditions can be, for $t = 0$,

$$\begin{aligned} u(x,y,t) &= u_0(x,y) \\ v(x,y,t) &= v_0(x,y) \\ \rho(x,y,t) &= \rho_0(x,y) \\ T(x,y,t) &= T_0(x,y) \end{aligned} \tag{21}$$

where the functions on the right-hand side of (21) must satisfy the conditions on the surface of the body (19). They may represent the parameters of undisturbed field of flow or a field of flow determined in the course of the iteration process.

The computation region G assumed on the physical plane /x, y/ in the form shown in Fig.1 can now be transformed on the auxiliary plane of variables (q_1, q_2) into the square of unit side length.

The transformation will be performed by means of the functions

$$x_1 = \frac{x}{x_k};\ y_1 = \frac{y - G_1(x)}{G_2(x) - G_1(x)}$$

$$\text{for } x_1 \leq x_{1p} \qquad q_1 = x_{1p} - \frac{\ln[1 + \alpha c(x_{1p} - x_1)]}{\alpha \ln(1+c)}$$

$$\text{for } x_1 \geq x_{1p} \qquad q_1 = x_{1p} + \frac{\ln[1 + \beta d(x_1 - x_{1p})]}{\beta \ln(1+d)} \tag{22}$$

$$q_2 = \frac{\ln(1 + b y_1)}{\ln(1+b)}$$

$$\alpha = \frac{1}{x_{1p}}, \qquad \beta = \frac{1}{1 - x_{1p}}$$

This transformation enables us also to condense difference meshes in the physical plane /x, y/ within the region of the boundary layer and in the vicinity of the sharp edge of the wedge by selecting appropriate values of the constants b, c, d. The meshes remain uniform in the computation plane $/q_1, q_2/$.

4. Method of Solution

Numerical solution to the set of equations of the problem can be found by transforming those relations according to /22/ and expressing them in terms of the coordinates q_1, q_2. Then the set of equations takes the form

$$\frac{\partial U}{\partial t} = W \qquad (23)$$

The solution algorithm is based on the method of decomposition of the set of equation.[9,10] To apply it let us rewrite Eq. /23/ in the form

$$\frac{\partial f}{\partial t} + \Omega f = F_p \qquad (24)$$

where the vector of unknown gas-dynamic functions is

$$f = \begin{Bmatrix} \rho \\ u \\ v \\ T \end{Bmatrix} \qquad (25)$$

and Ω is a differential matrix operator of a separated part of Eq./23/ which will be approximated by an implicit difference scheme and F_p is the remaining part of Eq. /23/ that will be approximated in an explicit manner.

On transforming Eqs./23/ and /24/, we obtain the relation

$$\frac{\partial f}{\partial t} = F_p - \Omega f = \left(\frac{\partial U}{\partial f}\right)^{-1} W \qquad (26)$$

The left-hand member of Eq. /24/ will be replaced with an implicit two-layer difference scheme of the Crank-Nicholson type. We obtain the difference equation

$$\frac{f^{m+1} - f^m}{\tau} + \Omega_h f^{m+1} = F_p \qquad (27)$$

which approximates Eq. /24/ with an accuracy to $O(\tau + h^k)$, where τ is the time step, h - spatial mesh pitch, k - order of the operators involved in /24/.

Making use of /26/, Eq. /27/ can be expressed in the form of the two-layer iteration formula

$$\left(I + \tau\Omega_h\right)\frac{f^{m+1} - f^m}{\tau} = \left[\left(\frac{\partial U}{\partial f}\right)^{-1} W\right]_h^m \qquad (28)$$

which can be used as a basis for devising an algorithm for numerical solution of the problem and I is the unit matrix.

In order to simplify the algorithm of solution let us first apply the method of multiple decomposition of Eq. /28/. To this end we present the operator Ω_h in the form

$$\Omega_h = \sum_{l=1}^{4} \Omega_{hl} \qquad (29)$$

The separation /29/ is performed with respect to the directions q_1 and q_2 as well as processes and fields. In this connection derivatives with respect to q_1 are involved in the operators Ω_{h1} and Ω_{h3} only and derivatives with respect to q_2 - in Ω_{h2} and Ω_{h4}.

The mixed derivatives are transferred to the right-hand member of Eq. /28/, which simplifies in an essential manner the solution procedure. In the difference scheme the first derivatives $\partial/\partial q_1$ and $\partial/\partial q_2$ are approximated by asymmetric difference operators of the first order and the second derivatives - by symmetric difference operators. It can be shown that

$$I + \tau\sum_{l=1}^{4}\Omega_{hl} = \prod_{l=1}^{4}\left(I + \tau\Omega_{hl}\right) + O(\tau^2) \qquad (30)$$

In this connection the iteration formula /28/ can be transformed into

$$\prod_{l=1}^{4}\left(I + \tau\Omega_{hl}\right)\frac{f^{m+1} - f^m}{\tau} = \left[\left(\frac{\partial U}{\partial f}\right)^{-1} W\right]_h^m \qquad (31)$$

The formula /31/ makes it possible to device a recurrence algorithm for computing the vector f of gas dynamic parameters of flow for the $n+1$ time layer, if the flow parameters in the n-th layer are known.

To do this let us introduce auxiliary vectors $\varphi^{n+m/4}$ for m = 0,1,2,3,4 and write the following sequence of relations

$$\varphi^{m+0} = \left[\left(\frac{\partial U}{\partial f}\right)^{-1} W\right]_h^m \qquad (32)$$

$$(I+\tau\Omega_{nl})\varphi^{n+l/4}=\varphi^{n+(l-1)/4} \quad \text{for } l=1,2,3,4 \tag{33}$$

and
$$f^{n+1}=f^{n}+\tau\varphi^{n+1} \tag{34}$$

The algorithm based on this sequence of equations enables us to determine the required vector f^{n+1} by solving much simpler sets of equations than in the case of direct application of Eq./28/. It should be added that intermediate, fractional time layers n+1/4, n+1/2, n+3/4, n+4/4 have been introduced in Eq./33/ to solve consecutive simplified sets of elgebraic equations.

Equations /32/-/34/ have been used for developing computational programs and numerical analysis of supersonic viscous flow past a wedge.[7,8]

5. Results of Numerical Analysis

Numerical analysis of the field of supersonic flow past a wedge was performed making use of some computers and programs written by the present authors. A testing analysis was performed for wedges with edge angle $2\Theta = 30^{\circ}$, 40°, 60° and supersonic flows with Mach number $M_{\infty} = 2; 3; 3.5$ and Reynolds numbers Re = 250 to 4000, the coefficients characterizing the gas being $k = 1.41$, $\omega = 0.75$, $Pr = 0.71$.

The analysis was made for various values of spatial steps h_1, h_2 and time step τ. The effect of the initial conditions on the flow stabilizing process was also studied.

Some characteristic results of the analysis are presented in a diagramatic form in Figures 2 to 6, where all the flow parameters and coordinates are taken in the dimensionless form, making use of Eqs. /15/.

Figure 2 shows variation of the velocity components, u, v, in the front of the wedge depending on Reynolds number Re, for $M_{\infty} = 3$ and $\Theta = 20^{\circ}$. It is seen that Re has an essential effect on the course of velocity profiles and, therefore, on the thickness of the shock wave.

Figure 3 shows variations of p_2/p_1 for y = 0 and on the surface of the wedge, where p_1 is the pressure in the undisturbed flow and p_2 - the disturbed value of the pressure. In Fig.3a we can see the course of the ratio p_2/p_1 for several values of the Mach number M_{∞} and Re = 500, $\Theta = 20^{\circ}$, in Fig.3b the course of the ratio p_2/p_1 is presented for several values of the Reynolds number Re and $M_{\infty} = 3$; $\Theta = 20^{\circ}$, while in Fig.3c we have the course of the ratio p_2/p_1 for several values of Θ and $M_{\infty} = 3$, Re = 500.

Figure 4 shows variations of the density ρ and temperature T in the front of the wedge and on the surface of the wedge, for several values of Re and $M_{\infty} = 3$, $\Theta = 20^{\circ}$.

It is seen that gradients of the pressure ratio p_2/p_1 in the vicinity of the front of the wedge increase rapidly with increasing Re, M_{∞}, Θ and, therefore, the shock wave becomes thinner.

The profile of the density ρ in the vicinity of the front of the wedge is similar to that of the pressure ratio p_2/p_1, while the course of the temperature T is different.

Figure 5 shows variations of the flow parameters for three values of x = 0.25, 0.5, 0.75 and $M_{\infty} = 3$, Re = 500, $\Theta = 20^{\circ}$ and in Fig.6 there are shown, by way of example, the profiles of the pressure p and the angle Θ_w of the inclination of the vector of flow velocity w for the coordinate x = 1/2.

It is seen that in the vicinity of the front of the wedge /Fig.5a/ there is an interaction between the boundary layer and the shock wave. In the farther sections /Figs. 5b,c/ those structures are separated and there is a region of almost homogeneous flow between them, in which, however, density ρ and temperature T are variable.

The pressure p remains constant in the region between the surface of the wedge and the shock wave /Fig.6/, while the vector of flow velocity changes its direction Θ_w.

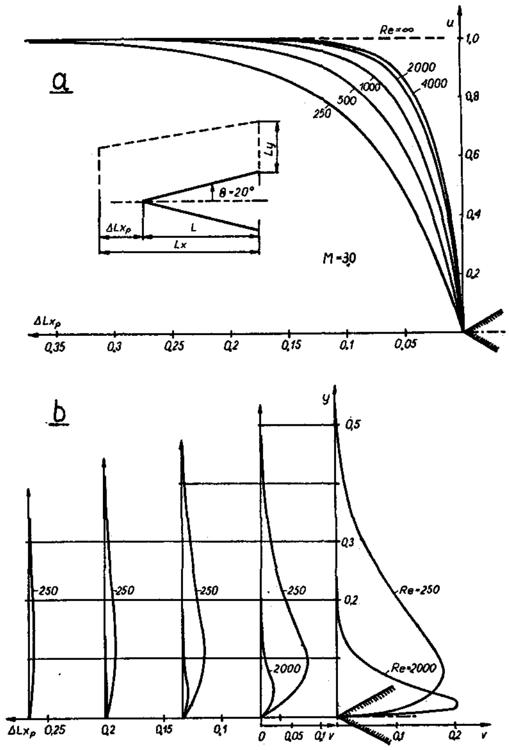


Fig.2 Velocity components,u,v, in front of the wedge

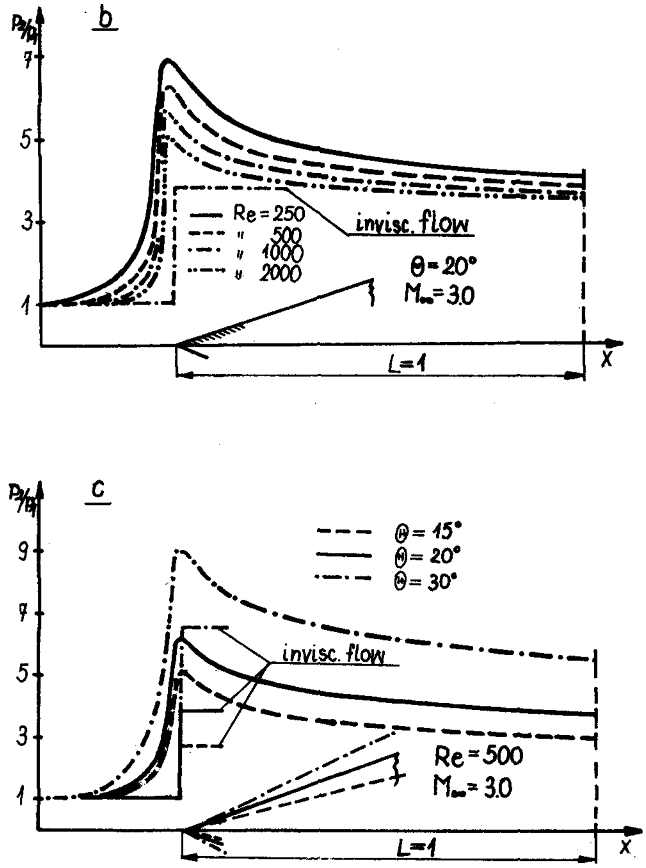


Fig.3b,c Pressure ratio p_2/p_1 in front and on the surface of the wedge

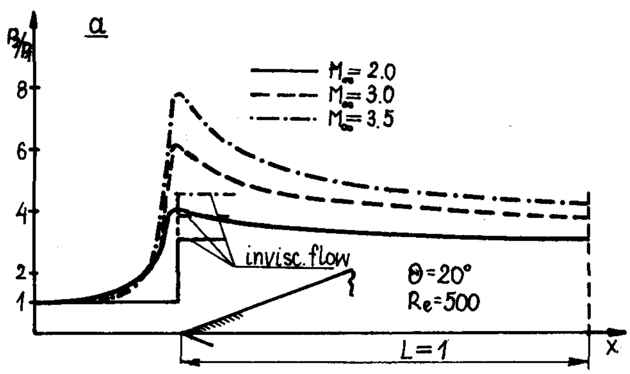


Fig.3a Pressure ratio p_2/p_1 in front and on the surface of the wedge

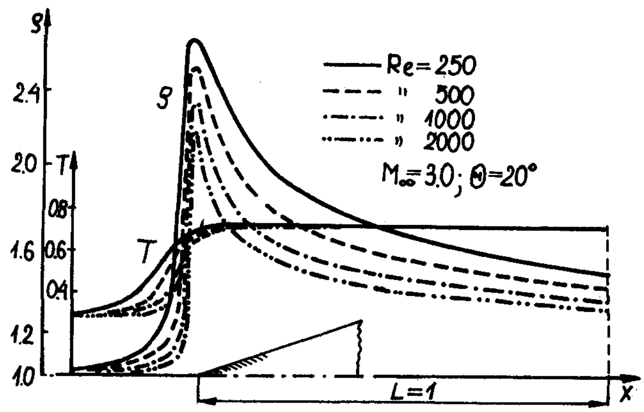
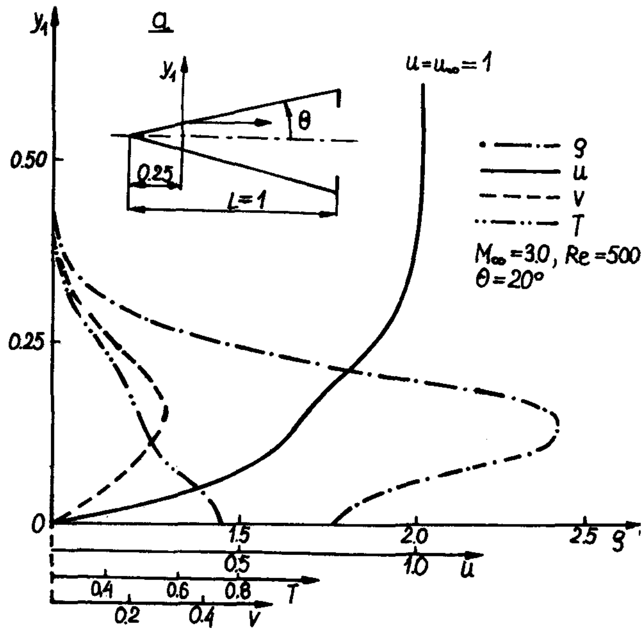


Fig.4 Temperature T and density ϱ in front and on the surface of the wedge



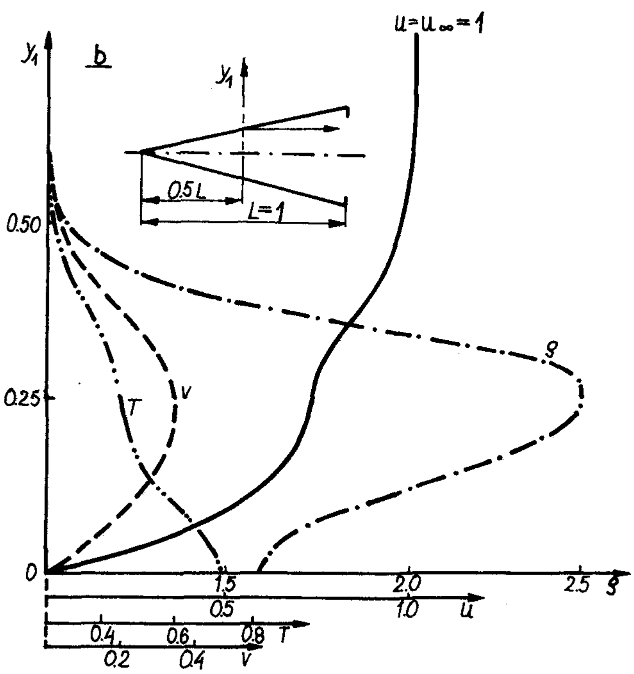


Fig.5a,b Profiles of the flow parameters for x = const

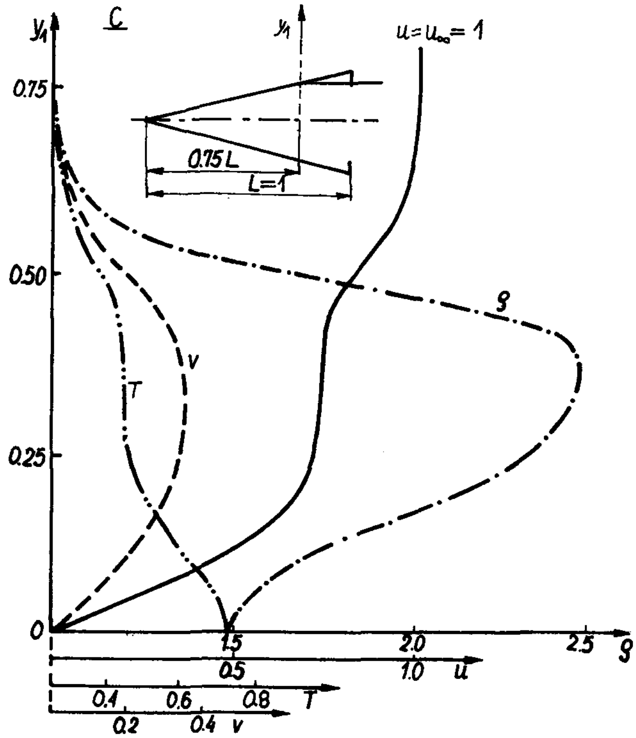


Fig.5c Profiles of the flow parameters for x = const

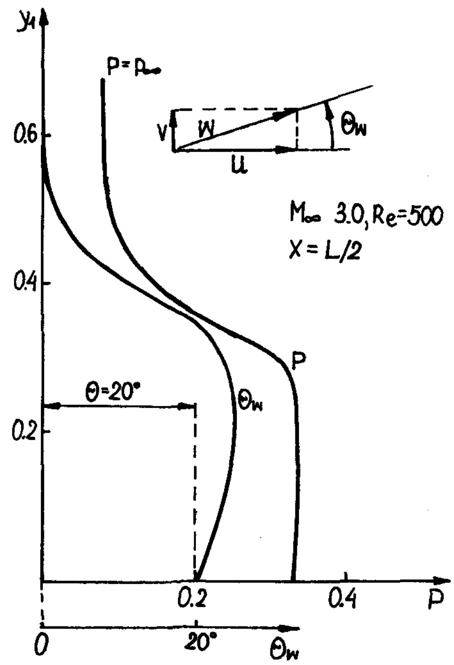


Fig.6 Profiles of pressure p and angle Θ_w for x = const

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