METHODS TO ENHANCE THE ACCURACY OF FINITE VOLUME SCHEMES*

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Abstract

Several high–order finite volume schemes to solve the Euler equations are constructed. The stability behaviour of the methods is analyzed and the usefulness of the schemes is demonstrated.

1. Introduction

Finite volume methods are today widely spread methods to solve the Euler and Navier–Stokes equations of gasdynamics. Most of the finite volume methods which are in use today are of at most first–order accuracy in space. We will describe where discretization errors during the approximation process are coming in and how they can be decreased. The main goal of this paper is to present some new techniques to increase the accuracy of the finite volume schemes by using high–order approximations. The development of high–order schemes is important for two reasons. The first one is the increasing interest in more accurate approximations. The second one is the necessity to reduce the computer time needed to solve numerically the problem under consideration.

In the following we give a brief outline on the approximation steps usually used in finite volume methods and we will point out the sources for discretization errors.

The starting–point for every finite volume method is the integral form of the conservation law for the quantities describing the fluid flow. This conservation law has to be fulfilled in each sub-domain V_{ijk} of the whole flow domain Ω under consideration. Making use of the Gauß' theorem the conservation laws follows

$$\int_{V_{ijk}} \partial_t f \, dx \; + \; \int_{\partial V_{ijk}} H(f) \, n \, ds \; = \; 0, \quad \forall \, V_{ijk} \in \Omega, \quad (1)$$

where n is the outward pointing unit normal vector to ∂V_{ijk} and the vector f is defined by $f := (\rho, \rho u^1, \rho u^2, \rho u^3, e)^T$.

In connection with the stability theory we will also use the primitive variable formulation

$$\int_{V_{ijk}} \partial_t g \, dx \; + \; \int_{\partial V_{ijk}} (A_1 g_x + A_2 g_y + A_3 g_z) \, n \, ds \; = \; 0, \quad (2)$$

$\forall \, V_{ijk} \in \Omega$, with $g := (\rho, u^1, u^2, u^3, p)^T$, and where the matrices are given by

$$A_1 := \begin{pmatrix} u^1 & \rho & 0 & 0 & 0 \\ 0 & u^1 & 0 & 0 & \rho^{-1} \\ 0 & 0 & u^1 & 0 & 0 \\ 0 & 0 & 0 & u^1 & 0 \\ 0 & \gamma(\gamma-1)e & 0 & 0 & u^1 \end{pmatrix},$$

$$A_2 := \begin{pmatrix} u^2 & 0 & \rho & 0 & 0 \\ 0 & u^2 & 0 & 0 & 0 \\ 0 & 0 & u^2 & 0 & \rho^{-1} \\ 0 & 0 & 0 & u^2 & 0 \\ 0 & 0 & \gamma(\gamma-1)e & 0 & u^2 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} u^3 & 0 & 0 & \rho & 0 \\ 0 & u^3 & 0 & 0 & 0 \\ 0 & 0 & u^3 & 0 & 0 \\ 0 & 0 & 0 & u^3 & \rho^{-1} \\ 0 & 0 & 0 & \gamma(\gamma-1)e & u^3 \end{pmatrix}.$$

Using the mean value theorem the volume integral in (1) writes

$$\int_{V_{ijk}} \partial_t f \, dx \; = \; \partial_t f(x^*_{ijk}, t) \int_{V_{ijk}} dx, \quad (3)$$

with $x^*_{ijk} \in V_{ijk}$ as a certain unknown point in the sub-volume V_{ijk}.

The first step toward obtaining a large coupled system of ordinary differential equations is to approximate the values $\partial_t f(x^*_{ijk}, t)$ by $\partial_t f(x_{ijk}, t)$, where x_{ijk} is the center point of the cell V_{ijk}. This approximation is of first order accuracy in space and here we have the first source for errors. We will not say anything how to get a better approximation order for this step, but we will explain some high–order methods to solve the resulting system of equations which now is given by

$$\partial_t f(x_{ijk}, t) \int_{V_{ijk}} dx \; = \; - \int_{\partial V_{ijk}} H(f) \, n \, ds, \quad \forall \, V_{ijk} \in \Omega. \quad (4)$$

In the following x_{ijk} denotes the center and y_{ijk} the vertex of the cell V_{ijk} which is the lower left corner point of the sub-surface common to the cells V_{ijk} and $V_{i-1,jk}$.

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Now, the surface integral in (4) is usually approximated by

$$\int_{\partial V_{ijk}} H(f)\, n\, ds \approx \sum_{l=1}^{6} \int_{S^l_{ijk}} H(f)\, n\, ds \approx \sum_{l=1}^{6} H(f(x^l_{ijk},t))\, |S^l_{ijk}| n, \quad (5)$$

where S^l_{ijk} denote approximations to the six sub-surfaces of the cell V_{ijk}, which often are calculated by $S^l_{ijk} = |d_1 \times d_2|/2$, with d_1 and d_2 as the diagonalvectors connecting opposite points on the sub-surfaces. x^l_{ijk} denotes a point on the sub-surface S^l_{ijk}. Since the values of $H(f)$ are unknown on the sub-surfaces S^l_{ijk} they are usually approximated by, e.g.,

$$H(f(x^l_{ijk},t)) \approx \frac{H(f(x_{i-1,jk},t)) + H(f(x_{ijk},t))}{2}. \quad (6)$$

In (5) and (6) the second and third sources for errors due to approximations arise. The approximation of the surface integrals in (5) is of at most first–order accuracy whereas the approximation in (6) is of at most second–order accuracy.

Using the common abbreviation $f^t_{ijk} := f(x_{ijk},t)$ the above approximations can be summarized to the following system of spatial discretized Navier–Stokes equations

$$\partial_t f^t_{ijk} \int_{V_{ijk}} dx = -L f^t_{ijk}, \quad \forall\, V_{ijk} \in \Omega, \quad t > 0, \quad (7)$$

where L is the operator approximating the surface integrals in the way mentioned above. Several finite volume schemes, explicit as well as implicit, based on (7) can be found, e.g., in [3].

Again, we point out that the approximation (7) is an discretization of (1) of at most first–order accuracy. The first of our new schemes is a simple upgrading of the existing finite volume methods in order to obtain second–order accuracy in space.

2. Explicit Finite Volume Approximations of High–Order

In order to construct our finite volume schemes in a first step we derive schemes to approximate the surface integrals with high–order accuracy. Next we use Taylor like expansions with respect to time, up to a certain order where the occuring time derivatives are replaced by the new spatial discretized finite volume equation (7). The idea to replace time derivatives by space derivatives was introduced by Lax and Wendroff [1,2]. Furthermore, we make the usual assumption that the matrices occuring in equation (2) are independant on both the time and space variables. It is mentioned that all the schemes are consistent.

Time step sizes are defined by $\Delta t_l := t_{l+1} - t_l$, $l = 0,1,2,3,\ldots..$, and by $\Sigma := \{t_0, t_1, t_2, \ldots..\}$, with $t_0 < t_1 < t_2 < \ldots..$, we denote the corresponding partition of the time axis. The precise step sizes Δt_l result from the restriction for stability.

2.1. A Second–Order Spatial Discretization

In this chapter we propose a simple way to calculate numerically the surface integrals in (4) with second–order accuracy. Let a, b, c, and d be the vertices common to the cells V_{ijk} and $V_{i-1,jk}$, where a is the lower left vertex and a, b, c, and d are counted clockwise (see Figure 1). We then define two triangles S^1_{ijk} and S^2_{ijk} in the way that S^1_{ijk} is the plane triangle with corner points a, b, and c, and S^2_{ijk} the corresponding one with corner points c, d, and a. In a similar way we can define triangles F^1_{ijk} and F^2_{ijk} in j-direction and G^1_{ijk} and G^2_{ijk} in k-direction.

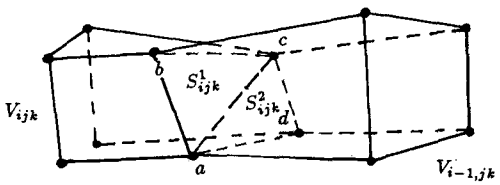

Figure 1.

Instead of the common approximation, we use the exact sub-surfaces consisting of the triangles defined above. In order to calculate the surface integrals we will use the function values of $H(f)$ at the cell vertices. The function values of $H(f)$ at these points are unknown, but they can be approximated with second–order accuracy by using the function values of $H(f)$ at the eight cell center points surounding the corner point under consideration. Therefore, the surface integral over ∂V_{ijk} can now be written as

$$\begin{aligned}
&\int_{\partial V_{ijk}} H(f)\, n\, ds \\
&= \sum_{l=1}^{2} \{ \int_{S^l_{ijk}} H(f)\, n\, ds - \int_{S^l_{i-1,jk}} H(f)\, n\, ds \} \\
&\quad + \sum_{l=1}^{2} \{ \int_{F^l_{ijk}} H(f)\, n\, ds - \int_{F^l_{i,j-1,k}} H(f)\, n\, ds \} \quad (8) \\
&\quad + \sum_{l=1}^{2} \{ \int_{G^l_{ijk}} H(f)\, n\, ds - \int_{G^l_{ij,k-1}} H(f)\, n\, ds \}.
\end{aligned}$$

Here, the surface integrals on the right-hand side can be approximated with second–order accuracy by, e.g.,

$$\begin{aligned}
&32 \int_{S^1_{ijk}} H(f)\, n\, ds \approx \\
&8\{H(f(a,t) + 2H(f(b,t) + H(f(c,t)\}|S^1_{ijk}|n(S^1_{ijk}) \approx \\
&\quad \{H(f^t_{i,j+1,k-1}) + H(f^t_{ij,k-1}) + H(f^t_{i,j+1,k}) + \\
&\quad H(f^t_{ijk}) + H(f^t_{i+1,j+1,k-1}) + H(f^t_{i+1,j,k-1}) \\
&\quad + H(f^t_{i+1,j+1,k}) + H(f^t_{i+1,jk})\} \\
&\quad + 2\{H(f^t_{i,j+1,k}) + H(f^t_{ijk}) + H(f^t_{ij,k+1}) + \quad (9) \\
&\quad H(f^t_{ij,k+1}) + H(f^t_{i+1,j+1,k}) + H(f^t_{i+1,jk}) \\
&\quad + H(f^t_{i+1,j,k+1}) + H(f^t_{i+1,j+1,k})\} \\
&\quad + \{H(f^t_{i,j-1,k}) + H(f^t_{i,j-1,k+1}) + H(f^t_{ijk}) + \\
&\quad H(f^t_{ij,k+1}) + H(f^t_{i+1,j-1,k}) + H(f^t_{i+1,j-1,k+1}) \\
&\quad + H(f^t_{i+1,jk}) + H(f^t_{i+1,j,k+1})\} \\
&\quad \cdot |S^1_{ijk}| n(S^1_{ijk})
\end{aligned}$$

Note that the unit normal vectors $n(S^l_{ijk})$ are constant on the triangles S^l_{ijk} and that, e.g., the normal vector on S^1_{ijk} is given by $n(S^1_{ijk}) = (c-b)\times(a-b)/|(c-b)\times(a-b)|$. Furthermore, the area of S^1_{ijk} is given by $|S^1_{ijk}| = |(c-b)\times(a-b)|/2$. Implementing this into (4), our second-order spatial discretization can be summarized to

$$\partial_t f^t_{ijk} + \int_{V_{ijk}} ds \;=\; -L_1 f^t_{ijk}, \quad \forall\, V_{ijk} \in \Omega, \quad t>0, \quad (10)$$

where L_1 is the operator approximating the surface integrals in the way mentioned above.

Denoting by $|V_{ijk}| := \int_{V_{ijk}} dx$ the volume of the cell V_{ijk}, we have the following

Lemma 1: *If $H(f(\cdot,t)) \in C^2(\Omega)$, $\forall t>0$, then the spatial finite volume discretization*

$$|V_{ijk}|\;\partial_t f^t_{ijk} = \; -L_1 f^t_{ijk}\;\forall\, V_{ijk}\in\Omega, \quad t>0, \qquad (11)$$

is a discretization of second-order accuracy of the system (4).

2.2. The Explicit Finite Volume Schemes EFV2*

The Taylor like expansion

$$f(x,t+\Delta t) = f(x,t) + \Delta t\partial_t f(x,t) + \alpha_1(\Delta t)^2\partial_t^2 f(x,t) \qquad (T)$$
$$+\alpha_2(\Delta t)^3\partial_t^3 f(x,t) + \alpha_3(\Delta t)^4\partial_t^4 f(x,t) + O((\Delta t)^p),$$

which is an expansion backward in time up to a certain order p, can be used to obtain an **explicit** finite volume scheme. When specifying the coefficients α_n in the way that $\alpha_n = \frac{1}{(n+1)!}$, one yields the usual Taylor expansion. By choosing the coefficients α_i in a special way, we are able to increase the largest possible time step, but at the same time the accuracy in time is decreased by a certain order. The exponential p takes the values 2, 3, 4, or 5, depending on the choice of the α_i's.

In the following the abbreviation $f^n_{ijk} := f(x_{ijk}, t_n)$ is employed. Using the spatial discretized finite volume equation (11) in order to replace the time derivatives $\partial_t^i f$ in the expansion (T) by the corresponding finite difference operator L_1 we obtain our basic explicit finite volume scheme of second order accuracy in space, from which our schemes result by specifying the coefficients α_i in a certain way:

$$f^{n+1}_{ijk} = f^n_{ijk} - \frac{\Delta t}{|V_{ijk}|}L_1 f^n_{ijk} + \frac{\alpha_1\Delta t^2}{|V_{ijk}|^2}L_1^2 f^n_{ijk} - \frac{\alpha_2\Delta t^3}{|V_{ijk}|^3}L_1^3 f^n_{ijk}$$
$$+\frac{\alpha_3\Delta t^4}{|V_{ijk}|^4}L_1^4 f^n_{ijk} + O(\Delta t^5) + O(\Delta x^2) + O(\Delta y^2). \quad (12)$$

Specifying now the coefficients α_i in (12) to $\alpha_1 = 1/2$, and $\alpha_i = 0$, for $i = 2,3$, leads to

$$f^{n+1}_{ijk} := f^n_{ijk} - \frac{\Delta t_n}{|V_{ijk}|}L_1 f^n_{ijk} + \frac{(\Delta t_n)^2}{2|V_{ijk}|^2}L_1^2 f^n_{ijk}, \quad (EFV2a)$$

and we note that this scheme is of third–order accuracy in time.

In order to obtain our second explicit finite volume scheme (EFV2b) we specify the parameters α_1 and α_2 in (12) to $\alpha_1 = 1/2$ and $\alpha_2 = 1/6$ obtaining

$$f^{n+1}_{ijk} := f^n_{ijk} - \frac{\Delta t_l}{|V_{ijk}|}L_1 f^n_{ijk} + \frac{(\Delta t_l)^2}{2|V_{ijk}|^2}L_2 f^n_{ijk} - \frac{(\Delta t_l)^3}{6|V_{ijk}|^3}L_3 f^n_{ijk}, \quad (EFV2b)$$

and we note that this scheme is of fourth–order accuracy in time.

3. Von Neumann Stability Analysis of the Schemes EFV2*

Since we are interested in von Neumann stability assertions for the finite–volume schemes (EFV2*), we now restrict ourselves to rectangular grids. A typical gridpoint x_{ijk} is then defined by

$$x_{ijk} := (i\cdot\Delta x, j\cdot\Delta y, k\cdot\Delta z)^T,$$

the volume $|V_{ijk}|$ of a sub-domain V_{ijk} can be expressed by

$$|V_{ijk}| = \Delta x\cdot\Delta y\cdot\Delta z,$$

and the area of, e.g., S^1_{ijk} is given by $|S^1_{ijk}| = \Delta y\cdot\Delta z/2$.

Now, by straightforward calculations it follows that in the case of the Euler equations and equi–spaced orthogonal grids the operator L_1 comes out to be

$$2L_1 f^t_{ijk} \;=\; \Delta x\,\Delta y\,\Delta z\,\{A_1\delta_i(4+2\gamma_j+2\gamma_k+\gamma_j\gamma_k)+ \quad (13)$$
$$A_2\delta_j(4+2\gamma_i+2\gamma_k+\gamma_i\gamma_k) + A_3\delta_k(4+2\gamma_i+2\gamma_j+\gamma_i\gamma_j)\}f^t_{ijk}$$

where the operators δ_* and γ_* are defined by

$$\delta_i f^t_{ijk} := \frac{f^t_{i+1,jk} - f^t_{i-1,jk}}{2\Delta x},\; \delta_j f^t_{ijk} := \frac{f^t_{i,j+1,k} - f^t_{i,j-1,k}}{2\Delta y},$$
$$\delta_k f^t_{ijk} := \frac{f^t_{ij,k+1} - f^t_{ij,k-1}}{2\Delta z},\; \gamma_i f^t_{ijk} := f^t_{i+1,jk} + f^t_{i-1,jk},$$
$$\gamma_j f^t_{ijk} := f^t_{i,j+1,k} + f^t_{i,j-1,k},\; \gamma_k f^t_{ijk} := f^t_{ij,k+1} + f^t_{ij,k-1}.$$

In order to study the stability of the scheme $(EFV2a)$ it is applied on a typical Fourier mode, given by

$$G = G_0 e^{i(k_1 x + k_2 y + k_3 z)},$$

where $k_1, k_2, k_3 \in \mathbb{R}$ with $k_1^2 + k_2^2 + k_3^2 = 1$, and where G_0 is a constant vector, and one obtains

$$G^{n+1} = Q_n G^n.$$

Defining the matrix E_1 by

$$E_1(\kappa_n) := -\{r(n)_x \sin\theta_x \cos^2(\theta_y/2)\cos^2(\theta_z/2)A_1$$
$$+r(n)_y \sin\theta_y \cos^2(\theta_x/2)\cos^2(\theta_z/2)A_2$$
$$+r(n)_z \sin\theta_z \cos^2(\theta_x/2)\cos^2(\theta_y/2)A_3\},$$

with the used abbreviations $\theta_x := k_1\Delta x$, $\theta_y := k_2\Delta y$, $\theta_z := k_3\Delta z$, $r(n)_x := \Delta t_n/\Delta x$, $r(n)_y := \Delta t_n/\Delta y$, $r(n)_z := \Delta t_n/\Delta z$, and $\kappa_n := (r(n)_x \sin\theta_x\cos^2(\theta_y/2)\cos^2(\theta_z/2),\; r(n)_y\sin\theta_y$ $\cos^2(\theta_x/2)\cos^2(\theta_z/2),\; r(n)_z\sin\theta_z\cos^2(\theta_x/2)\cos^2(\theta_y/2))$, the amplification matrix Q_n for the n-th time step is given by

$$Q_n := I + \frac{i}{2}E_1(\kappa_n) - \frac{1}{8}E_1^2(\kappa_n). \qquad (14)$$

The matrices A_1, A_2, and A_3 can simultaneously be symmetrized by using matrices consisting of the left and right eigenvectors to a linear combination of A_1, A_2, and A_3, see [3]. The sets of eigenvalues are given by $EV(E_1) :=$

$$\{\Lambda^j\}_{j=1}^5 := \{-\kappa\cdot u, -\kappa\cdot u, -\kappa\cdot u, -\kappa\cdot u - \|\kappa\|\sqrt{\gamma(\gamma-1)e/\rho},$$
$$-\kappa\cdot u + \|\kappa\|\sqrt{\gamma(\gamma-1)e/\rho}\}.$$

Since the eigenvalues and the linearly independent eigenvectors of the matrices $E_1(\kappa_n)$ are already known [3], the eigenvalues of the matrices $I + \frac{i}{2}E_1(\kappa_n) - \frac{1}{8}E_1^2(\kappa_n)$ are known, too. Therefore, the amplification matrices Q_n can be diagonalized [3],

$$P_n^{-1}Q_nP_n = \mathrm{diag}\{1 + \frac{i}{2}\Lambda(n)_l - \frac{1}{8}\Lambda(n)_l^2\} =: \mathrm{diag}\{\mu_l(n)\},$$

where the $\Lambda(n)_l$'s, $l = 1,...,5$, are the eigenvalues of the matrix $E_1(\kappa_n)$.

Since the Euler equations are of hyperbolic type, it follows that the von Neumann condition is necessary and sufficient for stability of the scheme (EFV2a). Therefore, denoting by $\rho(Q_n)$ the spectral radius of the matrix Q_n, the condition $\rho(Q_n) \le 1$, is necessary and sufficient for the stability of the scheme (EFV2a). So, the condition to be satisfied is

$$|\mu_l(n)| = |1 + \frac{i}{2}\Lambda(n)_l - \frac{1}{8}\Lambda(n)_l^2| \le 1, \text{ for } l = 1,...,5, \ \forall n \ge 1,$$

from which one yields $|\Lambda(n)_l| \le 2$, $l = 1,...,5, n = 0,1,2,3,..$ With the notation $|u^n|$, e^n, and ρ^n being the maximum values of the solution on the $n-$th time level, one gets the following

Lemma 2: *The explicit finite volume scheme (EFV2a) is stable, if*

$$\Delta t_n \le \frac{2\Delta x\ \Delta y\ \Delta z}{\Delta y\Delta z|u_1^n\sin\theta_x| + \Delta x\Delta z|u_2^n\sin\theta_y| + \Delta x\Delta y|u_3^n\sin\theta_z| + 2\sqrt{\gamma(\gamma-1)e^n/\rho^n}\Gamma},$$

where the abbreviation

$$\Gamma := \sqrt{\Delta y\Delta z\sin^2\theta_x + \Delta x\Delta z\sin^2\theta_y + \Delta x\Delta y\sin^2\theta_z}$$

has been used.

Note that this estimate is conservative.

Remark 1: For $\Delta x = \Delta y = \Delta z$, the above lemma simplifies to

$$\Delta t_n \le \frac{2\Delta x}{|u^n|_1|\sin\theta_x| + \sqrt{3}\sqrt{\gamma(\gamma-1)e^n/\rho^n}|\sin\theta_x|},$$

where the 1–norm of a vector u^n is defined by $|u^n|_1 := |u_1^n| + |u_2^n| + |u_3^n|$.

In [3] we obtained the same estimate but for a finite volume scheme which was of at most first order accuracy in space. The most favorable advantage with our scheme can be described in a simplified way as follows. Assume we have to solve a boundary value problem on the unit interval $x \in [0,1]$. In order to get an error of order e.g. 10^{-6} one has to use $N \approx 1/10^{-3} = 1000$ gridpoints when using a finite difference method of second–order accuracy. This corresponds to a system of 1000 equations. Note, that for a corresponding problem in two dimensions we would have to solve a system of 10^6 equations. Using a method of fourth–order accuracy we only need $N \approx 1/10^{-6/4} \approx 31$ gridpoints, which correspond to a system of only 31 equations.

Next we study the stability of the scheme EFV2b. The corresponding amplification matrix is given by

$$Q_n := I + \frac{i}{2}E_1(\kappa_n) - \frac{1}{8}E_1^2(\kappa_n) - \frac{i}{48}E_1^3(\kappa_n).$$

However, the amplification matrix Q_n can be diagonalized by $P_n^{-1}Q_nP_n =$

$$\mathrm{diag}\{1 - \frac{1}{8}\Lambda(n)_l^2 + i(\frac{1}{2}\Lambda(n)_l - \frac{1}{48}\Lambda(n)_l^3)\} =: \mathrm{diag}\{\mu_l(n)\}.$$

The condition for stability of the scheme comes out to be

$$|1 - \frac{1}{8}\Lambda(n)_l^2 + i(\frac{1}{2}\Lambda(n)_l - \frac{1}{48}\Lambda(n)_l^3)| \le 1, \text{ for } l = 1,...,5,$$

$\forall n \ge 1$, from which follows that $|\Lambda(n)_l| \le 4$, $l = 1,...,5$, $n = 0,1,2,3,....$.

Therefore, for the largest possible time step we have the following

Lemma 3: *The explicit finite volume scheme (EFV2b) is stable, if*

$$\Delta t_n \le \frac{4\Delta x\ \Delta y\ \Delta z}{\Delta y\Delta z\ |u_1^n\sin\theta_x| + \Delta x\Delta z\ |u_2^n\sin\theta_y| + \Delta x\Delta y\ |u_3^n\sin\theta_z| + \sqrt{\gamma(\gamma-1)e^i/\rho^i}\Gamma},$$

where the abbreviation

$$\Gamma := \sqrt{\Delta y\Delta z\sin^2\theta_x + \Delta x\Delta z\sin^2\theta_y + \Delta x\Delta y\sin^2\theta_z}$$

has been used.

Remark 2: For $\Delta x = \Delta y = \Delta z$, the above lemma simplifies to

$$\Delta t_n \le \frac{4\Delta x}{|u^n|_1|\sin\theta_x| + \sqrt{3}\sqrt{\gamma(\gamma-1)e^n/\rho^n}|\sin n\theta_x|}.$$

4. Future Work

In this proceeding we have given a simple upgrading of the existing finite volume methods in order to obtain second–order accuracy in space. We have started to develop schemes of high–order accuracy (higher than order two) by using overlapping control volumes. These schemes together with actual numerical results obtained for the three–dimensional Euler equations will be presented during the conference.

5. References

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