ON THE EFFECTS OF SHEAR FLOW ON SOUND TRANSMISSION ACCROSS BOUNDARY LAYERS

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Abstract

The transmission of sound accross a boundary layer, e.g. in the atmosphere near the ground, or into the fuselage of an aircraft, cannot be predicted acurately from ray tehory. We obtain the exact solution of the acoustic wave equation in an exponential shear flow. It is shown that there exists a critical level, which acts as an acoustic 'valve', i.e. amplifies outward propagating waves, and attenuates inward propagating sound. The sound fields near a critical level cannot be adequately described by ray theory; the solution of the wave equation in a linear velocity profile (Goldstein & Rice 1973) is not general, and overlooks the existence of the critical level. Yet the critical level absorption may be the physical mechanism whereby sound attenuation in a boundary layer exceeds significantly the predictions of ray theory; a confirmation of this conjecture depends on the detailed computation of the analytical solutions given here.

1. Introduction

Novel propulsion systems like the 'profan' produce significant noise levels in the near field, so that passenger confort in an aircraft cabin depends not only on sound damping in the fuselage wall, but also on attenuation in the boundary layer. Flight tests have shown that sound attenuation in the fuselage boundary layer at high subsonic Mach numbers can exceed the predictions of ray theory by as much as 10 dB. The high attenuation is a welcome effect, but its understanding or explanation seems to escape existing theories. In the present paper we do not use the ray approximation, since it applies only to wavelengths short compared with the lengthscale of flow variation, and this condition may be violated in the boundary layer near the wall. In other words, the adequate modelling of sound transmission accross a boundary layer may require an exact solution of the acoustic wave equation in the presence of shear flow.

There exists in the literature only one exact solution of the acoustic wave equation in a shear flow, viz. for the case of a llinear velocity profile [1]. The latter solution (which we designate GR for brevity) uses an independent variable ξ which is, to within a constant factor, the Doppler shifted frequency $\omega_*=\omega-k\,U(y)$, where ω is the wave frequency, k the horizontal wavenumber, and U(y) the velocity profile, viz. U(y)=ay for a linear shear flow. A change of dependent variable is made which transforms the acoustic wave equation from the second to the third order:

$$(d/d\xi)\left\{\xi^2 e^{-b\xi^2/2}\left[U''-(\xi^2/4+\beta)\,U\right]'\right\}=0. \qquad \text{(A)}$$

The equation (A) may be objectionable since its solution requires 3 boundary conditions, whereas there are only two acoustic boundary conditions.

This difficulty does not arise in GR, because they do not solve the third-order differential equation (A), but only the term in square brackets:

$$U'' - (\xi^2/4 + \beta)\,U=0, \qquad \text{(B)}$$

which is a second order equation, requiring only two boundary conditions. The solution of (B) is known in terms of parabolic cylinder functions, but it should be borne in mind that (B) is a particular case of (A). It is worth noting that (A) has a singularity at $\xi=0$ which is missed out in (B). The singularity $\xi=0$ occurs at the point in the shear flow where the Doppler shifted frequency vanishes $\omega_*=0$, which specifies the location $\omega=k\,U(y_c)$ of the critical level $y=y_c$. A linear shear flow U(y)=ay always has a critical level, at $y_c=\omega/ka$, and this important point is overlooked in GR.

It goes without saying that attempts to model sound transmission accross a boundary layer using the wave equation in an uniform stream [2] omit altogether the existence of a critical level. The critical level [3] appears in the equation [4] for the acoustic pressure in a shear flow:

$$\{\omega-k\,U\}\,p'' + 2kU'p' + (\omega-kU)$$

$$\{(\omega-kU)^2/c^2-k^2\}\,p=0, \tag{1}$$

as the singularity $\omega-kU=0$. Although the linear shear $U(y)=ay$ used in GR is the simplest, and is an acceptable approximation near the wall, the flow velocity diverges as $y \to \infty$, so that matching to an uniform free stream is not possible. Although we can solve (1) exactly for a linear shear flow [5], we do not do so here, because in that case the acoustic pressure cannot be matched to a plane wave in a free stream.

2 - The acoustic critical level in a shear flow

In the present communication we consider sound propagation in an exponential shear flow:

$$U(y) = V(1 - e^{-y/L}), \tag{2}$$

where we can adjust independently the free stream velocity V and shear layer thickness L. This corresponds [6] to the boundary layer with uniform suction at high Reynolds number, viz. the asymptotic suction profile. Although the velocity profile (2) is rather simple, it is preferable to the linear velocity profile, i.e. more suitable for the calculation of acoustic fields, on at least two counts: (i) since there is a free stream velocity, the acoustic field must match itself to a plane wave far from the wall:

$$\lim_{y\to\infty} p(y;k,\omega) \sim e^{\upsilon y}, \tag{3}$$

where υ may be complex; (ii) the vorticity:

$$dV/dy = (V/L)\,e^{-x/L}, \tag{4}$$

is concentrated in the boundary layer, and increases to a maximum V/L at the wall. The sound field is not sinusoidal in the boundary layer, and the main thrust of the present paper is to calculate the exact acoustic field in this vortical region.

The Doppler shifted frequency:

$$\omega_*(y) = \omega-k\,U(y), \tag{5}$$

is given in the case of the exponential shear flow (2) by:

$$\omega_*(y) = \omega-kV + kV\,e^{-y/L}, \tag{6}$$

and varies from the wave frequency $\omega=\omega_*(0)$ at the wall, to a minimum $\omega_*(\infty) = \omega-kV$ in the free stream. Thus two cases arise: (i) if the Doppler shifted frequency is negative in the free stream $\omega<kV$, since it is positive at the wall, it vanishes $\omega_*(y_c)=0$ at a critical level in the boundary layer, located at:

$$y_c= -L\log(1- \omega/kV); \tag{7}$$

(ii) if the Doppler shifted frequency is positive in the free stream $\omega>kV$, then it is positive in the whole flow region, and no critical level exists, viz. y_c is complex in (7) for $\omega>kV$. In the case $\omega=kV$, which is intermediate between (i) and (ii), the critical level would be located in the free stream $y_c=\infty$.

When considering the exact theory of sound propagation in a shear flow, the acoustic pressure p at position x,y and time t, is given by a Fourier integral for time and horizontal coordinate:

$$P(x,y,t) = \iint_{+\infty}^{-\infty} p(y;k,\omega)\,e^{i(kx-\omega t)}\,dk\,d\omega, \tag{8}$$

where $p(y;k,\omega)$ is the pressure spectrum, for a wave of frequency ω, and horizontal wavenumber k, at a distance y from the wall (*Figure 1*). The pressure is in general not a sinusoidal function of y, i.e. it is obtained exactly by solving the second-order differential equation (1), which has the critical level (5) as a singularity at finite distance (7).

We may attempt to interpret the existence of the critical level in terms of ray theory, even *though the ray approximation breaks down* in its vicinity. In the case of ray theory the waves are sinusoidal both in the direction parallel and orthogonal to the wall, and there exists a two-dimensional wave vector $\vec{K}$ with (*Figure 2*) horizontal component:

$$k = K\cos\theta, \tag{9}$$

where θ is the angle of the direction of propagation with the wall. The condition of existence of a critical level $\omega<kV$, states that:

$$\omega/K \equiv u < V\cos\theta, \tag{10}$$

that the phase speed u of sound is smaller than the free stream velocity projected on the direction of propagation. In other words if the free stream is supersonic relative to the phase speed of sound, acoustic signals are received in inverse order of emission, and a critical level exists in the boundary layer. If we define the local Mach cone by its aperture $\cos\alpha=1/M$, where $M \equiv V/u$ is the Mach number relative to the phase velocity of sound, then (10) implies $\cos\theta > \cos\alpha$; thus, critical level exists in the boundary layer if sound rays lie within the Mach cone. If sound rays lie outside the Mach cone there is no critical level, and if they lie on the Mach cone the critical level is in the free stream.

3 - Acoustic 'valve' effect and sound attenuation

In order to understand the effect of the existence of the critical level on the sound field, we return to the exact acoustic theory. The latter is based on the acoustic wave equation (1), for the boundary layer profile (2), viz.:

$$p'' + (2kV/(\omega-kV + kV e^{-y/L}) L^{-1} p' +$$

$$+ \{(\omega-kV + kV e^{-y/L})^2/c^2 - k^2\} p=0, \qquad (11)$$

which is a second-order differential equation, whose coefficients involve exponentials of the distance from the wall. This suggests the change of independent variable:

$$\zeta = e^{-y/L}/(1 - \omega/kV), \qquad (12)$$

which places the free stream $y=\infty$ at the origin $\zeta=0$; the constant factor in (12) is such that the critical level, when it exists, is placed at the point $y=y_c$ unity $\zeta_c=1$. With the change of variable (12), the coefficients of the differential equation (11) become polynomials of the third-degree:

$$(1-\zeta)\,\zeta^2\,\Phi'' + \zeta(1+\zeta)\,\Phi' +$$

$$+ (1-\zeta)\,\{\Lambda^2(1-\zeta)^2 - \Gamma^2\}\,\Phi=0, \qquad (13)$$

where we have introduced the dimensionless Doppler shifted frequency:

$$\Lambda=(\omega-kV)\,L/c, \qquad (14)$$

and the compactness:

$$\Gamma \equiv kL, \qquad (15)$$

and $\Phi(\zeta) = p(y;k,\omega)$.

If we re-write the differential equation (13) in the form:

$$\zeta^2\,\Phi'' + \zeta\,\{(1+\zeta)/(1-\zeta)\}\,\Phi' +$$

$$+ \{\Lambda^2(1-\zeta)^2 - \Gamma^2\}\,\Phi=0, \qquad (16)$$

it is clear that the terms in curly brackets are analytic functions of ζ, in a circle with centre at $\zeta=0$ and radius unit $|\zeta|<1$. Thus $\zeta=0$ is a regular singularity [7] of the differential equation, and within the unit circle in the ζ-plane, a solution exists as a power series:

$$\Phi(\zeta) = \sum_{n=0}^{\infty} A_n\,\zeta^{n+\upsilon} \qquad (17)$$

with coefficients a_n and index υ to be determined by the Frobenius-Fuchs method (see §5). In the absence of a critical level $\omega<kV$, the variable ζ in (12) is negative $\zeta<0$, i.e. (17) is power series with alternating sign, whose sum tends to be small; thus, in the absence of a critical level, acoustic pressure changes accross the boundary layer are small, i.e. there is no significant attenuation or amplification of sound. In the presence of a critical level $\omega<kV$, the variable ζ in (12) is positive $\zeta>0$, and the power series (17) tends to have a large sum, so that in this case there are significant pressure changes accross the boundary layer, implying that: (i) an incoming wave, from the free stream, suffers considerable attenuation, near the critical level, before reaching the wall; (ii) conversely, a wave outward propagating from the wall, is considerably amplified towards the free stream. Thus the critical level in a shear flow acts as an acoustic 'valve', attenuating incoming and amplifying outgoung waves. The attenuation effect (i) for incoming sound in a boundary layer has been observed experimentally, and the converse (ii) amplification effect for outgoing waves should be amenable to experimental verification.

Re-writing the equation (13) in the form:

$$(1-\zeta)^2\,\Phi'' + (1-\zeta)\,(1+1/\zeta)\,\Phi' +$$

$$+ (1-1/\zeta)^2\,\{\Lambda^2(1-\zeta)^2 - \Gamma^2\}\,\Phi=0, \qquad (17)$$

it is clear that the terms in curly brackets are analytic functions of ζ in a circle of centre at $\zeta=1$ and radius unity $|\zeta-1|<1$, and hence the critical layer $\zeta=1$ is a regular singularity. The remaining singularity is the point at infinity $\zeta=\infty$, which corresponds to the origin $\eta=0$ after the change of variables $\eta=1/\zeta$, viz. for:

$$\psi(\eta) = \Phi(1/\eta), \qquad (18)$$

we obtain from (13):

$$\eta^2\,\psi'' + \eta\{(\eta-3)/(\eta-1)\}\,\psi' +$$

$$+ \{\Lambda^2(1-1/\eta)^2 - \Gamma^2\}\,\psi=0. \qquad (19)$$

The second term in curly brackets is singular at the origin, and thus $\eta=0/\zeta=\infty$ is an irregular singularity of the differential equation (19)/(13). Thus the differential equation has one pair of linearly independent solutions in the neighbourhood of each singularity, according to the *Table I*. Since the regions of convergence of each pair of solutions overlap (*Figure 3*), analytic continuation is possible, i.e., any solution is a linear combination of any pair, e.g.

$$p_*(y;k,\omega) = C_+ \, p_+(y;k,\omega) + C_- \, p_-(y;k,\omega), \quad (20)$$

where $C_\pm$ are constants. We start the derivation of the solutions with that about the free stream.

4 - Evanescent, divergent or propagating waves

The solution (17) suggests that we make the change of dependent variable:

$$\Phi(\zeta) = \zeta^\upsilon f(\zeta); \quad (21)$$

if υ is so chosen that $f(0) \neq 0,\infty$ is finite, then the acoustic field in the free stream $y \to \infty$ or $y \to 0$, is specified by $\zeta^\upsilon \sim e^{-\upsilon y/L}$, viz. leading to propagating waves for complex υ, and evanescent or divergent waves for real υ. Substitution of (21) into (16) yields:

$$(1-\zeta)\,\zeta^2 f'' + \zeta\{(1+2\upsilon) + \zeta(1-2\upsilon)\} f' +$$

$$+ \{\Lambda^2(1-\zeta)^2 + (\upsilon^2-\upsilon-\Gamma^2)(1-\zeta) + \upsilon(1+\zeta)\} f=0. (22)$$

If we choose υ so as to cancel the terms in the last curly brackets which are independent of ζ:

$$\Lambda^2 + \upsilon^2 - \Gamma^2 = 0, \quad (23)$$

then the equation (22) can be divided throughout by ζ, viz.:

$$(1-\zeta)\,\zeta f'' + \{(1+2\upsilon) + \zeta(1-2\upsilon)\} f' +$$

$$+ \{(2\upsilon - \Lambda^2(1-\zeta)\,(2-\zeta)\} f=0, \quad (24)$$

we obtain a differential equation whose coefficients are quadratic polynomials, instead of cubic polynomials in (16).

The value of υ is specified by the roots of (23), viz.:

$$\pm\,\upsilon = \sqrt{\Gamma^2-\Lambda^2} = L\sqrt{k^2-(\omega-kV)^2/c^2}, \quad (25)$$

where we have used (24) and (15). The acoustic field in the free stream:

$$p(y;k,\omega) \sim \zeta^\upsilon \sim e^{-\upsilon y/L}, \quad (26)$$

consists of: (i) propagating waves, for complex υ, i.e. $(\omega-kV)^2 > k^2c^2$, i.e. in the frequency ranges:

$$\omega > k(V+c), \quad (27a)$$

which is of interest for any free stream velocity, or:

$$\omega > k(V-c), \quad (27a)$$

which is of interest only for supersonic free stream; (ii) evanescent or divergent waves which correspond to real υ, viz. $(\omega-kV)^2 < k^2c^2$, i.e. in the frequency range:

$$k(V+c) > \omega > k(V-c), \quad (28)$$

which is the complement of (27a,b). We will show next that (26) is the leading term of the acoustic field in the free stream, by obtaining the exact solution of (21) in the neighbourhood of the regular singularity $\zeta=0$; the latter solution is specified by the Frobenius-Fuchs method as a power series:

$$f_\sigma(\zeta) = \zeta^\sigma \sum_{n=0}^{\infty} a_n(\sigma)\,\zeta^n, \quad (29)$$

with coefficients and index to be determined.

Substituting (29) into (22) and equating to zero the coefficient of each power of $\zeta^{n+\sigma}$ we obtain a recurrence formula for the coefficients:

$$(n+\sigma+1)\,(n+\sigma+1+2\upsilon)\,a_{n+1} =$$

$$= \{(n+\sigma)(n+\sigma-2-2\upsilon) +$$

$$+ 2(\Lambda^2-\upsilon)\}\,a_n - 3\Lambda^2\,a_{n-1} + \Lambda^2\,a_{n-2}. \quad (30)$$

Setting $n=-1$, and noting that $a_{-1}=a_{-2}=a_{-3}=0\neq a_0$, we obtain $\sigma(\sigma+2\upsilon)=0$, which is the indicial equation, showing that the index can take two values $\sigma=0, -2\upsilon$, corresponding to particular solutions with leading terms:

$$p_\pm(y;k,\omega) =$$

$$\zeta^\upsilon f_{0,-2\upsilon}(\zeta) \sim \zeta^{\upsilon+\sigma} \sim \zeta^{\pm\upsilon} \sim e^{\pm\upsilon y/L}, \quad (31)$$

in agreement with (25) and (26). The complete series solution (29) in the variable (12) is given by:

$$p_\pm(y;k,\omega) =$$

$$= e^{\pm\upsilon y/L} \sum_{n=0}^{\infty} a_n(0,-2\upsilon)\,(1-\omega/kV)^{-n}\,e^{-ny/L}, \quad (32)$$

where the recurrence formula (30) for the coefficients $a_n(\sigma)$ is applied with $\sigma=0$ for p_+ and $\sigma=-2\upsilon$ for p_-. These two particular solutions are linearly independent, and thus the general solution is a linear combination of them:

$$p(y;k,\omega) = C_+ \, p_+(y;k,\omega) + C_- \, p_-(y;k,\omega), \quad (33)$$

where $C_\pm$ are constants. In the frequency ranges (27a,b) υ is complex, and if we take the root of (25) with $\mathrm{Im}(\upsilon) < 0$, then (31) shows that p_+ is an outward and p_- an inward propagating wave, in the free stream; in the complementary frequency range (28) υ is real and if we choose the root of (25) such that $\upsilon>0$, then (31) shows that p_+ is an evanescent and p_- a divergent wave, in the free stream. Thus if we set $C_-=0$ in (33), we obtain an evanescent acoustic field in the frequency range (28), and an outward propagating sound wave in the remaining frequency ranges (27a,b).

5 - Solutions for all frequency ranges and flow regions

We note in passing that choosing one or the other values of (25) merely interchanges the particular solutions (31), and leaves the general solution (33) unchanged. This solution is valid for $|\zeta|<1$, i.e. in a flow region extending from the free stream $\zeta=0$, $y=\infty$ to the critical level $\zeta=1$, $y=y_c$. The solution (33) converges for $|\zeta|<1$, and thus is limited by the critical level $\zeta=1$. In order to obtain the acoustic field near the critical level, we must expand in powers of the variable:

$$\xi = 1 - \zeta, \tag{34}$$

so that the differential equation (24) is replaced $f(\zeta) \equiv g(\xi)$ by:

$$\xi(1-\xi)\, g'' - \{2-\xi(1-2\upsilon)\}\, g' + \\ + \{2\upsilon - \Lambda^2\, \xi(1+\xi)\}\, g=0, \tag{35}$$

which has a regular singularity at $\xi=0$ or $\zeta=1$. Hence a solution exists by Frobenius-Fuchs method, as a power series:

$$g_\sigma(\zeta) = \xi^\sigma \sum_{n=0}^{\infty} b_n(\sigma)\, \xi^n, \tag{36}$$

whose coefficients satisfy the recurrence formula:

$$(n+\sigma+1)\ (n+\sigma-2)\ b_{n+1} = \{(n+\sigma)\ (n+\sigma-2\upsilon) \\ - 2\upsilon\, b_n + \Lambda^2(b_{n-1} + b_{n-2}), \tag{37}$$

which is obtained by substituting (36) into (35), and equating to zero the coefficients of powers of g.

Setting $n=-1$ in (37) and noting that $b_{-1}=b_{-2}=b_{-3}=0\neq b_0$, we obtain the indicial equation $\sigma(\sigma-3)=0$, showing that the index can take the values $\sigma=0,3$ differing by an integer. Thus one particular solution is:

$$g_3(\zeta) = \sum_{n=0}^{\infty} b_n(3)\,(1-\zeta)^{n+3}, \tag{38}$$

which vanishes at the critical level. The index $\sigma=0$ would lead to $b_3=\infty$ in (37), and because the difference between the indices is an integer; multiplying by σ-3 and letting $\sigma \to 3$ leads to the same particular solution (38) as before. In this case, a second linearly independent particular solution is given [7] by:

$$\bar{g}_3(\xi) = \lim_{\sigma\to 3} \partial g_\sigma(\xi)/\partial\sigma = \\ = \log\xi\, g_3(\xi) + \sum_{n=1}^{\infty} \{\partial b_n(\sigma)/\partial s\}_{\sigma=3}\, \xi^{n+3}. \tag{39}$$

Thus the general solution is:

$$p(y;k,\omega) = C_1\, p_1(y;k\omega) + C_2\, p_2(y;k,\omega), \tag{40}$$

where C_1, C_2 are arbitrary constants multiplying the acoustic fields: (i) vanishing like a cubic:

$$p_1(y;k,\omega) = \\ = \sum_{n=0}^{\infty} b_n(3)\,\{1 - e^{-y/L}/(1-\omega/kV)\}^{n+3}, \tag{41}$$

at the critical level; (ii) with a logarithmic singularity at the critical level:

$$p_2(y;k,\omega) = \log\{1 - e^{-y/L}/(1-\omega/kV)\}\, p_1(y;k,\omega) + \\ + \bar{p}(y;k,\omega), \tag{42}$$

which is dominated by the triple zero in p_1, and the zero of fourth-order in $\bar{p}$:

$$\bar{p}(y;k,\omega) = \sum_{n=1}^{\infty} \{\partial b_n(\sigma)/\partial\sigma\}_{\sigma=3} \\ \{1 - e^{-y/L}/(1-\omega/kV)\}^{n+3}. \tag{43}$$

Thus the acoustic field always vanishes at the critical level, and it may rise to quite dissimilar values at each side.

We have obtained two solutions of the acoustic wave equation in an exponential boundary layer, viz. (33) near the free stream (§4) and (40) near the critical level (§5). The question arises: do these two solutions cover the whole flow region, or do we need also the third solution near the singularity below the wall?

The answer depends on the frequency range in consideration, and the complete frequency spectrum is considered in *Table II*: (i) the range $\omega>2kV$, we have $-1<\zeta<0$ in (12) so that the solution I about the free stream covers the whole flow region $0<y<\infty$; (ii) in the frequency range $kV<\omega<2kV$ the solution I converges for $y>y_1$, where:

$$y_1 = -L \log(\omega/kV - 1), \tag{44}$$

and for $y<y_1$ we need solution II near the critical level; (iii) in the frequency range $kV>\omega>kV/2$, the solution II converges for $y>y_2$, where:

$$y_2 \equiv -L \log\{2 (\omega/kV-1)\} = y_c - L \log 2, \tag{45}$$

and thus covers the region of convergence $y>y_c$ of solution I, but for $y<y_c$ we need the solution III near the third singularity; (iv) in the last frequency range $\omega<kV/2$, the solution II converges over the whole flow region, so we do not need solutions I and III, which converge over disjoint regions $y>y_c$ and $y<y_c$. Over each of the four frequency ranges, we can have either propagating (27a,b) or evanescent/divergent (28) acoustic fields in the free stream, as indicated in *Table III*.

6 - Acoustic fields in the limit of strong vorticity

The Table II shows that in the frequency range $1/2\ kV<\omega<kV$ the solutions I and II, obtained by expansion around the free stream, do not cover the whole flow region, and apply to distances from the wall greater than y_2 in (42). In order to obtain the acoustic field in the region $0<y<y_2$, we need the solution III, which is an expansion about the singularity at infinity $\zeta=\infty$, i.e. below the wall $y=-\infty$ in (12), where the vorticity (4) is infinite. The singularity at infinity for $\zeta=\infty$, corresponds to the origin $\eta=0$ for:

$$\eta = 1/\zeta, \tag{46}$$

and the differential equation (24) transforms to:

$$(\eta-1)\ \eta\ j'' + \{(2\upsilon-3) + \eta(1-2\upsilon)\}\ j' +$$
$$+ \{(2\upsilon-\Lambda^2)/\eta + 3\Lambda^2/\eta^3\}\ j=0, \tag{47}$$

for $j(\eta) = f(\zeta) = f(1/\eta)$. Since (47) is no simpler than (19) for $\psi(\eta)=\eta^{-\upsilon}\, j(\eta)$, we use the latter. The origin $\eta=0$ is an irregular singularity of (19), and thus the solution appears as a series of ascending and descending powers:

$$\psi_\sigma(\eta) = \sum_{n=-\infty}^{+\infty} d_n(\sigma)\eta^{\sigma+n}, \tag{48}$$

where the coefficients satisfy the infinite system of linear homogeneous equations:

$$\sum_{n=-\infty}^{+\infty} D_{n,m}(\sigma)\, d_n(\sigma) = 0\ , \tag{49}$$

involving a four-banded diagonal matrix, whose non-zero elements are:

$$D_{n,n} = (n+\sigma)^2 + \Lambda^2 - \Gamma^2 - 1, \tag{50a}$$

$$D_{n,n+1} = 3\Lambda^2 - (n+\sigma)\ (n+\sigma+4) - \Gamma^2\ , \tag{50b}$$

$$D_{n,n+2} = -\ 3\Lambda^2\ , \tag{50c}$$

$$D_{n,n+3} = -\Lambda^2\ . \tag{50d}$$

Note that (49), unlike (30) and (37), is not a recurrence formula, because there is no finite lower or upper value for n.

A necessary condition for (49) to have non-trivial solution is the vanishing of the infinite determinant:

$$\mathrm{Det}\{D_{n,m}(\sigma)\} = 0; \tag{51}$$

this plays the role of indicial equation, and has two roots σ_1, σ_2 specifying two particular integrals:

$$p_*,\ p_{**}(y;k,\omega) = \psi_{\sigma_1,\sigma_2}((1-\omega/kV)\ e^{y/L}), \tag{52}$$

whose linear combination is the general integral:

$$p(y;k,\omega) = C_*\ p_*(y;k,\omega) + C_{**}\ p_{**}(y;k,\omega), \tag{53}$$

where C_*, C_{**} are constants. For each value of the index (48) may be solved as a infinite non-homogeneous system of equations:

$$\sum_{\substack{n=-\infty \\ n\neq\infty}}^{+\infty} D_{n,m}\ d_n/d_o = -D_{o,m}\ , \tag{54}$$

for the ratio of the coefficients to d_o, where we may put $d_o=1$ by incorporating d_o into C_* or C_{**}. The solution of (48) by means of infinite determinants suggests that it can be transformed to [8] Hill's equation:

$$h''(\theta) + \{\ \sum_{n=0}^{\infty} c_n \cos(2n\theta)\}h(\theta) = 0. \tag{55}$$

This can be done via changes of independent:

$$\eta = \cos^2\theta, \tag{56}$$

and dependent:

$$h(\theta) = e^{4\theta} \sin\theta \, \psi(\cos^2 \theta), \tag{57}$$

variable, which transforms (19) to:

$$h(\theta) + \{\Gamma^2-\Lambda^2 + (3\Lambda^2-\Gamma^2+1) \sec^2\theta - 3\Lambda^2 \sec^4\theta +$$

$$+ \Lambda^2 \sec^6\theta - 4 \sec^2 2\theta +$$

$$+ (4 + \tan\theta + 2 \cot 2\theta)^2\} h(\theta) = 0, \tag{58}$$

where the term in curly brackets has a series of cosines of even argument as in Hill's equation (55).

We conclude our discussion of solutions of the acoustic wave equation in an exponential boundary layer, by showing that it is possible to obtain one particular solution valid in the neighbourhood of the irregular singularity [9], without using infinite determinants. The latter solution appears in the form [10] of a normal integral:

$$\psi(\eta) = \exp\{\chi (1/\eta)\} q(\eta), \tag{59}$$

where, if the function $\chi(1/\eta)$ represents accounts for essential singularity, then $q(\eta)$ should be an ascending power series obtainable by the Frobenius-Fuchs method:

$$q(\eta) = \sum_{n=0}^{\infty} c_n(\theta)\eta^{\sigma+n}. \tag{60}$$

Substitution of (59) into (19) yields:

$$\eta(\eta-1) q'' + \{2 \chi' \eta(\eta-1) + \eta-3\} q' +$$

$$+ \{(\chi'' + \chi'^2) \eta(\eta-1) + (\eta-3) \chi'$$

$$+ \Lambda^2-\Gamma^2 + (\Gamma^2-3\Lambda^2)/\eta$$

$$+ 3\Lambda^2/\eta^2 - \Lambda^2/\eta^3\} q = 0. \tag{61}$$

Choosing the function χ in the form:

$$\chi(\eta) = i\Lambda/\eta^2) - (i\Lambda + 1/2)/\eta, \tag{62}$$

eliminates all singular terms of the second curly brackets in (61), which simplifies to:

$$\eta^2(\eta-1) q'' + 2\eta\{-i\Lambda/\eta + 2i\Lambda-1-i\Lambda\eta\} q'$$

$$+ \{(1/4+3i\Lambda-\Gamma^2) \eta + (3/4-i\Lambda+3\Gamma^2 - 6\Lambda^2\} q = 0. \tag{63}$$

Although the singularity at $\eta=0$ is still irregular, the Frobenius-Fuchs method is partially successful, since (60) leads to a recurrence relation:

$$i\Lambda(\eta+\sigma+1) c_{n+1} =$$

$$= \{3/4-i\Lambda+3\Gamma^2-6\Lambda^2 + (\eta+\sigma) (2i\Lambda-n-\sigma)\} c_n +$$

$$+ \{(n+\sigma-1) (n+\sigma-2-i\Lambda) + 1/4 + 3i\Lambda - \Gamma^2\} c_{n-1}, \tag{64}$$

which for $n=1$ implies the indicial equation $\sigma=0$, of first degree. Thus we obtain from (59), (60) and (62):

$$\psi(\eta) = e^{-\Lambda/\eta} \eta^{5/2-\Lambda} \sum_{n=0}^{\infty} c_n(0)\eta^n, \tag{65}$$

as one solution of the differential equation in the neighbourhood $|\eta|<1$ or $|\zeta|>1$ of the limit of infinite vorticity.

7 - Conclusion

The sound pressure fields calculated by the preceding methods can be normalized to the wall value and plotted in logarithmic scale.

$$Q(X) = \log \{p(y;k,\omega)/p(0;k,\omega)\}, \tag{66}$$

versus dimensionless distance from the wall:

$$0 \le X \equiv y/L \le 5. \tag{67}$$

In the case of evanescent/divergent acoustic fields (66) is real, and in the case of propagating fields:

$$Q(X) = \log |p(y;k,\omega)/p(0;k,\omega)| +$$

$$+ i \{\arg p(y;k,\omega) - \arg p(0;k,\omega)\}, \tag{68}$$

the real part is the logarithm of the ratio of amplitudes, and the imaginary part is the difference of phases. The acoustic field is affected by three dimensionless parameters: (i) the frequency:

$$\Omega \equiv \omega/kV; \tag{69}$$

(ii) the Mach number of the free stream:

$$M = V/c; \tag{70}$$

(iii) the compactness (15). The plots show the degree of attenuation of incoming and amplification of outgoing waves. There is a large number of combinations needed to cover all cases in Tables II and III, and their exploration is continuing at present.

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TABLE I

Solutions of the acoustic wave equation in an exponential shear flow

Singularity	$\zeta=0$	$\sigma=1$	$\zeta=\infty$
Location	$y=\infty$	$y=y_c$	$y=-\infty$
Meaning	free stream	critical level	below wall
Solution	p_+, p_-	p_1, p_2	p_*, p_{**}
in powers of	ζ	$1-\zeta$	$1/\zeta$
valid for	$-1<\zeta<+1$	$0<\zeta<2$	$\zeta<-1$ or $\zeta>+1$

TABLE II

Regions of convergence of solutions for four frequency ranges

Solution	I	II	III
Acoustic pressure	p_+, p_-	p_1, p_2	p_*, p_{**}
Spectral range:			
$\omega>2kV$	$0<y<\infty$	—	—
$2kV>\omega>kV$	$y>y_1$	—	$y<y_c$
$kV>\omega>\frac{1}{2}kV$	$y>y_c$	$y>y_2$	$y<y_c$
$\omega<\frac{1}{2}kV$	$y>y_c$	$0<y<\infty$	$y<y_c$

TABLE III

Conditions for evanescent/divergent or propagating acoustic fields in the free stream

Frequency range	Acoustic field in free stream	
	Evanescent/divergent	Propagating
$\omega>2kV$	$c>V$	—
$2kV>\omega>kV$	$V<c$	$V>c$
$kV>\omega>\frac{1}{2}kV$	—	$V>2c$
$\omega>\frac{1}{2}kV$	$V<c$	—

Legends for the figures

Figure 1 — Sound wave of frequency ω and horizontal wavenumber k in a boundary layer.

Figure 2 — Position of sound rays relative to the Mach cone, as criterion for the existence of a critical level.

Figure 3 — Regions of convergence of the three pairs of solutions of the acoustic wave equation in an exponential shear flow.

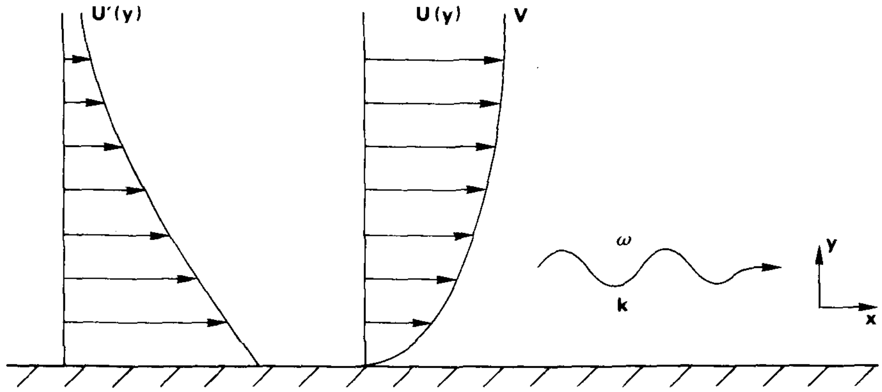

FIGURE 1

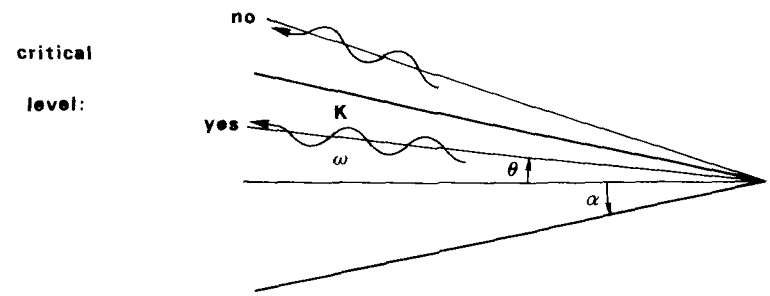

FIGURE 2

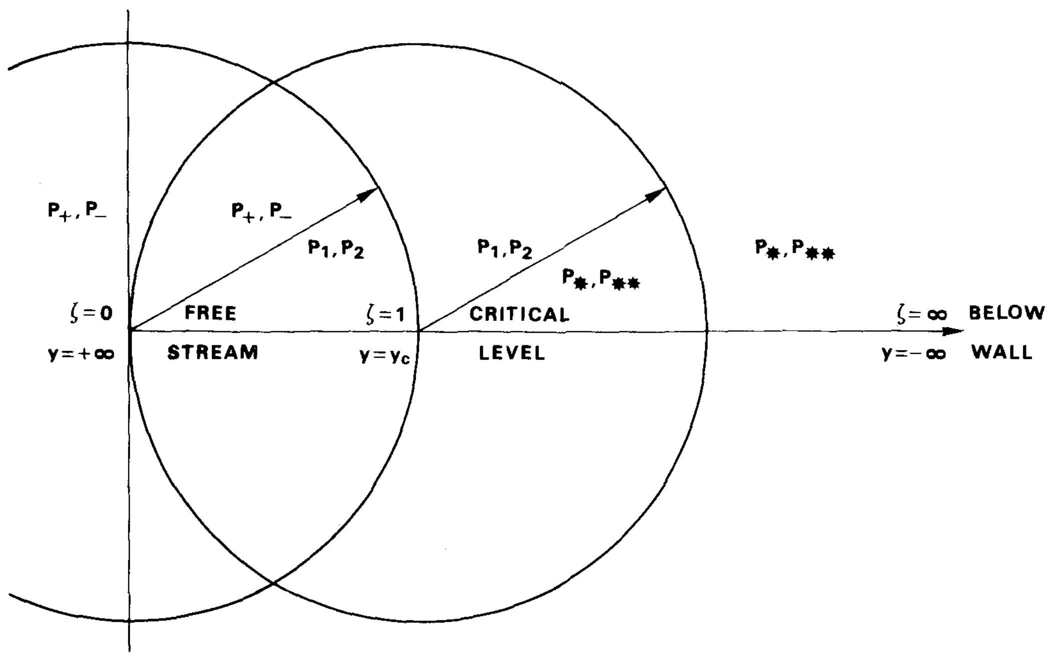

FIGURE 3