A NEW THEORY OF TURBULENCE BASED ON THE CONCEPT OF FRACTAL ANALYSIS

Ge Gao*
Beijing University of Aeronautics and Astronautics
Beijing 100083, P. R. CHINA

Wen L. Chow**
Department of Mechanical Engineering
Florida Atlantic University
Boca Raton, FL 33431

ABSTRACT

In meeting the demand for a practical turbulence modeling describing the nonisotropic character of turbulence and the turbulent stresses rapidly and accurately, an attempt of building up a turbulence modeling based on the concept of fractal analysis, has been carried out. Instead of the conventional Reynolds-averaging method, a general analytical-averaging method is proposed, which is widely applicable to obtain a perturbation equation for any nonlinear equation describing its averaged chaotic behavior. The perturbation of the Navier-Stokes equation, obtained by superposing a noise onto the flow field, gives a concrete expression of the turbulent stresses with clear physical meanings. In this sense, the perturbation equation may be considered as the physical equation of turbulence, which has been sought after since the beginning of this century. The corresponding turbulence modeling has been applied to a variety of complex turbulent flows and the results show that the modeling successfully predicts the extra-stresses, and may apply to any complex turbulent flows and replaces the existing industry-standard k-ε and Baldwin-Lomax modelings. The new modeling may also compete with any Reynolds-stress transport modelings.

Nomenclature

$c, c_0, c_1, c_2, c_3, c_\mu$ constants
e_i, e_l unit vectors
$\vec{F}_g$ generalized perturbation force
G_k generation rate of the kinetic energy of turbulence.
G_ε Generation rate of the dissipation of turbulence energy.
k kinetic energy of turbulence
p pressue
p_E mathematically expected pressure obtained by discrete method
p' perturbation value of pressure
$\doubleaccent{\tilde}{s}$ tensor of strain rate
t time
u mean velocity, equal to $\tilde{v}$
$\vec{v}$ instantaneous velocity
$\vec{v}_E$ mathematically expected velocity obtained by discrete method
$\vec{v}'$ perturbation value of velocity
$\tilde{v}$ mean velocity
$x, y,$ coordinates
δE virtual work
$\delta\vec{r}$ perturbation displacement
ε dissipation rate of the kinetic energy of turbulence
ϵ_{ijk} alternative tensor
λ heat conductivity
μ_L laminar kinetic viscosity
μ_T turbulent kinetic viscosity
ν, ν_L laminar kinematic viscosity
ν_T turbulent kinematic viscosity
ρ density
$\sigma_k, \sigma_\varepsilon$ coefficients
τ_{ij} turbulence stress tensor
ϕ dissipation function
$\nabla\phi$ velocity of strain
ψ dispersion function
Ω verticity

INTRODUCTION

The problem of turbulence, along with the strong nonlinearity of its governing equations, has been recognized as one of the most perplexing areas within the realm of classical mechanics. The major barrier for solving this problem is that a convincing understanding on physics of turbulence is lacking and a powerful mathematical method is needed to handle its nonlinear equations. There are two basic approaches for statistical turbulence modeling: first to average then to analyze; or precisely the reverse. The first way is the well-known Reynolds-averaging approach, which results in an unclosed set of equations. Turbulence modelings for the closure of the Reynolds equation, so far, are not entirely satisfactory. As for the second approach, owing to the intractability of the nonlinear hydrodynamic equations, hardly anyone has followed it.

The Reynolds-average method leaves all the effects of nonlinearity of turbulence stresses in terms of Reynolds stresses. If a more precise

* Professor of Aerospace Engineering

** Professor of Mechanical Engineering

expression for these turbulence stresses can be found with definite physical meanings, some universal modelings with fixed constant coefficients would be possible. The trouble is that the concrete content of these turbulence stresses is never known. Therefore, modelings with incorrect stresses terms have to struggle with coefficient adjustment. The concept of zonal modeling, suggested by Kline (1981) at the 1980/1981 Standord Conference on Computation of Complex Turbulent Flows, implies that there is no universal turbulence modeling and the coefficients may depend on the flow type, which in turn implies that a user has to choose the qualitative flow type and its corresponding coefficients. By definition, however, a modeling should have coefficients which will be applied without case-to-case adjustment by the user. Otherwise the modeling loses its generality and creditability.

Despite many modelers are strongly biased towards Reynolds-stress modeling, stress-transport modelings have not lived up to their promise. It seems that along the first approach, the road is endless so long as the problem of nonlinearity is imminent.

THEORETICAL DEVELOPMENT

In the last decade, chaos and fractal have become the frontiers of research in non-linear dynamics. It has been learned that the seemingly chaotic events resulting from the orderly physical laws, are not truly formless chaos, but exhibit underlying coherent patterns and possess fractal structures. This modern interpretation of chaos has been gradually replacing the conventional view that the order emerges from the underlying formless chaos. Along with the discovery of chaotic phenomena in all areas of nonlinear dynamics, it is also recognized that nonlinear difference and differential equations can admit bounded, nonperiodic solutions which exhibit randomized behavior even though no random parameters appear in the original governing equations (e.g. Moon, 1987) Turbulence is one of the few remaining unsolved problems in classical physics. The recent discovery of deterministic systems exhibiting chaotic oscillations has created much optimism about understanding the mysteries of turbulence. Recent studies on fractals have been focused on fractal geometry dimension and attractors (e.g. Orbach, 1986 and Moon, 1987). So far as the fractal dynamics is concerned, it has not reached it pragmatic stage of analyzing the dynamic process of a fractal system.

Since a turbulent field is governed by a giant system of multiple strange attractors, the concept and the method of attractors seem far away to give an accountable description of turbulence. It is necessary to try to find some other alternatives within the concept of fractals. It is indeed the intention of the present effort to introduce a general analytical-averaging method for nonlinear equations, developed within the fractal concept. Application of this technique to turbulence opens a new way for turbulence modeling, indicating the fruitfulness of the above mentioned second approach.

A GENERAL FRACTAL ANALYTICAL METHOD FOR NONLINEAR EQUATIONS

In dealing with the generalities of all disordered systems, fractal claims that random recurrent behavior gives rise to a family of remarkably similar patterns, which appear on different scales at the same time with abrupt boundary and sudden transitions. Despite the fact that accumulation of recurrent patterns gives apparent chaos, the quasi-self-similar patterns imply that the recurrent behavior must be governed by the same nonlinear equation, but with different scales at the same time.

Fractal has unforeseen importance on the nonlinear science. It points out that the spatio-temporal dimension is not an integer, but a fraction, and there is hidden orderliness behind the apparent disorder phenomena. The change of outlook on space and time would have dramatic impact both in the Natural science and the Philosophy. On the side of the nonlinear science, it lays a physical foundation of analytical-averaging method for the nonlinear equation. After several bifurcations, an unstable nonlinear equation usually produces chaotic solutions. Among numerous chaotic solutions, only their average behavior is meaningful and controllable. But the control equation describing an averaged chaotic phenomenon differs from its original nonlinear equation. Only after necessary mathematical treatment can the average control equation be obtained. Ordering or quasi-ordering is determined by the original nonlinear equation. Accumulation of recurrent patterns with different scales presents apparently disordered chaotic patterns. Based on this concept, a general analytical method seeking for the control equation of chaos is suggested as follows:

1. From the D'Alembert's principle, any nonlinear equation may be considered as a generalized dynamically balanced system. Upon multiplying the original nonlinear equation by a small scaled displacement of perturbation, it yields the corresponding terms of perturbation work controlled by the equation within the small scaled space.

2. Through the virtual work analysis on the perturbation work, distinguish the active and passive work terms; the active terms being responsible for the perturbation work production the passive ones for the work redistribution.

3. A new equation is obtained by adding the corresponding active perturbation force terms to the original equation. This new equation is able to describe the average behavior of the chaotic system with a second order of accuracy.

4. Repeat the procedures 1-3 with even smaller scales to obtain a new equation with a higher order of accuracy.

Nonlinear functions do not have additivity. But after discarding the passive terms, the active generalized perturbation forces are addible. In the meantime, when the active terms are added to the original equation, all dependent variable of the new equation became averaged quantities and an average equation has been obtained.

THE PERTURBATION EQUATION OF THE NAVIER-STOKES EQUATION

Turbulence is a typical fractal phenomenon. The Navier-Stokes equation isusually applied to the total time varying velocity field. For a laminar flow with a single scale, the Navier-Stokes equation gives a single determinate solution. For a turbulent flow with multiple scales, it gives chaotic solution. Chaotic flow field usually has infinite layers of scales. Therefore, the Navier-Stokes equation can give accurate solutions for turbulent flows only when the smallest scale valid under the concept of continuum, is adopted. When discrete mathematical method is used to solve the Navier-Stokes equation, any of its determinate solution is only a mathematically expected distribution upon which there must always be added some stochastic fluctuations with scales less than the adopted size of grid. In other words, there always exists an accompanying noise or perturbation field along with the solution of the Navier-Stokes equation. In the sense of fractal, there must always be some self-similar subpatterns with even smaller scales within the flow simulating and confirming the flow patterns of reality. This is the reason why turbulence models must be performed instead of employing the Navier-Stokes equation for practical engineering calculations. Nevertheless, the Navier-Stokes equation is still the starting point for the analysis of turbulence. The aim of the analysis is to find out the precise expression of the effect of the perturbation field on the average equation of turbulence, i.e. the alternative expression of Reynolds stress terms.

Let

$$\hat{V} = \hat{V}_E + \hat{V}', \quad P = P_E + P' \tag{1}$$

where $\hat{V}_E$, P_E are mathematically expected values of the velocity and the pressure, $\hat{V}'$, P' are their perturbation values and $\hat{V}$, P are their real values. Both the real and the expected values satisfy the Navier-Stokes equation. They belong to the infinitely numerous chaotic solutions of the nonlinear equation. In fact, both $\hat{V}$ and $\hat{V}_E$ correspond to two distinct realizations of the fractal dynamical system, each obeying the Navier-Stokes equation. From the Navier-Stokes equation, one obtains

$$\frac{D\hat{V}_E}{Dt} = -\frac{\nabla P_E}{\rho} + \nu \nabla^2 \hat{V}_E \tag{2}$$

$$\frac{D\hat{V}}{Dt} = \frac{D(\hat{V}_E+\hat{V}')}{Dt} = -\frac{1}{\rho}\nabla (P_E+P') + \nu \nabla^2 (\hat{V}_E+\hat{V}') \tag{3}$$

Substracting Eq. (2) from (3) yields

$$\frac{\partial \hat{V}'}{\partial t} + (\hat{V}_E.\nabla)\,\hat{V}' + (\hat{V}'.\nabla)\,\hat{V}_E + (\hat{V}'.\nabla)\,\hat{V}' = -\frac{1}{\rho}\nabla P' + \nu\nabla^2\hat{V}' \tag{4}$$

Equation (4) and the Poisson equation of the fluctuating pressure given by

$$-\frac{1}{\rho}\nabla^2 P' = \nabla.[(\hat{V}_E.\nabla)\,\hat{V}' + (\hat{V}'.\nabla)\,\hat{V}_E + (\hat{V}'.\nabla)\,\hat{V}']$$

yield

$$\frac{\partial \hat{V}'}{\partial t} - \nu\nabla^2\hat{V}' = C \tag{5}$$

where C is the constant of integration determined by the boundary conditons of the fluctuating velocity and pressure. Equation (4) may be expressed as

$$C + \nu\nabla^2\hat{V}' + (\hat{V}_E.\nabla)\,\hat{V}' + (\hat{V}'.\nabla)\,\hat{V}_E + (\hat{V}'.\nabla)\,\hat{V}' = -\frac{1}{\rho}\nabla P' + \nu\nabla^2\hat{V}' \tag{6}$$

Equation (2) with its boundary condition may have a determinate solution for a group of streamlines. From the viewpoint of analytical mechanics, the equation and its boundary conditions are still a set of incomplete constraints for the chaotic state. Upon taking the natural frame of the streamline group as the generalized coordinates, the perturbation force may be interpreted as the generalized force acting within the generalized system of coordinates. Therefore, through the virtual work analysis, and subsequently adding these perturbation forces to Eq. (2) may directly give the perturbation equation of the Navier-Stokes equation.

According to the rate of deformation tensor given by

$$\frac{\partial u_i}{\partial x_j} = \hat{\hat{S}} + 1/2\ (\nabla x \hat{V}) \tag{7}$$

where $\hat{\hat{S}}$ is the rate of strain tensor, and $1/2\ (\nabla x \hat{V})$ is the rotation tensor. One may use a method analogous to the Prandtl's mixing length concept $(v' = l\frac{\partial u}{\partial y})$ to obtain

$$\hat{V}' = \frac{\partial u_i}{\partial x_j}\delta\hat{r} = \hat{\hat{S}}.\delta\hat{r} + 1/2\ (\nabla x \hat{V}) \times \delta\hat{r}$$

or

$$\hat{V}' = \nabla\phi + 1/2\ (\nabla x\hat{V}) \times \delta\hat{r} \qquad (8)$$

where $\delta\hat{r}$ is a perturbation displacement and $\nabla\phi$ is the velocity of strain. Due to the local discrete character of turbulence (the particle character of turbulent eddies), $\delta\hat{r}$ may be considered as a local spatio-temporal constant vector much larger than the differential scales. This assumption introduces discreteness into the treatment of turbulence. Upon multiplying Eq. (6) by $\delta\hat{r}$ and substituting Eq. (8) into it, one may obtain the virtual work as

$$\delta E = \hat{F}_g.\delta\hat{r} = \delta\hat{r}.[-\frac{1}{\rho}\nabla p' + \nu\nabla^2\hat{V}]$$

$$= \delta\hat{r}\ .\ [c+\nu\nabla^2(\nabla\phi) + (\hat{V}_E.\nabla)\ \nabla\phi + (\nabla\phi.\nabla)\ \nabla\phi$$

$$+ 1/2\ (\hat{\Omega}x\delta\hat{r}.\nabla)\nabla\phi\ - \frac{\nu}{2}\ (\delta\hat{r}.\nabla)\ \nabla x\hat{\Omega}] \qquad (9)$$

where $\hat{F}_g$ represents the generalized perturbation force. After discarding the passive terms, the generalized perturbation force is addible to the right side of Eq. (2) and then let the velocity variables have new values to obtain a new equation, which includes the interaction between the two distinct fractal flows of different scales and describes their average behavior. Upon adding $\hat{F}_g$ to the right side of Eq. (2) and treating all parameters as the ensemble average, one obtains the perturbation equation of the Navier-Stokes equation as:

$$\frac{D\tilde{V}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\tilde{V} + (\tilde{V}.\nabla)\ \nabla\phi + \frac{1}{2}\ (\tilde{\Omega}x\delta\tilde{r}.\nabla)\ \nabla\phi$$

$$+ (\nabla\phi.\nabla)\ \nabla\phi + \nu\nabla^2(\nabla\phi) - \frac{\nu}{2}\ (\delta\tilde{r}.\nabla)\ \nabla x\tilde{\Omega}+c \qquad (10)$$

where the tildes, "~", denotes the mean values. The indefinite integral constant c has no influence on the discrete numerical calculation of turbulent flows. It is thus negligible in the differential treatment of turbulence.

Obviously, Eq. (10) is an alternative to the Reynolds equation, and provides a concrete expression of turbulent stress terms. The physical interpretation of the extra turbulence stresses has been precisely described by the five extra terms on the right side of Eq. (10). In fact, the equation is the physical equation of turbulence with a second order accuracy, which has been sought after for a long time. A detailed discussion on this equation has been reported elsewhere (Gao, 1989a).

THE ESSENCE OF EXTRA NONISOTROPIC STRESSES

Shearing is the most obvious reason of production of turbulent kinetic energy. However, the curvature of streamline, the stretching or compression of streamline and the interaction between the large and small turbulent eddies may also play important roles in producing turbulent stresses. All non-shearing stresses are named extra stresses. They must be appropriately considered for complex turbulent flows. The momentum equation based on the concept of eddy viscosity is

$$\frac{D\hat{V}}{Dt} = -\frac{1}{\rho}\nabla p + (\nu_L+\nu_T)\ \nabla^2\hat{V} \qquad (11)$$

This equation is a result of analogous simulation of turbulent eddy viscosity to molecular viscosity. A rigorous derivation of the equation is lacking.

To simplify the perturbation Eq. (10), one assumes that the perturbation velocity $\nabla\phi$ and the perturbation displacement $\tilde{\delta r}$ are locally constant and con-tribute as part of the eddy viscosity, $\nu_T = c'\nabla\phi\delta\tilde{r}$, where c' is an empirical coefficient. Equation (10) now becomes

$$\frac{D\tilde{V}}{Dt} = -\frac{1}{\rho}\nabla p + (\nu_L+\nu_T)\ \nabla^2\tilde{V} + \frac{1}{2}\ (\tilde{\Omega}x\delta\tilde{r}.\nabla)\ \nabla\phi$$

$$+ (\tilde{V}.\nabla)\ \nabla\phi - \frac{\nu_L}{2}\ (\delta\tilde{r}.\nabla)\ (\nabla x\tilde{\Omega}) \qquad (12)$$

with $\nu_T\ \nabla^2 V = (\nabla\phi.\nabla)\tilde{\nabla}\phi + \nu\nabla^2\ (\nabla\phi)$

$\nu\nabla^2(\nabla\phi)$, representing the effect of positive dispersion on the turbulent stresses which is one of the most important features of turbulence (Karweit, 1985), has been absorbed into the viscosity coefficient, and

$$(\nabla\phi.\nabla)\ \nabla\phi = \nabla\phi\ \delta\tilde{r}\ \nabla^2\tilde{V}$$

Comparing Eq. (12) with Eq. (11), one recognizes that there are three extra turbulent stresses in Eq. (12): the vorticity term in which the effect of streamline curvature is included; the convective term of perturbation velocity in which the effect of streamline stretching or compression is included; and the last term consisting of the perturbation displacement and the secondary derivative of the vorticity, which reflects the effect of interaction between large and small eddies. If a correct modeling of these extra terms can be accomplished, problems with these extra nonisotropic stresses can be properly dealt with. These three extra terms also reveal the physical essence of these extra stresses, which are the direct

results of the convective perturbation velocity and the interaction between vorticity and perturbation parameters. They also simultaneously show the convective transportation (expressed by the two preceding terms) and the discrete character (expressed by the last term) of turbulence.

A brief observation of Eq. (12) may also reveal the shortcomings of the prevailing turbulence modelings. For instance, in the eddy-viscosity models, only $\nu_T \nabla^2 \tilde{V}$ is considered; in the large eddy simulation, only $\nu\nabla^2(\nabla\phi)$ is considered; in the Reynolds stress modeling, the discreteness term,

$$-\frac{\nu}{2}(\delta\tilde{r}.\nabla)\ \nabla x\tilde{\Omega},$$

has not been considered.

Anisotropy, as a direct result of nonlinearity, is a basic characteristic of turbulent flows, and is not the property of the fluid. Adopting the Boussinesq's eddy-viscosity concept, a traditional treatment for anisotropy is to look for an anisotropic viscosity coefficient. But from the viewpoint of random movement, it is known that the local isotropy of turbulent eddies prevails. The movement of eddies has a strong tendency to isotropy due to the isotropic stochastic collision, except for those occasions where the extra stresses cause eddies moving in a particular direction. Therefore, in a general sense, adoption of an isotropic turbulent viscosity coefficient μ_T at any point of the flow field is more convincing. The assumption of isotropic μ_T is in consistence with stochastic property of turbulence. Anisotropy should be expressed by the extra stresses in the momentum equation. From Eq. (10), the corresponding turbulence stress tensor can be interpreted as

$$\tau_{ij} = \tilde{V}\nabla\phi + \frac{1}{2}\ (\tilde{\Omega}x\delta\tilde{r})\ \nabla\phi + \nabla\phi\ \nabla\phi + \nu\nabla(\nabla\phi)$$

$$-\frac{\nu}{2}\ \delta\tilde{r}\ (\nabla x\tilde{\Omega}) \qquad (13)$$

It is obvious that with the effects of the ordinary and the extra stresses, Eq. (13) will display non-isotropic features.

TURBULENCE MODELING

The perturbation equation of the Navier-Stokes Eq. (10) contains two turbulent parameters: the perturbation velocity $\nabla\phi$ and the perturbation displacement $\delta\tilde{r}$. If both of them can be correctly described by differential equations, complex turbulent flow problems can be dealt with completely within the realm of the two-equation modeling. There exist many ways to modeling the Eq. (10). As a simulation of molecular motion, turbulent viscosity usually consists of a length scale, a velocity scale and an experimentally determined coefficient, i.e. $\nu_T = c\tilde{v}\tilde{l}$. In order to model the perturbation equation, the velocity of strain $\nabla\phi$ may be chosen as the velocity scale and the perturbation displacement $\delta\tilde{r}$ may be chosen as the length scale. Changing the positions of $\nabla\phi$ and $\delta\tilde{r}$ to bring them together to form the turbulent viscosity ν_T, one may effectively consider that they are locally constants and neglect their spatial differentials. One of these attempts of modeling is given by Gao (1989b). In this scheme a set of new control equations for incompressible turbulent flow, i.e., continuity, momentum, turbulent kinetic energy, its dissipation and the energy equations, are formulated as:

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (14)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial(\nu_L+\nu_T)}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$+ \Omega_{ij}\frac{\partial \nu_T}{\partial x_j} + C_3\, e_i\, u_j \frac{\partial\sqrt{k}}{\partial x_j}$$

$$- C_0\, \nu_T \frac{k^{3/2}}{\epsilon}\ e_l\ \epsilon_{ijk}\ \frac{\partial^2\Omega_k}{\partial x_j \partial x_l} \qquad (15)$$

$$\frac{\partial k}{\partial t} + u_j\frac{\partial k}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\sigma k}\frac{\partial k}{\partial x_j}\right) + \frac{G_k}{\rho} - \epsilon \qquad (16)$$

$$\frac{\partial \epsilon}{\partial t} + u_j\frac{\partial \epsilon}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right)$$

$$+ C_1 \frac{\epsilon G_\epsilon}{k+(2\nu_L\epsilon)^{1/2}} - C_2 \frac{\epsilon^2}{k+(2\nu_L\epsilon)^{1/2}} \qquad (17)$$

$$\frac{\partial h}{\partial t} + u_j\frac{\partial h}{\partial x_j} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}u_j\frac{\partial p}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial T}{\partial x_j}\right) + \frac{\phi}{\rho} + \frac{\psi}{\rho} \qquad (18)$$

$$G_k = \max\ (I\text{-}II\text{-}III,\ 0) \quad , \quad G_\epsilon = \max\ (I - II,\ 0)$$

$$I = \mu_T \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} + \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}\right]$$

$$= \mu_T \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} - \frac{1}{2}\mu_T\,\Omega_{ij}^2 \quad ,$$

$$II = C_3\, e_i u_j \sqrt{k}\, \frac{\partial u_i}{\partial x_j}\, \rho \quad ,$$

$$III = C_0\, \mu_T\, \frac{k^{3/2}}{\epsilon}\, e_l\, \epsilon_{ijk}\, \frac{\partial \Omega_k}{\partial x_j}\, \frac{\partial u_i}{\partial x_l} \quad ,$$

$$\Omega_{ij} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \quad , \quad \mu_T = C_\mu \frac{\rho k^2}{\epsilon} \quad ,$$

when $u_i \geq 0$, $e_i = e_l = 1$
$u_i < 0$, $e_i = e_l = -1$

For the dissipation function ϕ, ϕ = max (I-II,0)

$$+ \mu_L \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \frac{\partial u_i}{\partial x_j}$$

For the dispersion function ψ, ψ = - III.

Various constants and coefficients related to this modeling are given in Table I.

Table 1. The coefficients for the Rectangular system of coordinates.

C_0	C_1	C_2	C_3
III ≥ 0, 0.30 III < 0, 0.0	1.44	1.92	0.16

σ_k	σ_ϵ	C_μ
1.0	1.3	0.09

As discussed previously (Gao, 1989b) due to the spatio-temporal instability, intermittency of turbulent flow is the basic reason that the principle directions of stress and strain are non-coincident with each other. This is why the Boussinesq's eddy-viscosity model fails. The task of turbulence modeling is to resolve the incompatibility of the constitutive relations of the viscous flow with the non-Newtonian property of the intermittent turbulent flow. In this sense, all the extra stresses, added to the momentum equations as presented in Eq. (10), cannot be interpreted as standard tensors. This present modeling by Gao (1989b), however, is a modification of the original k-ε modeling by Launder and Spalding (1974) by taking advantage of their previous tremendous achievement. In order to bring all the extra terms into tensorial expression, an effort to use the summation subscripts has been applied to them.

Based on the nonlinear fractal analysis, the new modeling has extra terms of stretching, rotation and negative dispersion, which reflect the non-linear coherent property of turbulence. On the other hand, the empirical coefficients reflect the stochastic nature of turbulence. Many perplexing problems of turbulence, such as anisotropy, turbulence energy inversion (Sommeria, 1986), cut-off of the turbulence energy spectra, intermittency and coherent structure may be explained theoretically and simulated numerically by this nonlinear modeling.

DISCUSSION OF COMPUTED RESULTS

By adopting the procedures suggested by Patankar (1981) numerical computations for many complex turbulent flows have been performed with the improved turbulence modeling. The results of these calculations are respectively discussed in the following:

1. The Anomaly of the Round Jet/Plane Jet

The computed and measured behavior for the plane and the round jets are compared in Figs. 1-3, and in Table 2. The corresponding kinetic energy profile, shown in Figs. 1 and 2, are in satisfactory accord. The difference in the shape of the turbulence kinetic energy profiles between the round and plane jets is correctly reporduced by these numerical computations. The improvement of the k profile of the round jets, especially in the region near the axis, is mainly attributable to the extra stretching term in the momentum equation. In the case of the round jet, the strength of the velocity decay in this region is far greater than that in the plane jet. Since the velocity decay effectively increases the kinetic energy production, the k profile of the round jets exhibits a monotonic decrease from the axis while the k profile of the plane jets exhibits a hump with its maximum at y = 0.65 $y_{1/2}$. The most interesting

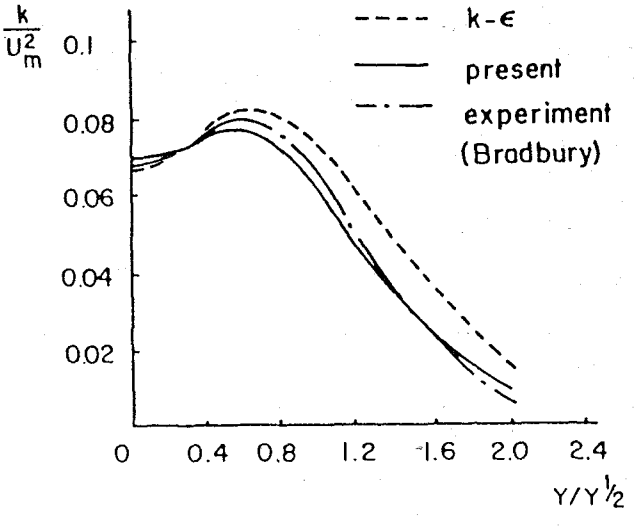


Fig.1. Turbulence Kinetic Energy Profile of a Plane Jet in Stagnant Surrounding

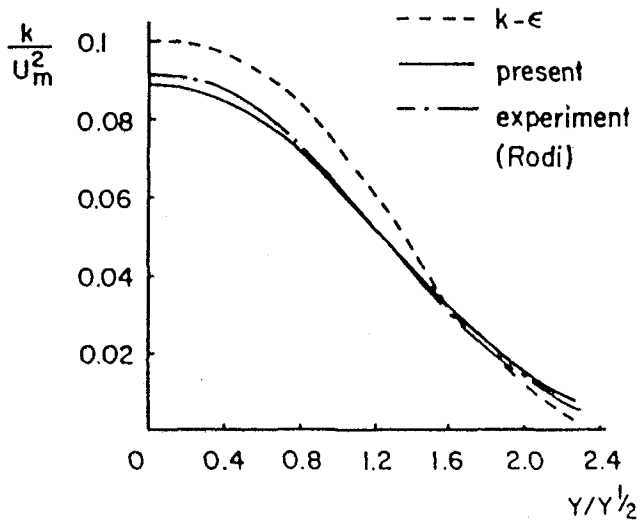


Fig.2. Turbulence Kinetic Energy Profile of a Round Jet in Stagnant Surrounding

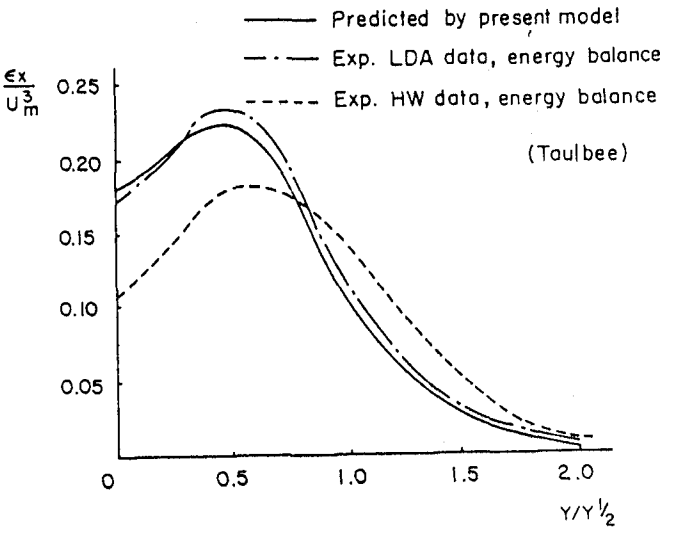


Fig.3. The Measured and the Predicted Profiles of a Round Jet (Dissipation Rate of the Turbulence Kinetic Energy)

results appear in Table II, which lists the rates of spread of the two flows including that of the experimental data. It also includes the results from the Algebraic stress model and the Reynolds stress transport model. Obviously the anomaly of the round jet/plane jet has been adequately resolved (Hanjalic, 1988). The reason for the improvement of the spread rate is traced to the extra stretching term in the momentum equation. The component of the stretching term in the y-direction is $C_3 V \frac{\partial \sqrt{k}}{\partial y}$. From Figures 1 and 2, it is shown that this term always suppresses the rate of spread for the round jet, while it is positive within the region from the axis to $y = 0.65y_{1/2}$ for the plane jet, which helps to increase the rate of spread. Shown in Figure 3 are the measured and predicted dissipation profiles of the round jets. Since there is no extra stretching term in the dissipation production G_ε of the present modeling, a hump-like profile of dissipation has been obtained. If this extra stretching term is also included into G_ε, the level of ε decreases monotonically from the axis, similar to the k profile in the round jets. This gives an important support to the opinion that G_k and G_ε should not contain the same element, and the extra stretching term should appear only in G_k.

2. Near Wall Flow

Since the modeled differential equation for free turbulence yields incorrect results near a wall, various ad hoc functions (there are at least five functions of this kind in the k-ε modeling) are conventionally added in an effort to eliminate this shortcoming. Since the present modified k-ε method is based upon a thorough consideration of the non-linearity of turbulent flow, the appearance of the proposed modeling undergoes a substantial change. Group I of G_k always substracts the term $1/2\ \mu_T \Omega_{ij}^2$, which accounts for a significant improvement of the shortcoming of the original k-ε modeling. It is well known that the original k-ε modeling invariably predicts a higher turbulence intensity and turbulent viscosity in the near wall region ($Y^+ < 40$). The cut-off state of G_k and G_ε in the region of $Y^+ = 10$ also offers an important contribution to the correct prediction in the near-wall region. An interesting discovery arising from this improvement is the modified k-ε modeling is effective for both the free flow and the near-wall flow. No additional function is needed in this improved modeling.

Table II Rates of Spread of Plane and Round Jets

Flow	Experiment	Original k-ε Model	ASM	RST	Present Model
Plane jet	0.110	0.108	0.114	0.108	0.111
Round jet	0.086-0.093	0.114	0.149	0.121	0.092

3. Three-Dimensional Calculation of a Centrifugal Compressor

With this Gao's modeling, Bo and Bosman (1989) at the University of Manchester Institute of Science and Technology, computed a three-dimensional flow field of a centrifugal compressor, and achieved a complete success. The grid of calculated flow field is shown in Fig. 4. The computed results at the section No. 25 by the original k-ε model and the present model are shown in Figs. 5 and 6, respectively. Fig. 7 presents the experimentally measured results in the same section. In comparing these sets of results, it is obvious that the present modeling gives very good prediction of the secndary flow vortices in scale, form, position and the flow direction, very similar to the experimental data. However, for the original k-ε modeling, the vortex on the suction side almost disappears. Both the turning points of the flow direction and the velocity profile of the secondary flow vortices are qualitatively incorrect, and the poor results cannot be corrected despite wide adjustment on its empirical coefficients. It has been found that the extra stretching and rotation terms of the new modeling are responsible for the correct prediction, which exhibits the characteristic of strong stretching and sharp rotation in the centrifugal compressor, (Cumpsty, 1989).

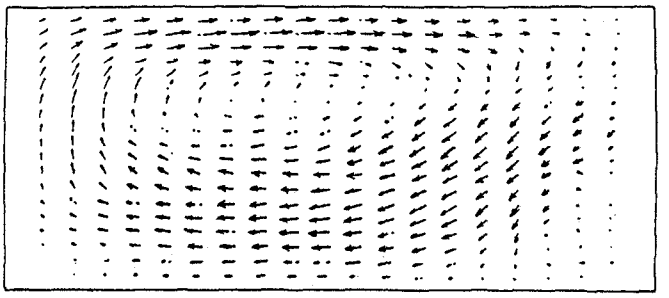


Fig.5. Comouted Results from the Ordinary K-E Moel

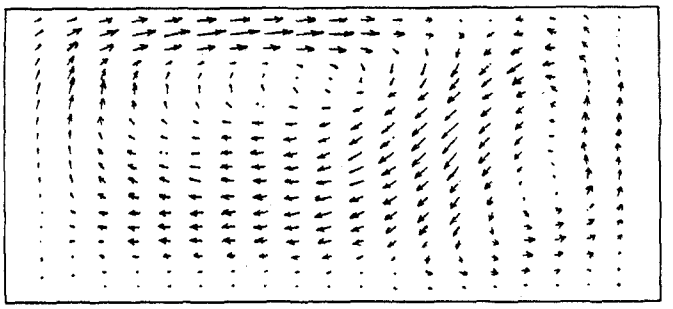
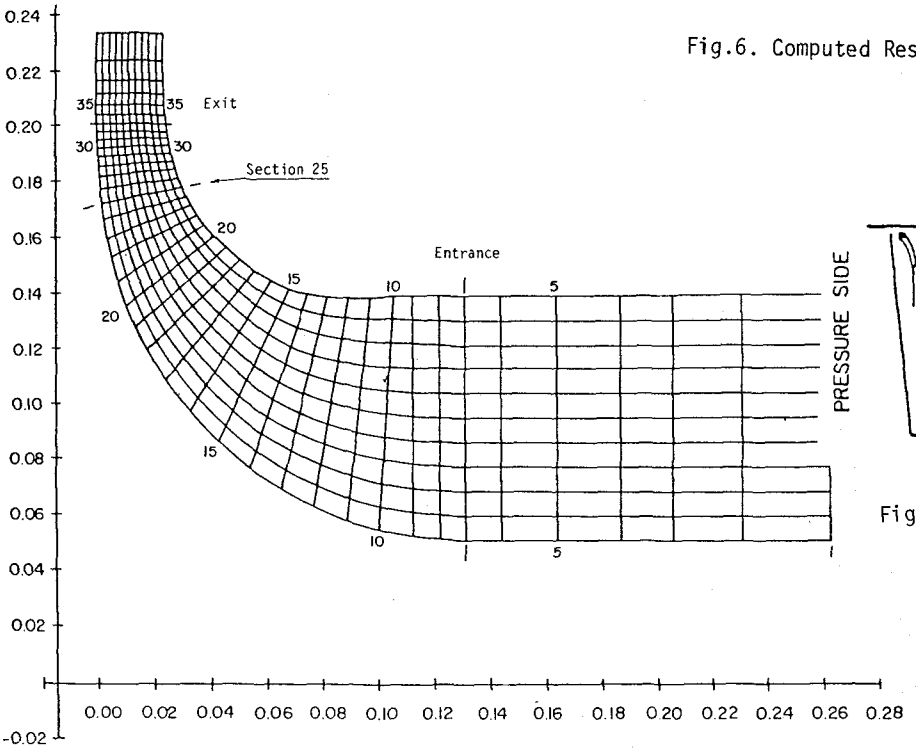


Fig.6. Computed Results from the Improved Model



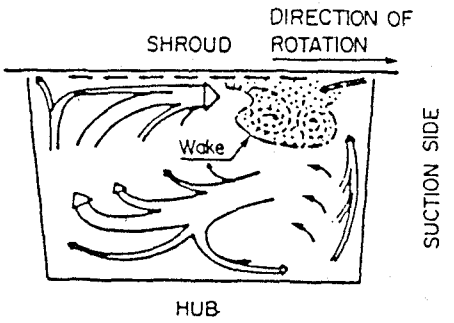


Fig.7.Experimental Data for the Eckardt Compressor

Fig.4. The Computational Grid for the Eckardt Compressor

4. Swirling Flows

Another importany improvement was observed in the confined sudden-expansion swirling flows by Xu and Gao (1990). With the new modeling, the form of the recirculating zone, the composite vortex profile of the tangential velocity, the proceeding vortex roll behind the recirculating zone, and the profile of the axial velocity within the recirculating zone are all correctly reporduced through numerical computations. Fig. 8 shows the comparison of the axial and tangential velocity profiles between the experimental and the numerically predicted results. The original k-ε modeling failed to give the proceeding vortex roll (Fig. 8a), the outer composite vortex (Fig. 8b) and the deficit of the axial velocity near the central axis

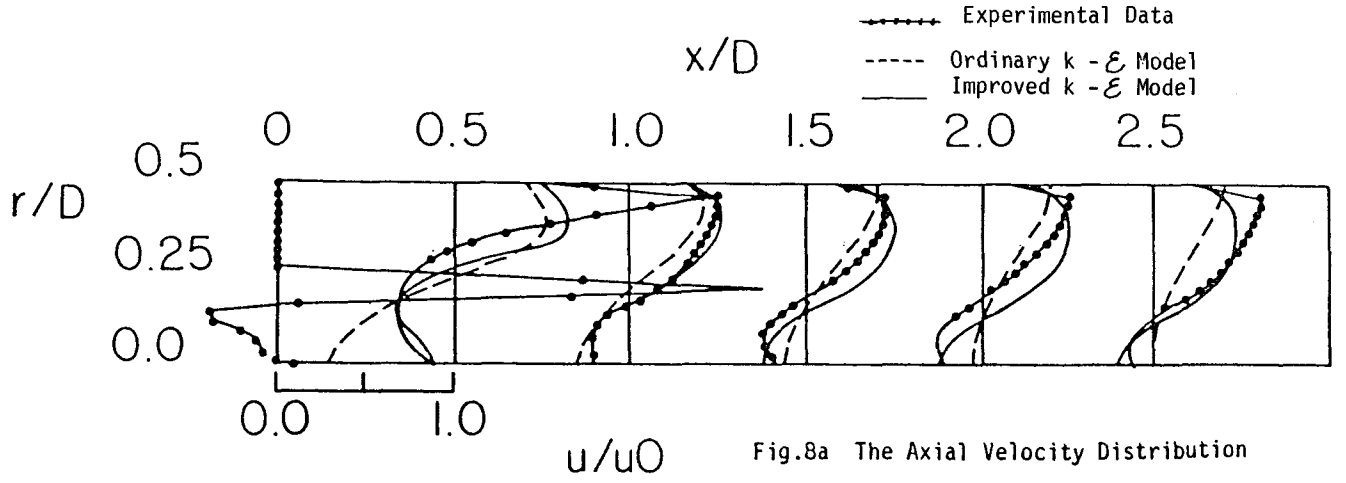


Fig.8a The Axial Velocity Distribution

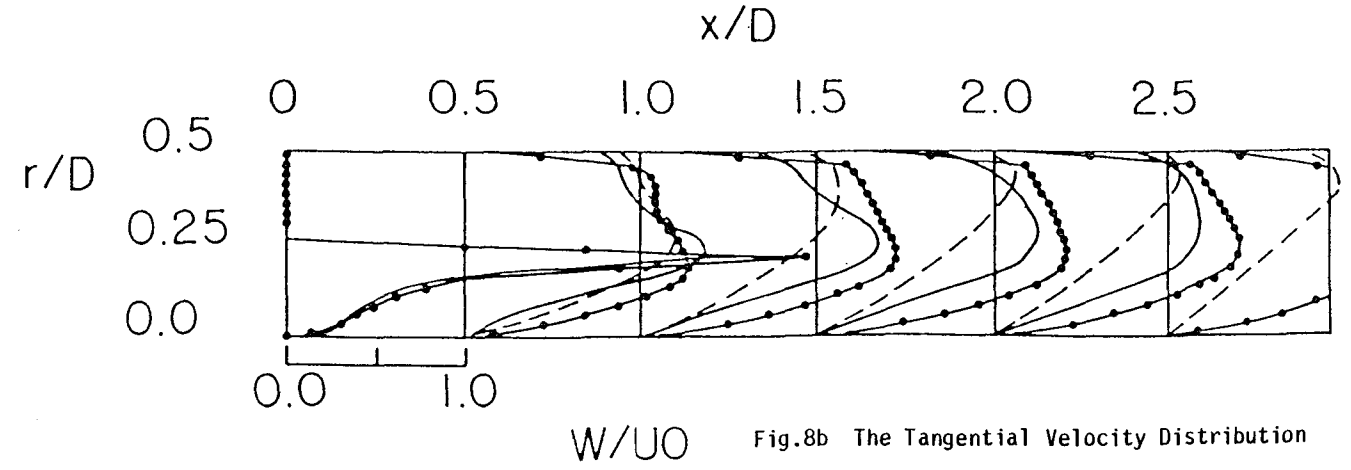


Fig.8b The Tangential Velocity Distribution

Fig.8 The Flow Field Within a Confined Sudden Expansion Swirling Flow

within the recirculating zone (Fig. 8a). These defects, however, cannot be corrected by adjusting the coefficients. The results by the new model do not show the deficiencies.

It has been found that the extra rotation term and the third-order negative dispersion term play the important roles in obtaining the correct tangential velocity profile and the proceeding vortex roll behind the recirculating zone. The stretching term makes the deficient axial velocity profile within the recirculating zone possible. The two modelings were also employed to predict the non-swirling and moderate swirling flows. Different sets of constants were required to deal with different levels of swirl for the ordinary k-ε modeling. Even under this condition, the results showed the maximum reverse flow always occurred on the axis and it never reproduced the structure of the downstream proceeding vortex roll-composite vortex structure. For the improved k-ε model all these features have been correctly and adequately reproduced.

DISCUSSION AND SUMMARY

With this improved modeling, computations have been carried out to study the vortex field behind a bluff body, diffusive flows, and the near wall flows. All results showed improved qualities. The success of predicting complex turbulent flows so far instills the confidence of future capabilities of dealing with complex three-dimensional turbulent flows, with accurate prediction of extra stresses, but within a reasonable amount of computing time. In this sense, it is a strong competitor for the industry-standard turbulence modeling.

It has been learned that the new coefficients C_0, C_3 need some additional adjustment. Although the physical Eq.(10) has the second order accuracy, some terms have been omitted in the modeling process for the sake of reducing the amount of computing time. Some errors must have been unavoidably introduced. For example, for the convection term of the perturbation velocity, $(\tilde{V}.\nabla)\nabla\phi$, it has three components in each momentum

equation. Only the component in the i direction has been retained. This treatment, in turn, introduced an amount of uncertainty of the coefficient C_3. Another example is in the extra rotation

term 1/2 $(\tilde{\Omega}x\delta\tilde{r}.\nabla)\ \nabla\phi$. $\delta\tilde{r}$ was considered as a local constant and was combined with $\nabla\phi$ to form the turbulent viscosity μ_T, which introduced a certain amount of error to the coefficient C_0.

Besides the present modeling, new modeling for the physical equation of turbulence is possible. Additional attempt for improving the capability of the non-linear modeling is worth exploring. Since the intermittent coherent structures are produced by the interactions of dissipation and dispersion, it is necessary to keep the positive dispersion term $\nabla^2(\nabla\phi)$ in the modeling equation if the function of simulating the coherent behavior of turbulent flow is emphasized. However, for any attempt of modeling, it should be kept in mind that a compromise of accuracy and simplicity should be considered.

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