THREE DIMENSIONAL DYNAMIC AND STATIC STABILITY OF VORTEX SYSTEMS AND THEIR OBSERVABILITY AND CONTROLLABILITY

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Abstract

A general theory for the three-dimensional long-wave linear stability of a general system of straight potential vortices immersed in an incompressible inviscid potential external flow field and in the vicinity of solid surfaces, is presented. The flow field equations are governed by the streak-line equations for the vortices and by the satisfaction of the tangency boundary condition on the solid surfaces. In their basic state, the vortices are assumed to be straight parallel filaments, near infinite two-dimensional steady surfaces that are represented by a potential source surface distribution. Small three-dimensional long wave perturbations, constructed of two orthogonal waves, are imposed on the vortex filaments, concurrently with fluctuations in the intensity of the source distribution, and with known unsteady perturbations along the shape of the solid surfaces. The singularity in the calculation of the self-induced velocity of a curved vortex line is treated by the "Cut-off distance" model. A linearization of the flow-field equations followed by a Fourier transformation, results in a system of zero-order equations, for the two-dimensional dynamics of the vortices in their basic state, and first-order equations, that are constructed of Fredholm integral equations of the second kind for the source fluctuations distribution, and of first-order linear differential equations that describe the time-history of the vortex perturbations. When the basic state is steady, or quasi-steady, the first-order equations result in an eigenvalue problem. Analysis shows that the perturbations development in time can never be asymptotically stable. The outcome of the model are the new concepts of "Generalized Damping of a Vortex System" that govern its dynamic stability, and "Rigidity" that governs the static divergence of a vortex system, which was never considered before. Also, a control approach is presented, together with the new concepts of "Controllability" and "Observability", that lead to the challenging idea of "Active Control" of vortex stability.

1. Introduction

Modern flight vehicles, and especially highly maneuverable modern fighter aircraft and missiles, generate highly complex, three dimensional flow fields that contain concentrated vortices. The vortical flows strongly affect the aerodynamic characteristics of the vehicles. In the case of military maneuverable vehicles, as well as in several novel designs of general aviation vehicles, the vortices contribute to the vehicle performance and extend its operational flight envelope to super maneuverability and increased agility[1]. In more conventional aircraft, such as large transports, the vortical wakes induce drag on the aircraft and can endanger small vehicles that inadvertently happen to cross these wakes[2].

The utilization of the vortex fields to enhance flight-vehicle performance is strongly limited by vortex instabilities. These are usually accompanied by strong unsteady phenomena that can endanger the stability of the flight vehicle itself[1]. On the other hand, destabilization of the tip vortices of heavy transport aircraft could reduce their drag penalty, and could alleviate the air-traffic controllers' problem of keeping a small craft out of the wakes of the larger ones[2]. The ability to predict the flow-field conditions that destabilize the vortices, and even better, the ability to control this stability would, therefore, be beneficial for future utilization of vortex flows in the aerodynamic design of flight vehicles. Future vortex-stability control (either stabilization or destabilization), could greatly enhance the aerodynamic performance of flight vehicles.

The investigation of the stability of vortex systems has challenged researchers in fluid mechanics for over a hundred years. Among the best known works on this subject are the classic analyses of Lord Kelvin[3], J. J. Thomson[4], von- Kármán[5], Föppl[6], Havelock[7] and of Lamb[8]. Most classic approaches to vortex stability were limited to particular cases of discrete straight vortices in a two-dimensional, steady or quasi-steady flow, under two-dimensional perturbations only.

Of the small number of three-dimensional long-wave stability analyses known to date, none is of a general character. They all deal with specific problems of self-preserving configurations. These include: a single discrete straight line vortex (Crow[9]), a pair of parallel identical counter-rotating vortices, or trailing wing-tip vortices (Crow[9]), a pair of parallel identical co-rotating vortices (Jimenez[10]), the von-Kármán symmetric and asymmetric vortex streets (Robinson and Saffman[11]), the Föppl vortices (Widnall[12]), and a discrete line vortex running along the axis of a straight circular tube (Rusak and Seginer[13]). The classic approach is to study the linear stability of the perturbed vortex system, by determining the character of the eigenvalues of the linearized equations that describe the development of the perturbations. However, this approach had never been extended to a general theory for the three-dimensional linear stability of vortex systems. Also all of the above-mentioned analyses, be they two- or three- dimensional, strove to identify the conditions under which a partic-

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ular vortex system became unstable. They all identify the limits of stability only, without explaining physical reasons or mechanisms, that were responsible for the stability, or instability, of a vortex system. Furthermore, all the known stability analyses dealt with dynamic instabilities only, by examining the character of the eigenvalues of the given problem. In none was the possibility of a *static instability (divergence) ever found* or even considered.

Also, despite the large number of analyses of two-dimensional stability problems[3-8] and several three- dimensional studies [9-13] the classic linear control approach that could predict the observability and the controllability of vortex systems has never been proposed. Such a prediction could lead to the measurement of the development of the perturbations in vortex systems and to their control. *Controlling vortex instabilities could open* the gate to a new technology of stabilizing or destabilizing vortices at will.

Presented in this paper is the general theory of Rusak[14] for the calculation of the three-dimensional long-wave linear stability of systems of straight potential vortices immersed in an external inviscid and incompressible flow field and in the vicinity of solid surfaces (Section 2, 3). To the best of the authors knowledge, *this theory is the first of its kind. The outcome of this* general model are new ideas that were recently presented by the authors of this paper, on the dynamic and static stability of vortex systems and the parameters that govern it (see Sections 4, 5 and Rusak and Seginer[15]), and on the control possibilities of vortex sys- (*see* Section 6 and Rusak and Seginer[16]).

2. Mathematical Model

The model developed by Rusak[14] considers a configuration of N concentrated potential vortex filaments with different, but known, circulations $\Gamma_n (n = 1, \ldots, N)$, in an inviscid and incompressible, potential external flow field. The velocity- potential field (ϕ) of the flow, $\mathbf{u} \equiv \nabla\phi$, is described by the Laplace equation,

$$\nabla^2\phi = 0 \qquad (1)$$

The solution of Eq. (1) is governed by:

(i) The streak-line equations that describe each of the N *vortex lines* $\mathbf{R}_n(S_n, t)$,

$$\frac{D\mathbf{R}_n}{Dt} = \mathbf{u}_n(S_n, t) \qquad (n = 1, \ldots, N) \qquad (2)$$

where $\mathbf{R}_n(S_n, t)$ is the position vector to a point S_n on the n-th vortex filament at time t, and $\mathbf{u}_n(S_n, t)$ is the *velocity* induced by the flow field on this point.

(ii) When solid but flexible surfaces, given by their time-dependent shapes $F_R(\mathbf{R}_R, t) = 0$, also are presented in the flow field (where $\mathbf{R}_R$ is the position vector to a point on the surface), the flow-tangency boundary condition has to be satisfied at each point $\mathbf{R}_R$ and time t,

$$\frac{DF_R}{Dt} \equiv \frac{\partial F_R}{\partial t} + \mathbf{u}_R \cdot \nabla F_R = 0 \quad on\ F_R = 0 \qquad (3)$$

where $\mathbf{u}_R$ is the velocity induced by the flow field on *any point on the surfaces*, and ∇F_R is a vector normal to the surface $F_R = 0$, pointing into the flow field.

In order to satisfy the tangency condition (3), the solid surfaces $F_R = 0$ are represented by a potential source distribution, Q, over them where,

$$Q = Q(\mathbf{R}_R, t) \qquad (4)$$

(iii) For unbounded flow fields the velocity perturbations vanish at large distances r from the vortices and the solid surfaces like $\frac{1}{r^2}$ and,

$$\nabla\phi \rightarrow \mathbf{U}_\infty \ \text{as}\ r \rightarrow \infty \qquad (5)$$

where $\mathbf{U}_\infty$ is a given flow velocity vector at infinite.

(iv) Initial conditions that describe the vortex lines at time $t = t_0$ are also given for every n by, $\mathbf{R}_n(S_n, t_0)$ $(n = 1, \ldots, N)$.

The total velocity $\mathbf{u}_p$ at a point $\mathbf{R}_p$ in the flow field is given by the sum,

$$\mathbf{u}_p = \mathbf{U} + \mathbf{u}_p^V + \mathbf{u}_p^S \qquad (6)$$

where,

(i) $\mathbf{U}$ is a given external incompressible and inviscid, potential flow field,

$$\nabla \cdot \mathbf{U} = 0, \qquad \nabla \times \mathbf{U} = 0 \qquad (7)$$

(ii) $\mathbf{u}_p^V$ is the velocity induced by all the vortex lines on a point $\mathbf{R}_p$ and is given by the *Biot-Savart law* (Batchelor[17]),

$$\mathbf{u}_p^V = \sum_{m=1}^{N} \frac{\Gamma'_m}{4\pi} \int_{\mathbf{R}'_m} \frac{\mathbf{R}'_{mp} \times d\mathbf{R}'_m}{|\mathbf{R}'_{mp}|^3} \qquad (8)$$

in terms of the relative positions $\mathbf{R}'_{mp} \equiv \mathbf{R}'_m - \mathbf{R}_p$, length elements $d\mathbf{R}'_m$, and circulation strengths Γ'_m. The primes are used here to designate points on a vortex filament and terms related to them when $p = n$.

(iii) $\mathbf{u}_p^S$ is the velocity induced by the source distribution Q on a point $\mathbf{R}_p$ and is given by (Batchelor[17]),

$$\mathbf{u}_p^S = -\iint_{F'_R} \frac{Q'\mathbf{R}'_{Rp} dF'_R}{4\pi|\mathbf{R}'_{Rp}|^3} \qquad (9)$$

in terms of the relative position $\mathbf{R}'_{Rp} = \mathbf{R}'_R - \mathbf{R}'_p$, the area of the surface element of dF'_R and the source strength Q' at a point $\mathbf{R}'_R$. The primes are used here to designate points lying on the surfaces $F_R = 0$ and quantities related to them when $p = R$.

All three components of the velocity $\mathbf{u}_p$ satisfy Eqs. (1)-(5) identically.

Using Eq. (8) to calculate the velocity ($\mathbf{u}_n^V$), raises a difficulty with the self-induced term (when $m = n$). For a crooked vortex line it results in a logarithmic singularity of the self-induced velocity when $|\mathbf{R}'_{nn}| = 0$ (Batchelor[17]). This is a nonphysical singularity because the physical vortex has a finite rotational core where the velocities are bounded. In order to relax the singularity in the self-induction integral, the theory presented here uses the "cutoff-distance" model. The idea is to cut the integral off at an arc-length of ($\pm d$) around the point $|\mathbf{R}'_{nn}| = 0$. This idea has been used to calculate the self-induced velocity in other vortex stability analyses[9-13]. Widnall[18] proposed an analytical expression for estimating the cutoff-distance (d) as a function of the rotational core characteristics, where (d) is proportional to the diameter (c) of the vortex core. Widnall's expression is valid only when the wavelength and radius of curvature of the perturbations in the vortex line are larger than the diameter of the core by an order of magnitude, and when the vortex core is slender with respect to a characteristic distance (ℓ) between the vortices, $(c/\ell)^2 \ll 1$ (Moore and Saffman[19]).

The system of Eqs. (1) through (9) together with the "cutoff- distance" model of the self-induced velocity, are used to analyze the motion and stability of the vortex filaments under the interaction between the vortices, the interaction with the solid surfaces, and the influence of the external flow field. Although Eq. (1) (Laplace Eq.) is linear, the problem defined by Eqs. (1) $\div$ (9) is mathematically non-linear because of the unknown spatial trajectories of the vortex filaments that has to be determined as part of the solution.

3. Linearized Problem

3.1 Assumptions and Basic Equations

An orthogonal and inertial cartesian coordinate system (x, y, z) is assumed with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ respectively. The N vortices are assumed to be infinite filaments, each along its x_n-axis (which is parallel to the x-axis), and described by,

$$\mathbf{R}_n = \mathbf{e}_x x_n + \mathbf{e}_y(y_{0n} + y_n) + \mathbf{e}_z(z_{0n} + z_n) \qquad (n = 1, \ldots, N) \tag{10}$$

where x_n is the Lagrangian variable, $x_n \equiv S_n$ that goes from $(-\infty)$ to $(+\infty)$, (y_{0n}, z_{0n}) are the basic state position coordinates of the n-th vortex that depend on time (t) only, $y_{0n} = y_{0n}(t)$, $z_{0n} = z_{0n}(t)$, and (y_n, z_n) are two orthogonal displacement waves in the y and z directions, respectively, that are imposed on each of the vortex filaments and that depend on position (x_n) along the n-th vortex and on time (t), $y_n = y_n(x_n, t)$, $z_n = z_n(x_n, t)$. The initial conditions at time t_0 are assumed to be given for every n by,

$$y_{0n}(t_0),\ z_{0n}(t_0),\ y_n(x_n, t_0),\ z_n(x_n, t_0) \tag{11}$$

The solid surfaces $F_R = 0$ are defined as infinite surfaces along an x_R-axis (that is also parallel to the x-axis), and given at each point $\mathbf{R}_R = \mathbf{e}_x x_R + \mathbf{e}_y y_R + \mathbf{e}_z z_R$ by,

$$F_R(x_R, y_R, z_R, t) = f_R(y_R, z_R) - R_1(x_R, t) = 0 \tag{12}$$

where x_R goes from $(-\infty)$ to $(+\infty)$, $f_R(y_R, z_R) = 0$ is the basic state form of surfaces $f_R = 0$, defined as infinite two-dimensional surfaces, and $R_1(x_1, t)$ is an a-priori known unsteady lengthwise perturbation to the shape of the surfaces $f_R = 0$. The functions f_R and R_1 are given as non-dimensional functions.

The characteristic length (ℓ) of the problem is defined as the minimal distance in the basic state between the vortices or between the vortices and the surfaces $f_R = 0$,

$$\ell = \min_t \left\{ \begin{array}{l} \min\limits_{\substack{m,n \\ m \neq n}} \sqrt{(y_{0m} - y_{0n})^2 + (z_{0m} - z_{0n})^2}, \\ \min\limits_{R,n} \sqrt{(y_R - y_{0n})^2 + (z_R - z_{0n})^2} \end{array} \right\} \tag{13}$$

All the perturbations are characterized by sufficiently small amplitudes and slopes compared with the characteristic length (ℓ) of the problem, for every n and (x_n, t),

$$\left(\frac{y_n}{\ell}\right)^2 \ll 1, \quad \left(\frac{z_n}{\ell}\right)^2 \ll 1, \quad \left(\frac{\partial y_n}{\partial x_n}\right)^2 \ll 1, \quad \left(\frac{\partial z_n}{\partial x_n}\right)^2 \ll 1 \tag{14a}$$

and for every (x_R, t),

$$R_1^2 \ll 1, \qquad \ell^2 \left(\frac{\partial R_1}{\partial x_R}\right)^2 \ll 1 \tag{14b}$$

Each point on the surfaces $F_R = 0$ is assumed to be given at a specific cross-section x_R by,

$$y_R = y_{R0} + \delta y_R, \qquad z_R = z_{R0} + \delta z_R \tag{15}$$

where (y_{R0}, z_{R0}) are the (y, z) coordinates of a point on the basic- state surfaces, $f_R(y_{R0}, z_{R0}) = 0$, and $(\delta y_R, \delta z_R)$ are small displacements that depend on $R_1(x_R, t)$. The source distribution Q is also assumed to be constructed of a basic state term Q_0 that depends on (y_{R0}, z_{R0}, t), and a fluctuation Q_1 that depends on (x_R, y_R, z_R, t),

$$Q = Q_0(y_{R0}, z_{R0}, t) + Q_1(x_R, y_R, z_R, t) \tag{16a}$$

where for every n and $(\mathbf{R}_R, t)$,

$$\left(\frac{Q_1}{\Gamma_n / 2\pi\ell}\right)^2 \ll 1 \tag{16b}$$

The external flow field $\mathbf{U}$ is described by,

$$\mathbf{U} = \mathbf{e}_x U + \mathbf{e}_y V(y, z) + \mathbf{e}_z W(y, z) \tag{17}$$

where U is a uniform axial velocity and V, W are steady cross-flow velocities in the y and z directions. This is in principle a quasi two-dimensional flow, but despite the dynamic similarity it can be shown that under certain conditions the U component has a major influence on the static stability of the vortex system (For details see Rusak and Seginer[15] and Section 5 below).

Substituting Eqs. (6)(10) in the streak lines equations (Eq. (2)) results for every n and (x_n, t) in,

$$\frac{dy_{on}}{dt}+\frac{\partial y_n}{\partial t}+[U+(\mathbf{u}_n^V)_x+(\mathbf{u}_n^S)_x]\frac{\partial y_n}{\partial x_n}=$$
$$V(y_{0n}+y_n,z_{0n}+z_n)+(\mathbf{u}_n^V)_y+(\mathbf{u}_n^S)_y$$
$$\frac{dz_{on}}{dt}+\frac{\partial z_n}{\partial t}+[U+(\mathbf{u}_n^V)_x+(\mathbf{u}_n^S)_x]\frac{\partial z_n}{\partial x_n}= \tag{18}$$
$$W(y_{0n}+y_n,z_{0n}+z_n)+(\mathbf{u}_n^V)_z+(\mathbf{u}_n^S)_z$$

Substituting Eqs. (6)(12) in the flow-tangency boundary condition (Eq. (3)) results for every $(\mathbf{R}_R,t)$ in,

$$-\frac{\partial R_1}{\partial t}+[U+(\mathbf{u}_R^V)_x+(\mathbf{u}_R^S)_x]F_{Rx}+$$
$$+[V(y_R,z_R)+(\mathbf{u}_R^V)_y+(\mathbf{u}_R^S)_y]F_{Ry} \tag{19}$$
$$+[W(y_R,z_R)+(\mathbf{u}_R^V)_z+(\mathbf{u}_R^S)_z]F_{Rz}=0$$

where F_{Rx},F_{Ry},F_{Rz} are the (x,y,z) components of vector ∇F_R.

Eqs. (18)(19) together with Eqs. (8)(9)(10)(12)(15) initial conditions (Eqs. (11)) and the "cutoff-distance" model constitute a mathematical non-linear problem for the dynamics and stability of the vortex system. Following the assumptions in Eqs. (14)(16b) this problem can be linearized about the basic unperturbed state where all perturbations vanish, for every $n: y_n\equiv z_n\equiv 0$ and for every $\mathbf{R}_R$: $Q_1\equiv R_1\equiv 0$. In the basic state the vortices are straight parallel filaments and the surfaces are infinite two-dimensional surfaces. In the next sections the various velocity components in Eqs. (18)(19) will be calculated in their linearized form, and will be substituted back to obtain a linearized model.

3.2 Linearized components of velocities

3.2.1 Velocity induced by all the vortices on a specific vortex

Substituting $p=n$ in Eq. (8) defines the velocity $\mathbf{u}_n^V$ induced by all the vortices in the flow field on a point $\mathbf{R}_n$ on the n-th vortex. From Eqs. (10),

$$\mathbf{R}'_{mn}=\mathbf{e}_x(x'_m-x_m)+\mathbf{e}_y(y'_{0m}-y_{0n}+y'_m-y_n)$$
$$+\mathbf{e}_z(z'_{0m}-z_{0n}+z'_m-z_n)$$
$$d\mathbf{R}'_m=\left(\mathbf{e}_x+\mathbf{e}_y\frac{\partial y'_m}{\partial x'_m}+\mathbf{e}_z\frac{\partial z'_m}{\partial x'_m}\right)dx'_m \tag{20}$$

The integrals in Eq. (8) along the vortex lines are performed along each (x'_m) from $(-\infty)$ to $(+\infty)$. Linearization of the terms in the integrand in Eq. (8) for $p=n$ gives,

$$|\mathbf{R}_{mn}|^{-3}=\frac{1}{R^3_{0mn}} \tag{21a}$$
$$-\frac{3}{R^5_{0mn}}[(y'_{0m}-y_{0n})(y'_m-y_n)+(z'_{0m}-z_{0n})(z'_m-z_n)]$$

$$\mathbf{R}'_{mn}\times d\mathbf{R}'_m=\mathbf{e}_x\left[(y'_{0m}-y_{0n})\frac{\partial z'_m}{\partial x'_m}-(z'_{0m}-z_{0n})\frac{\partial y'_m}{\partial x'_m}\right]dx'_m$$
$$+\mathbf{e}_y\left[(z'_{0m}-z_{0n}+z'_m-z_n)-(x'_m-x_n)\frac{\partial z'_m}{\partial x'_m}\right]dx'_m \tag{21b}$$
$$-\mathbf{e}_z\left[(y'_{0m}-y_{0n}+y'_m-y_n)-(x'_m-x_n)\frac{\partial y'_m}{\partial x'_m}\right]dx'_m$$

where $R^2_{0mn}=(x'_m-x_n)^2+\ell^2_{mn}$ and $\ell^2_{mn}=(y'_{0m}-y_{0n})^2+(z'_{0m}-z_{0n})^2$. Substituting Eqs. (21) in Eq. (8) (for $p=n$) and using the following equations,

$$\sum_{m=1}^{N}\frac{\Gamma_m}{4\pi}\int_{-\infty}^{+\infty}\frac{z'_{0m}-z_{0n}}{R^3_{0mn}}dx'_m=\sum_{m=1}^{N}\frac{\Gamma_m}{2\pi}G_{mn}\equiv v^V_{n0},$$
$$\sum_{m=1}^{N}\frac{\Gamma_m}{4\pi}\int_{-\infty}^{+\infty}\frac{y'_{0m}-y_{0n}}{R^3_{0mn}}dx'_m=\sum_{m=1}^{N}\frac{\Gamma_m}{2\pi}H_{mn}\equiv w^V_{n0}, \tag{22a}$$
$$H_{mn}\equiv\frac{y'_{0m}-y_{0n}}{\ell^2_{mn}}\quad G_{mn}\equiv\frac{z'_{0m}-z_{0n}}{\ell^2_{mn}}$$

$$\int_{-\infty}^{+\infty}\frac{y'_m-y_n}{R^5_{0mn}}dx'_m=\frac{1}{3\ell^2_{mn}}\int_{-\infty}^{+\infty}\frac{2(y'_m-y_n)-(x'_m-x_n)\frac{\partial y'_m}{\partial x'_m}}{R^3_{0mn}}dx'_m$$
$$\int_{-\infty}^{+\infty}\frac{z'_m-z_n}{R^5_{0mn}}dx'_m=\frac{1}{3\ell^2_{mn}}\int_{-\infty}^{+\infty}\frac{2(z'_m-z_n)-(x'_m-x_n)\frac{\partial z'_m}{\partial x'_m}}{R^3_{0mn}}dx'_m \tag{22b}$$

results in the linearized form of the velocity $\mathbf{u}_n^V$

$$\mathbf{u}_n^V=\mathbf{e}_y v^V_{n0}-\mathbf{e}_z w^V_{n0}+\frac{\Gamma_n}{4\pi}(\mathbf{e}_y\bar{v}^V_{nn}-\mathbf{e}_z\bar{w}^V_{nn})$$
$$+\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m\ell^2_{mn}}{4\pi}(\mathbf{e}_x\bar{u}^V_{mn}+\mathbf{e}_y\bar{v}^V_{mn}-\mathbf{e}_z\bar{w}^V_{mn}) \tag{23}$$

where

$$\bar{w}^V_{nn}=\int_{-\infty}^{+\infty}\frac{y'_n-y_n-(x'_n-x_n)\frac{\partial y'_n}{\partial x'_n}}{|x'_n-x_n|^3}dx'_n, \tag{24a}$$

$$\bar{v}^V_{nn}=\int_{-\infty}^{+\infty}\frac{z'_n-z_n-(x'_n-x_n)\frac{\partial z'_n}{\partial x'_n}}{|x'_n-x_n|^3}dx'_n$$
$$\bar{u}^V_{mn}=\int_{-\infty}^{+\infty}\left(H_{mn}\frac{\partial z'_m}{\partial x'_m}-G_{mn}\frac{\partial y'_m}{\partial x_m}\right)\frac{dx'_m}{R^3_{0mn}} \tag{24b}$$
$$\bar{v}^V_{mn}=\int_{-\infty}^{+\infty}\left\{H^2_{mn}\left[z'_m-z_n-(x'_m-x_n)\frac{\partial z'_m}{\partial x'_m}\right]-G^2_{mn}(z'_m-z_n)\right.$$
$$\left.-G_{mn}H_{mn}\left[2(y'_m-y_n)-(x'_m-x_n)\frac{\partial y'_m}{\partial x'_m}\right]\right\}\frac{dx'_m}{R^3_{0mn}}$$

$$\bar{w}^V_{mn} = \int_{-\infty}^{+\infty} \left\{ G^2_{mn} \left[y'_m - y_n - (x'_m - x_n)\frac{\partial y'_m}{\partial x'_m} \right] - H^2_{mn}(y'_m - y_n) \right. \tag{24c}$$

$$\left. - G_{mn}H_{mn}\left[2(z'_m - z_n) - (x'_m - x_n)\frac{\partial z'_m}{\partial x'_m} \right] \right\} \frac{dx'_m}{R^3_{0mn}}$$

The velocity $\mathbf{u}^V_n$ induced by all the vortices on a point $\mathbf{R}_n$ on the n-th vortex is (Eq. (23)) constructed of basic state terms (v^V_{n0}, w^V_{n0}) that are the two-dimensional velocities induced by the vortices in their basic-state position, of the self-induced perturbation components $(\bar{v}^V_{nn}, \bar{w}^V_{nn})$ and of the perturbation velocities $(\bar{u}^V_{mn}, \bar{v}^V_{mn}, \bar{w}^V_{mn})$ induced by all the other perturbed vortices on n-th vortex.

As is mentioned in Section 2 the self-induced components $\bar{v}^V_{nn}, \bar{w}^V_{nn}$ (Eqs. (24a)) are calculated by the "cutoff distance" model in the form,

$$\int_{-\infty}^{+\infty} \frac{f(x'_n)dx'_n}{|x'_n - x_n|^3} \approx \int_{-\infty}^{-d_n} \frac{f(x'_n)dx'_n}{|x'_n - x_n|^3} + \int_{+d_n}^{+\infty} \frac{f(x'_n)dx'_n}{|x'_n - x_n|^3} \tag{25}$$

The cutoff-distance (d_n) in these integrals is given for every n as a small fraction C^*_n of the characteristic length (ℓ), $(C^{*2}_n \ll 1)$,

$$d_n = C^*_n \ell \tag{26a}$$

C^*_n is defined as the "core diameter" parameter of the n-th vortex and can be described by Widnall's[18] expressions,

$$C^*_n = \frac{c_n/2}{\ell} \exp\left[\frac{1}{2} - ln2 - A_n + C_n \right] \tag{26b}$$

$$A_n = \lim_{2r/c_n \to \infty} \left[\int_0^{2r/c_n} \bar{r}V^2_{0n} d\bar{r} - ln\frac{2r}{c_n} \right], C_n = \frac{8}{c^2_n}\int_0^{\infty} \bar{r}W^2_{0n} d\bar{r}$$

$$V_{0n} = \frac{V_{\theta n}(\bar{r})}{\Gamma_n/\pi c_n}, \qquad W_{0n} = \frac{V_{xn}(\bar{r})}{\Gamma_n/\pi c_n} \tag{26c}$$

where c_n is the core diameter of the n-th vortex, r is the radial distance from the vortex axis, $\bar{r} = 2r/c_n$, and V_{0n} and W_{0n} are the dimensionless circumferential and axial velocity distributions $(V_{\theta n}, V_{xn})$ respectively, in the n-th vortex slender, rotational core. The use of the "cutoff-distance" model limits the present theory to slender vortices where $(c_n/\ell)^2 \ll 1$ or $C^{*2}_n \ll 1$, and to long-wave perturbations where the wave number (k) of a sinusoidal perturbation is limited by $(k\ell)^2 \ll \frac{1}{C^{*2}_n}$.

The "cutoff-distance" model gives a simplified approximation to the self-induced velocity of slender vortices, without the need to calculate the details of the flow in the vortices core. In the present theory the functional formulation of the "cutoff-distance" model is the same for every vortex filament and in both the y and z directions, and is not affected by the presence of the solid surfaces. The influence of changing the "cutoff-distance" parameter C^*_n was examined in many examples (Rusak[14]), and it was found the basic character of the stability solutions are not significantly influenced when $C^*_n < 0.2$ (see also Saffman and Baker[20]).

3.2.2 Velocity induced by all the vortices on a point in the flow field and on the solid surfaces

Eq. (8) defines the velocity $\mathbf{u}^V_p$ induced by all the vortices on a point $\mathbf{R}_p$ in the flow field. Application of the same derivation of Eqs. (20) to (24) results in a similar equation for the linearized form of $\mathbf{u}^V_p$,

$$\mathbf{u}^V_p = \mathbf{e}_y v^V_{p0} - \mathbf{e}_z w^V_{p0} + \sum_{m=1}^{N} \frac{\Gamma_m \ell^2_{mp}}{4\pi}(\mathbf{e}_x \bar{u}^V_{mp} + \mathbf{e}_v \bar{v}^V_{mp} - \mathbf{e}_z \bar{w}^V_{mp}) \tag{27}$$

where the various velocity components in Eq. (27) are found by substituting $n = p$ and $y_n = z_n = 0$ in Eqs. (22a) (24b)(24c). The velocity $\mathbf{u}^V_p$ is also constructed of basic state terms and perturbations components.

Substituting $p = R$ in Eq. (27) gives the linearized form of the velocity $\mathbf{u}^V_R$ induced by all the vortices in the flow field on a point $\mathbf{R}_R$ on the surfaces $F_R = 0$. Linearization of this velocity has to be performed also in the perturbation $R_1(x_R, t)$ of the surfaces $F_R = 0$. A linearization of $F_R = 0$ about its basic state, using Eqs. (15), gives,

$$\delta F_R = f_{Ry0}(\delta y_R) + f_{Rz0}(\delta z_R) - R_1(x_R, t) = 0 \tag{28}$$

where f_{Ry0}, f_{Rz0} are the (y, z) components of $\nabla f_R \equiv \mathbf{e}_y\frac{\partial f_R}{\partial y_R} + \mathbf{e}_z\frac{\partial f_R}{\partial y_z}$ at a point (y_{R0}, z_{R0}). The solution of Eq. (28) results in,

$$\delta y_R = F_{Ry0}R_1(x_R, t), \qquad \delta z_R = F_{Rz0}R_1(x_R, t) \tag{29}$$

where

$$F_{Ry0} = \frac{f_{Ry0}}{f^2_{R0}}, \qquad F_{Rz0} = \frac{f_{Rz0}}{f^2_{R0}}, \qquad f^2_{R0} = f^2_{Ry0} + f^2_{Rz0} \tag{30}$$

From Eqs. (14b)(15)(29),

$$\ell'^2_{mR} = (y'_{0m} - y_R)^2 + (z'_{0m} - z_R)^2 = \ell'^2_{mR0}[1 - 2(H_{mR0}F_{Ry0} + G_{mR0}F_{Rz0})R_1(x_R, t)] \tag{31a}$$

where,

$$\ell'^2_{mR0} = (y'_{0m} - y_{R0})^2 + (z'_{0m} - z_{R0})^2,$$

$$H_{mR0} = \frac{y'_{0m} - y_{R0}}{\ell'^2_{mR0}}, \qquad G_{mR0} = \frac{z'_{0m} - z_{R0}}{\ell'^2_{mR0}} \tag{31b}$$

Also,

$$H_{mR} = \frac{y'_{0m} - y_R}{\ell'^2_{mR}} = H_{mR0} + \bar{w}^R_{mR0}R_1(x_R, t),$$

$$G_{mR} = \frac{z'_{0m} - z_R}{\ell'^2_{mR}} = G_{mR0} + \bar{v}^R_{mR0}R_1(x_R, t), \tag{32a}$$

where,

$$\bar{w}^R_{mR0} = 2H_{mR0}G_{mR0}F_{Rz0} - (G^2_{mR0} - H^2_{mR0})F_{Ry0}$$

$$\bar{v}^R_{mR0} = 2H_{mR0}G_{mRy}F_{Ry0} + (G^2_{mR0} - H^2_{mR0})F_{Rz0} \tag{32b}$$

substituting Eqs. (31)(32) in Eq. (27) for $p = R$ results in the full linearized form of the velocity $\mathbf{u}_R^V$,

$$\mathbf{u}_R^V = \mathbf{e}_y v_{R0}^V - \mathbf{e}_z w_{R0}^V + \sum_{m=1}^{N} \frac{\Gamma_m \ell'^2_{mR0}}{4\pi}(\mathbf{e}_z \bar{u}_{mR0}^V + \mathbf{e}_y \bar{v}_{mR0}^V - \mathbf{e}_z \bar{w}_{mR0}^V) + \sum_{m=1}^{N} \frac{\Gamma_m}{2\pi}(\mathbf{e}_y \bar{v}_{mR0}^R - \mathbf{e}_z \bar{w}_{mR0}^R) R_1(x_R, t) \tag{33}$$

where,

$$v_{R0}^V = \sum_{m=1}^{N} \frac{\Gamma_m}{2\pi} G_{mR0}, \quad w_{R0}^V = \sum_{m=1}^{N} \frac{\Gamma_m}{2\pi} H_{mR0} \tag{34a}$$

$$\bar{u}_{mR0}^V = \int_{-\infty}^{+\infty} \left(H_{mR0} \frac{\partial z'_m}{\partial x'_m} - G_{mR0} \frac{\partial y'_m}{\partial x'_m} \right) \frac{dx'_m}{R'^3_{0mR}} \tag{34b}$$

$$\bar{v}_{mR0}^V = \int_{-\infty}^{+\infty} \left\{ H^2_{mR0} \left[z'_m - (x'_m - x_R) \frac{\partial z'_m}{\partial x'_m} \right] - G^2_{mR0} z'_m - H_{mR0} G_{mR0} \left[2y'_m - (x'_m - x_R) \frac{\partial y'_m}{\partial x'_m} \right] \right\} \frac{dx'_m}{R'^3_{0mR}}$$

$$\bar{w}_{mR0}^V = \int_{-\infty}^{+\infty} \left\{ G^2_{mR0} \left[y'_m - (x'_m - x_R) \frac{\partial y'_m}{\partial x'_m} \right] - H^2_{mR0} y'_m - H_{mR0} G_{mR0} \left[2z'_m - (x'_m - x_R) \frac{\partial z'_m}{\partial x'_m} \right] \right\} \frac{dx'_m}{R'^3_{0mR}} \tag{34c}$$

where $R'^3_{0mR} = (x'_m - x_R)^2 + \ell'^2_{mR0}$. The velocity $\mathbf{u}_R^V$ induced by all the vortices on a point $\mathbf{R}_R$ on the surfaces $F_R = 0$ (Eq. (33)) is constructed of basic-state terms (v_{R0}^V, w_{R0}^V) that are the two-dimensional velocities induced by the vortices in their basic-state position on the basic-state surfaces, $f_R = 0$, of the perturbation velocities $(\bar{u}_{mR0}^V, \bar{v}_{mR0}^V, \bar{w}_{mR0}^V)$ that are induced by all the perturbed vortices on the basic-state surfaces $f_R = 0$, and of the perturbation velocities $(\bar{v}_{mR0}^R, \bar{w}_{mR0}^R)$ induced by all the vortices in their basic-state position on the perturbed surfaces $\delta F_R = 0$.

3.2.3 Velocity induced by the sources distributed on the surfaces

Substituting $p = n$ in Eq. (9) defines the velocity $\mathbf{u}_n^S$ induced by the sources distributed on the surfaces $F_R = 0$ on a point $\mathbf{R}_n$ on the n-th vortex. From Eq. (10),

$$\mathbf{R}'_{Rn} = \mathbf{e}_x(x'_R - x_n) + \mathbf{e}_y(y'_R - y_{0n} - y_n) + \mathbf{e}_z(z'_R - z_{0n} - z_n) \tag{35}$$

Assuming that at a specific section x_R the coordinates (y_R, z_R) of a point $\mathbf{R}_R$ on $F_R = 0$ are given by a Lagrangian parameter θ_R,

$$y_R = y_R(\theta_R), \qquad z_R = z_R(\theta_R) \tag{36}$$

where $\theta_{RL} \le \theta_R \le \theta_{RU}$, the surface element area dF'_R is given by,

$$dF'_R = S(\theta'_R) dx'_R d\theta'_R, \quad S(\theta'_R) \equiv \left[\left(\frac{\partial y'_R}{\partial \theta'_R} \right)^2 + \left(\frac{\partial z'_R}{\partial \theta'_R} \right)^2 \right]^{1/2} \tag{37a}$$

Using Eqs. (15)(36), where (y_{R0}, z_{R0}) are given by θ_{R0}, the linear approximation of dF'_R is given by,

$$dF'_R = S(\theta'_{R0}) dx'_R d\theta'_{R0}, S(\theta'_{R0}) \equiv \left[\left(\frac{\partial y'_{R0}}{\partial \theta'_{R0}} \right)^2 + \left(\frac{\partial z'_{R0}}{\partial \theta'_{R0}} \right)^2 \right]^{1/2} \tag{37b}$$

and the integration in Eq. (9) is performed over $-\infty < x'_R < +\infty$ and $\theta_{RL} \le \theta'_{R0} \le \theta_{RU}$. Also, from Eqs. (15)(16) linearization of $Q'(x'_R, \theta'_R, t)$ gives

$$Q'(x'_R, \theta'_R, t) = Q'_0(x'_R, \theta'_{R0}, t) + Q'_1(x'_R, \theta'_{R0}, t) \tag{38a}$$

and

$$Q' |\mathbf{R}'_{Rn}|^{-3} = \frac{Q'_0 + Q'_1}{R'^3_{Rn}} + \frac{3Q'_0}{R'^5_{Rn}} [y_n(y'_R - y_{0n}) + z_n(z'_R - z_{0n})] \tag{38b}$$

where $R'^2_{Rn} = (x'_R - x_n)^2 + \ell'^2_{Rn}$ and $\ell'^2_{Rn} = (y'_R - y_{0n})^2 + (z'_R - z_{0n})^2$, and $Q'_0 = Q_0(\theta'_{R0}, t)$, $Q'_1 = Q_1(x'_R, \theta'_{R0}, t)$. Using Eqs. (28), (30) (31),

$$\frac{1}{R^3_{Rn}} = \frac{1}{R'^3_{Rn0}} \left[1 + 3 \frac{H'_{nR0} F'_{Ry0} + G'_{nR0} F'_{Rz0}}{R'^2_{Rn0}} R'_1(x'_R, t) \right] \tag{39}$$

where $R'^2_{Rn0} = (x'_R - x_n)^2 + \ell'^2_{nR0}$ and $F'_{Ry0}, F'_{Rz0}, \ell'_{nR0}$, are calculated by Eqs. (30) (31b) respectively at point (y'_{R0}, z'_{R0}). Substituting Eq. (39) in (38) and using Eqs. (28)(30)(31)(35) results in,

$$\frac{Q' \mathbf{R}'_{Rn}}{|\mathbf{R}'_{Rn}|^3} = \frac{Q'_0 + Q'_1}{R'^3_{Rn0}} \mathbf{R}'_{Rn0} + \frac{Q'_0}{R'^3_{Rn0}} [(\mathbf{e}_y F'_{Ry0} + \mathbf{e}_z F'_{Rz0}) R_1(x'_R, t) - (\mathbf{e}_y y_n + \mathbf{e}_z z_n)] + - 3 \frac{Q'_0}{R'^5_{Rn0}} \ell'^2_{nR0} [(H'_{nR0} F'_{Ry0} + G'_{nR0} F'_{Rz0}) R_1(x'_R, t) - (y_n H'_{nR0} + z_n G'_{nR0})] \mathbf{R}'_{Rn0} \tag{40}$$

where $\mathbf{R}'_{Rn0} = \mathbf{e}_x(x'_R - x_n) + \mathbf{e}_y(y'_{R0} - y_{0n}) + \mathbf{e}_z(z'_{R0} - z_{0n})$. Using the following equations,

$$\int_{-\infty}^{+\infty} \frac{\mathbf{R}'_{Rn0}}{R'^3_{Rn0}} dx'_R = -2(\mathbf{e}_y H'_{nR0} + \mathbf{e}_z G'_{nR0}) \tag{41a}$$

$$\int_{-\infty}^{+\infty} \frac{dx'_R}{R'^3_{Rn0}} = \frac{2}{\ell'^2_{nR0}}, \qquad \int_{-\infty}^{+\infty} \frac{dx'_R}{R'^5_{Rn0}} = \frac{4}{3\ell'^4_{nR0}} \tag{41b}$$

$$\int_{-\infty}^{+\infty} \frac{R'_1(x'_R, t)}{R'^5_{Rn0}} dx'_R = \frac{1}{3\ell'^2_{nR0}} \int_{-\infty}^{+\infty} \frac{2R'_1(x'_R, t) - (x'_R - x_n) \frac{\partial R'_1}{\partial x'_R}}{R'^3_{Rn0}} dx'_R \tag{41c}$$

results in the linearized form of the velocity $\mathbf{u}_n^S$,

$$\mathbf{u}_n^S = \mathbf{e}_y v_{n0}^S + \mathbf{e}_z w_{n0}^S + \mathbf{e}_y v_{nQ}^S + \mathbf{e}_z w_{nQ}^S + \mathbf{e}_x(u_{nR1}^S + u_{nQ1}^S) + \mathbf{e}_y(v_{nR1}^S + v_{nQ1}^S) + \mathbf{e}_z(w_{nR1}^S + w_{nQ1}^S) \tag{42}$$

where

$$v^S_{n0} = \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi} H'_{nR0} S(\theta'_{R0}) d\theta'_{R0},$$

$$w^S_{n0} = \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi} G'_{nR0} S(\theta'_{R0}) d\theta'_{R0} \tag{43a}$$

$$v^S_{nQ} = \frac{1}{2\pi} \int_{\theta_{RL}}^{\theta_{RU}} [y_n(G'^2_{nR0} - H'^2_{nR0}) - 2z_n G'_{nR0} H'_{nR0}]\, Q'_0 S(\theta'_{R0}) d\theta'_{R0}$$

$$w^S_{nQ} = \frac{1}{2\pi} \int_{\theta_{RL}}^{\theta_{RU}} [z_n(H'^2_{nR0} - G'^2_{nR0}) - 2y_n G'_{nR0} H'_{nR0}]\, Q'_0 S(\theta'_{R0}) d\theta'_{R0} \tag{43b}$$

$$u^S_{nR1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{3Q'_0}{4\pi} \frac{\ell'^2_{nR0}}{R'^5_{Rn0}} (H'_{nR0} F'_{Ry0} + G'_{nR0} F'_{Rz0}) (x_n - x'_R) R'_1(x'_R, t) S(\theta'_{R0}) d\theta'_{R0} dx'_R$$

$$v^S_{nR1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0 \ell'^2_{nR0}}{4\pi R'^3_{Rn0}} \left[F'_{1n} R'_1(x'_R, t) - F'_{2n}(x'_R - x_n) \frac{\partial R'_1}{\partial x'_R}\right] S(\theta'_{R0}) d\theta'_{R0} dx'_R \tag{43c}$$

$$w^S_{nR1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0 \ell'^2_{nR0}}{4\pi R'^3_{Rn0}} \left[F'_{3n} R'_1(x'_R, t) - F'_{4n}(x'_R - x_n) \frac{\partial R'_1}{\partial x'_R}\right] S(\theta'_{R0}) d\theta'_{R0} dx'_R$$

$$u^S_{nQ1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_R, \theta'_{R0}, t)}{4\pi R'^3_{Rn0}} (x'_R - x_n) S(\theta'_{R0}) d\theta'_{R0} dx'_R$$

$$v^S_{nQ1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_R, \theta'_{R0}, t)}{4\pi R'^3_{Rn0}} \ell'^2_{nR0} H'_{nR0} S(\theta'_{R0}) d\theta'_{R0} dx'_R$$

$$w^S_{nQ1} = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_R, \theta'_{R0}, t)}{4\pi R'^3_{Rn0}} \ell'^2_{nR0} G'_{nR0} S(\theta'_{R0}) d\theta'_{R0} dx'_R \tag{43d}$$

where

$$F'_{1n} = F'_{Ry0}(H'^2_{nR0} - G'^2_{nR0}) + 2F'_{Rz0} G'_{nR0} H'_{nR0},$$

$$F'_{3n} = F'_{Rz0}(G'^2_{nR0} - H'^2_{nR0}) + 2F'_{Ry0} G'_{nR0} H'_{nR0},$$

$$F'_{2n} = F'_{Rz0} G'_{nR0} H'_{nR0} + F'_{Ry0} H'^2_{nR0} \tag{43e}$$

$$F'_{4n} = F'_{Ry0} G'_{nR0} H'_{nR0} + F'_{Rz0} G'^2_{nR0}$$

Eq. (42) shows that the velocity $\mathbf{u}^S_n$ induced by the sources over $F_R = 0$ on a point $\mathbf{R}_n$ on the n-th vortex is constructed of basic-state terms (v^S_{n0}, w^S_{n0}) that are the two-dimensional velocities induced by the basic-state sources Q_0 over $f_R = 0$ on the vortex basic-state position, of perturbation velocities (v^S_{nQ}, w^S_{nQ}) induced by Q_0 over $f_R = 0$ on the displaced vortex, of perturbation velocities $(u^S_{nR1}, v^S_{nR1}, w^S_{nR1})$ induced by Q_0 over perturbed surfaces $\delta F_R = 0$ on the n-th vortex basic-state position, and of perturbation velocities $(u^S_{nQ1}, v^S_{nQ1}, w^S_{nQ1})$ induced by the source fluctuations Q_1 over $f_R = 0$ on the n-th vortex basic-state position.

The velocity $\mathbf{u}^S_p$ induced by the sources over $F_R = 0$ on a point $\mathbf{R}_p$ in the flow field is defined by Eq. (9). Its linearized form can be found in a similar derivation of Eqs. (35) to (43) or by substituting (x_p, y_p, z_p) for (x_n, y_{0n}, z_{0n}) and $y_n = z_n = 0$ in Eqs. (31)(42)(43),

$$\mathbf{u}^S_p = \mathbf{e}_y v^S_{p0} + \mathbf{e}_z w^S_{p0} + \mathbf{e}_x(u^S_{pR1} + u^S_{pQ1}) + \mathbf{e}_y(v^S_{pR1} + v^S_{pQ1}) + \mathbf{e}_z(w^S_{pR1} + w^S_{pQ1}) \tag{44}$$

3.2.4 Velocity induced by the sources on a point on the surfaces

The velocity $\mathbf{u}^S_R$ induced by the sources over the surfaces $F_R = 0$ on a point $\mathbf{R}_R$ on the surfaces is calculated by,

$$\mathbf{u}^S_R = \lim_{\mathbf{R}_p \to \mathbf{R}_R} \left\{ - \iint_{F'_R} \frac{Q' \mathbf{R}'_{Rp} dF'_R}{4\pi |\mathbf{R}'_{Rp}|^3} \right\} \tag{45}$$

The singular limit in Eq. (45) was calculated by Hess and Smith[21],

$$\mathbf{u}^S_R = \frac{1}{2} Q(\mathbf{R}_R, t) \frac{\nabla F_R}{|\nabla F_R|} - \iint_{F'_R} \frac{Q' \mathbf{R}'_{RR} dF'_R}{4\pi |\mathbf{R}'_{RR}|^3},$$
$$\mathbf{R}'_{RR} \equiv \mathbf{R}'_R - \mathbf{R}_R \tag{46}$$

where the first term in Eq. (46) is the contribution of the sources in the surrounding of point $\mathbf{R}_R$ on $F_R = 0$, and the second integral term is the contribution of all the other sources over $F_R = 0$. Using Eqs. (15)(16)(30),

$$\frac{Q'}{|\mathbf{R}'_{RR}|^3} = \frac{Q'_0 + Q'_1}{R'^3_{RR0}} - \frac{3Q'_0}{R'^5_{RR0}} [(y'_{R0} - y_{R0})(F'_{Ry0} R'_1 - F_{Ry0} R_1) + (z'_{R0} - z_{R0})(F'_{Rz0} R'_1 - F_{Rz0} R_1)] \tag{47}$$

where $R'^2_{RR0} = (x'_R - x_R)^2 + \ell'^2_{RR0}$ and $\ell'^2_{RR0} = (y'_{R0} - y_{R0})^2 + (z'_{R0} - z_{R0})^2$, and $R'_1 = R_1(x'_R, t)$. From Eq. (47) the linearized form of the integrand in Eq. (46) is,

$$\frac{Q' \mathbf{R}'_{RR}}{|\mathbf{R}'_{RR}|^3} = \frac{Q'_0 + Q'_1}{R'^3_{RR0}} \mathbf{R}'_{RR0} + \frac{Q'_0}{R'^3_{RR0}} [\mathbf{e}_y(F'_{Ry0} R'_1 - F_{Ry0} R_1) + \mathbf{e}_z(F'_{Rz0} R'_1 - F_{Rz0} R_1)] - 3\frac{Q'_0}{R'^5_{RR0}} \ell'^2_{RR0} [H'_{RR0}(F'_{Ry0} R'_1 - F_{Ry0} R_1) + G'_{RR0}(F'_{Rz0} R'_1 - F_{Rz0} R_1)] \mathbf{R}'_{RR0} \tag{48}$$

where $\mathbf{R}'_{RR0} = \mathbf{e}_x(x'_R - x_R) + \mathbf{e}_y(y'_{R0} - y_{R0}) + \mathbf{e}_z(z'_{R0} - z_{R0})$. Using Eqs. (41) for $n = R$ and Eq (37b) results in the linearized form of the velocity $\mathbf{u}_R^S$,

$$\mathbf{u}_R^S = \frac{1}{2}[Q_0(\mathbf{R}_{R0}, t) + Q_1(\mathbf{R}_{R0}, t)]\frac{\nabla F_R}{|\nabla F_R|} + \mathbf{e}_y v_{R0}^S + \mathbf{e}_z w_{R0}^S + \mathbf{e}_x(u_{RR1}^S + u_{RQ1}^S) + \mathbf{e}_y(v_{RR1}^S + v_{RQ1}^S) + \mathbf{e}_z(w_{RR1}^S + w_{RQ1}^S) \tag{49}$$

where

$$v_{R0}^S = -\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi} H'_{RR0} S(\theta'_{R0}) d\theta'_{R0},$$

$$w_{R0}^S = -\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi} G'_{RR0} S(\theta'_{R0}) d\theta'_{R0} \tag{50a}$$

$$u_{RR1}^S = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{3Q'_0}{4\pi}\frac{\ell'^2_{RR0}}{R'^5_{RR0}}\Big[H'_{RR0}(F'_{Ry0}R'_1 - F_{Ry0}R_1) + G'_{RR0}(F'_{Rz0}R'_1 - F_{Rz0}R_1)\Big](x'_R - x_R)S(\theta'_{R0})d\theta'_{R0}dx'_R$$

$$v_{RR1}^S = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{4\pi}\frac{\ell'^2_{RR0}}{R'^3_{RR0}}\Big[F'_{1R}R'_1 - F'_{2R}R_1 - F'_{3R}(x'_R - x_R)\frac{\partial R'_1}{\partial x'_R}\Big] S(\theta'_{R0})d\theta'_{R0}dx'_R$$

$$w_{RR1}^S = \int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{4\pi}\frac{\ell'^2_{RR0}}{R'^3_{RR0}}\Big[F'_{4R}R'_1 - F'_{5R}R_1 - F'_{6R}(x'_R - x_R)\frac{\partial R'_1}{\partial x'_R}\Big] S(\theta'_{R0})d\theta'_{R0}dx'_R \tag{50b}$$

$$u_{RQ1}^S = -\int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_1, \theta'_{R0}, t)}{4\pi R'^3_{RR0}}(x'_R - x_R)S(\theta'_{R0})d\theta'_{R0}dx'_R$$

$$v_{RQ1}^S = -\int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_1, \theta'_{R0}, t)}{4\pi R'^3_{RR0}}\ell'^2_{RR0}H'_{RR0}S(\theta'_{R0})d\theta'_{R0}dx'_R \tag{50c}$$

$$w_{RR1}^S = -\int_{-\infty}^{+\infty}\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_1(x'_1, \theta'_{R0}, t)}{4\pi R'^3_{RR0}}\ell'^2_{RR0}G'_{RR0}S(\theta'_{R0})d\theta'_{R0}dx'_R$$

where,

$$H'_{RR0} = \frac{y'_{R0} - y_{R0}}{\ell'^2_{RR0}}, \qquad G'_{RR0} = \frac{z'_{R0} - z_{R0}}{\ell'^2_{RR0}}$$

$$F'_{1R} = (H'^2_{RR0} - G'^2_{RR0})F'_{Ry0} + 2G'_{RR0}H'_{RR0}F'_{Rz0},$$

$$F'_{2R} = (H'^2_{RR0} - G'^2_{RR0})F_{Ry0} + 2G'_{RR0}H'_{RR0}F_{Rz0},$$

$$F'_{3R} = H'^2_{RR0}F'_{Ry0} + G'_{RR0}H'_{RR0}F'_{Rz0}$$

$$F'_{4R} = (G'^2_{RR0} - H'^2_{RR0})F'_{Rz0} + 2G'_{RR0}H'_{RR0}F'_{Ry0},$$

$$F'_{5R} = (G'^2_{RR0} - H'^2_{RR0})F_{Rz0} + 2G'_{RR0}H'_{RR0}F_{Ry0},$$

$$F'_{6R} = G'_{RR0}H'_{RR0}F'_{Ry0} + G'^2_{RR0}F'_{Rz0} \tag{50e}$$

The velocity $\mathbf{u}_R^S$, that is induced by the sources over the surfaces $F_R = 0$ on a point $\mathbf{R}_R$ on these surfaces, is constructed (Eq. (49)) of the local influence of the sources, of two-dimensional velocities (v_{R0}^S, w_{R0}^S) induced by the basic-state sources Q_0 over the basic-state surfaces $f_R = 0$ on a point on these surfaces, of perturbation velocities $(u_{RR1}^S, v_{RR1}^S, w_{RR1}^S)$ induced by Q_0 over the perturbed surfaces $\delta F_R = 0$ on a point on $f_R = 0$, and of perturbation velocities $(u_{RQ1}^S, v_{RQ1}^S, w_{RQ1}^S)$ induced by the source fluctuations Q_1 over $f_R = 0$ on a point on these surfaces.

3.3 The linear approximation conserves flow-field equations

Eqs. (6)(27)(44) define the linearized approximation of the velocity $\mathbf{u}_p$ induced at a point $\mathbf{R}_p$ in the flow field. Using also Eqs. (22a)(24b)(24c) (for $n = p$ and $y_n = z_n = 0$) and Eqs. (31b)(43a)(43c)(43d) (for $n = p$), it can be shown (Rusak[14]) that for every $(\mathbf{R}_p, t)$ the linearized velocities $\mathbf{u}_p^V$ and $\mathbf{u}_p^S$ satisfy,

$$\nabla_p \cdot \mathbf{u}_p^V = 0, \qquad \nabla_p \cdot \mathbf{u}_p^S = 0$$
$$\nabla_p \times \mathbf{u}_p^V = 0, \qquad \nabla_p \times \mathbf{u}_p^S = 0 \tag{51}$$

where the derivation operator is: $\nabla_p \equiv \mathbf{e}_x\frac{\partial}{\partial x_p} + \mathbf{e}_y\frac{\partial}{\partial y_p} + \mathbf{e}_z\frac{\partial}{\partial z_p}$. Eqs. (51) are satisfied for each vortex circulation Γ_n, for every basic state solution of vortices positions $(y_{0n}(t), z_{0n}(t))$ and sources distribution $Q_0(\theta_{R0}, t)$, and for every perturbation function of $y_n(x_n, t), z_n(x_n, t), Q_1(x_R, y_R, z_R, t), R_1(x_R, t)$.

In proving Eqs. (51), a direct derivation of the different components was performed. Also the following theorem was used;

For continuous functions $\alpha(x), \beta(x)$ with continuous first derivatives, where $\frac{d\beta}{dx} = \alpha(x)$ and $\lim_{|x|\to\infty}\beta(x) = 0$, the following integral $\int_{-\infty}^{+\infty}[\alpha(x)h(x) + \beta(x)\frac{dh}{dx}]dx = 0$ for every continuous function $h(x)$ with continuous first derivatives.

For more details of proving Eqs. (51) see Rusak[14] (Appendix C). Eqs. (7)(51) show that the linear approximation of $\mathbf{u}_p$ satisfies both the continuity and irrotational equations of the flow-field,

$$\nabla_p \cdot \mathbf{u}_p = 0, \qquad \nabla_p \times \mathbf{u}_p = 0 \tag{52}$$

This means that the linearization process of Section 3.2 does not add any sources or vorticity to the flow field, and that the basic flow-field equations (Eq. (1)) are conserved in the linear approximation.

3.4 Linearized equations

The basic flow-field equations (Eqs. (18)(19)) also can be linearized by neglecting second-order terms. Developing V, W in Taylor series about (y_{0n}, z_{0n}),

$$V(y_{0n}+y_n, z_{0n}+z_n) = = V(y_{0n}, z_{0n}) + \left(\frac{\partial V}{\partial y}\right)_{0n} y_n + \left(\frac{\partial V}{\partial z}\right)_{0n} z_n$$

$$W(y_{0n}+y_n, z_{0n}+z_n) = = W(y_{0n}, z_{0n}) + \left(\frac{\partial W}{\partial y}\right)_{0n} y_n + \left(\frac{\partial W}{\partial z}\right)_{0n} z_n \quad (53)$$

where the derivatives are calculated at point (y_{0n}, z_{0n}), substituting Eqs. (23)(42) in Eqs. (18) and neglecting the second-order terms such as the longitudinal convection of the perturbations by their induced velocity $[(\mathbf{u}_n^V)_x + (\mathbf{u}_n^S)_x]\frac{\partial y_n}{\partial x_n}, [(\mathbf{u}_n^V)_x + (\mathbf{u}_n^S)_x]\frac{\partial z_n}{\partial x_n}$, results in the linearized equations for the vortex dynamics for every n and (x_n, t),

$$\frac{dy_{0n}}{dt} + \frac{\partial y_n}{\partial t} + U\frac{\partial y_n}{\partial x_n} = V(y_{0n}, z_{0n}) + v_{n0}^V + v_{n0}^S + \left(\frac{\partial V}{\partial y}\right)_{0n} y_n + \left(\frac{\partial V}{\partial z}\right)_{0n} z_n + \frac{\Gamma_n}{4\pi}\bar{v}_{nn}^V + \sum_{\substack{m=1\\ m\neq n}}^{N} \frac{\Gamma_m}{4\pi}\ell_{mn}^2 \bar{v}_{mn}^V + v_{nQ}^S + v_{nR1}^S + v_{nQ1}^S$$

$$\frac{dz_{0n}}{dt} + \frac{\partial z_n}{\partial t} + U\frac{\partial z_n}{\partial x_n} = W(y_{0n}, z_{0n}) - w_{n0}^V + w_{0n}^S + \left(\frac{\partial W}{\partial y}\right)_{0n} y_n + \left(\frac{\partial W}{\partial z}\right)_{0n} z_n - \frac{\Gamma_n}{4\pi}\bar{w}_{nn}^V - \sum_{\substack{m=1\\ m\neq n}}^{N} \frac{\Gamma_m}{4\pi}\ell_{mn}^2 \bar{w}_{mn}^V + w_{nQ}^S + w_{nR1}^S + w_{nQ1}^S \quad (54)$$

Developing V, W, F_{Ry}, F_{Rz} in Taylor series about the perturbed surfaces $f_R = 0$ and using Eqs. (15)(30) gives,

$$V(y_R, z_R) = V(y_{R0}, z_{R0}) + \left[\left(\frac{\partial V}{\partial y}\right)_{R0} F_{Ry0} + \left(\frac{\partial V}{\partial z}\right)_{R0} F_{Rz0}\right] R_1(x_R, t)$$

$$W(y_R, z_R) = W(y_{R0}, z_{R0}) + \left[\left(\frac{\partial W}{\partial y}\right)_{R0} F_{Ry0} + \left(\frac{\partial W}{\partial z}\right)_{R0} F_{Rz0}\right] R_1(x_R, t) \quad (55a)$$

and,

$$F_{Ry} = f_{Ry0} + F_{f1}R_1(x_R, t), \qquad F_{Rz} = f_{Rz0} + F_{f2}R_1(x_R, t) \quad (55b)$$

where all the derivatives are calculated at point (y_{R0}, z_{R0}) and,

$$f_{Ryy0} = \left(\frac{\partial^2 f_R}{\partial y_R^2}\right)_{R0}, \qquad f_{Ryz0} = \left(\frac{\partial^2 f_R}{\partial y_R \partial z_R}\right)_{R0},$$

$$f_{Rzz0} = \left(\frac{\partial^2 f_R}{\partial z_R^2}\right)_{R0}$$

$$F_{f1} = f_{Ryy0}F_{Ry0} + f_{Ryz0}F_{Rz0},$$

$$F_{f2} = f_{Ryz0}F_{Ry0} + f_{Rzz0}F_{Rz0} \quad (55c)$$

Using Hess and Smith[21] expression for the calculation of the self-induced normal velocity at a point on a source surface, together with Eqs. (30) (33)(49)(55), the tangency boundary condition, Eq. (9), becomes for every (x_R, θ_{R0}, t), in its linearized form,

$$-\frac{\partial R_1}{\partial t} - U\frac{\partial R_1}{\partial x_R} + \frac{1}{2}(Q_0+Q_1)f_{R0} + \left[V(y_{R0}, z_{R0}) + v_{R0}^V + v_{R0}^S + \sum_{m=1}^{N}\frac{\Gamma_m}{4\pi}(\ell'^2_{mR0}\bar{v}_{mR0}^V + 2\bar{v}_{mR0}^R R_1) + v_{RR1}^S + v_{RQ1}^S\right] f_{Ry0}$$
$$+ \left[W(y_{R0}, z_{R0}) - w_{R0}^V + w_{R0}^S - \sum_{m=1}^{N}\frac{\Gamma_m}{4\pi}(\ell'^2_{mR0}\bar{w}_{mR0}^V + 2\bar{w}_{mR0}^R R_1) + w_{RR1}^S + w_{RQ1}^S\right] f_{Rz0}$$
$$+ R_1\left[[V(y_{R0}, z_{R0}) + v_{R0}^V + v_{R0}^S]F_{f1} + [W(y_{R0}, z_{R0}) - w_{R0}^V + w_{R0}^S]F_{f2} + \left(\frac{\partial V}{\partial y}\right)_{R0} F_{f3} + \left(\frac{\partial V}{\partial z}\right)_{R0} F_{f4}\right] = 0 \quad (56)$$

where,

$$F_{f3} = \frac{1}{f_{R0}^2}(f_{Ry0}^2 - f_{Rz0}^2), \qquad F_{f4} = \frac{2}{f_{R0}^2} f_{Ry0} f_{Rz0} \quad (57)$$

Equations (54)(56) must be satisfied in the basic state when all the perturbations vanish, $y_n \equiv z_n \equiv 0$ for every n and $Q_1 \equiv R_1 \equiv 0$ for every (x_R, θ_R, t). The basic-state equations (or zero-order equations) result in,

$$\frac{dy_{0n}}{dt} = V(y_{0n}, z_{0n}) + v_{n0}^V + v_{n0}^S,$$
$$\frac{dz_{0n}}{dt} = W(y_{0n}, z_{0n}) - w_{n0}^V + w_{n0}^S$$
$$[V(y_{R0}, z_{R0}) + v_{R0}^V + v_{R0}^S]f_{Ry0} + [W(y_{R0}, z_{R0}) - w_{R0}^V + w_{R0}^S]f_{Rz0} + \frac{1}{2}Q_0(\theta_{R0})f_{R0} = 0 \quad (58)$$

where the various velocities in Eqs. (58) are defined by Eqs. (22a)(34a)(43a)(50a). Eqs. (58) represent a two-dimensional flow-field problem of the dynamics of N potential vortex points given near a surface $f_R = 0$. The solution of these equation gives the two-dimensional dynamics of the vortices, $y_{0n}(t), z_{0n}(t)$ for every (n,t), together with the source distribution $Q_0(\theta_{R0}, t)$ over the surfaces $f_R = 0$. This is referred to the present model as the "basic-state" of the problem. Eqs. (58) are nonlinear integro-differential equations, the solution of which is difficult and except for specific well-known solutions (see Lamb[8], Batchelor[17]) it requires the use of numerical methods(Moore[22], Leonard[23], Dyer et. al.[24], Smith[25]).

Assuming that the basic state solution of Eqs. (58) is known, the perturbation equations (or first-order equations) result from Eqs. (54)(56)(58) for every n and (x_n, t) in,

$$\frac{\partial y_n}{\partial t} + U\frac{\partial y_n}{\partial x_n} = \left(\frac{\partial V}{\partial y}\right)_{0n} y_n + \left(\frac{\partial V}{\partial z}\right)_{0n} z_n$$

$$+ \frac{\Gamma_n}{4\pi}\bar{v}^V_{nn} + \sum_{\substack{m=1\\m\neq n}}^{N} \frac{\Gamma_m}{4\pi}\ell^2_{mn}\bar{v}^V_{mn} + v^S_{nQ} + v^S_{nR1} + v^S_{nQ1}$$

$$\frac{\partial z_n}{\partial t} + U\frac{\partial z_n}{\partial x_n} = \left(\frac{\partial W}{\partial y}\right)_{0n} y_n + \left(\frac{\partial W}{\partial z}\right)_{0n} z_n$$

$$- \frac{\Gamma_n}{4\pi}\bar{w}^V_{nn} - \sum_{\substack{m=1\\m\neq n}}^{N} \frac{\Gamma_m}{4\pi}\ell^2_{mn}\bar{w}^V_{mn} + w^S_{nQ} + w^S_{nR1} + w^S_{nQ1} \tag{59}$$

and for every (x_R, θ_{R0}, t),

$$- \frac{\partial R_1}{\partial t} - U\frac{\partial R_1}{\partial x_R} + \frac{1}{2}Q_1(x_R, \theta_{R0}, t)f_{R0} +$$

$$+ \left[\sum_{m=1}^{N} \frac{\Gamma_m}{4\pi}(\ell'^2_{mR0}\bar{v}^V_{mR0} + 2\bar{v}^R_{mR0}R_1) + v^S_{RR1} + v^S_{RQ1}\right] f_{Ry0}$$

$$- \left[\sum_{m=1}^{N} \frac{\Gamma_m}{4\pi}(\ell'^2_{mR0}\bar{w}^V_{mR0} + 2\bar{w}^R_{mR0}R_1) - w^S_{RR1} - w^S_{RQ1}\right] f_{Rz0}$$

$$+ R_1\Bigg[[V(y_{R0}, z_{R0}) + v^V_{R0} + v^S_{R0}]F_{f1} + [W(y_{R0}, z_{R0}) - w^V_{R0} + w^S_{R0}]F_{f2} + \left(\frac{\partial V}{\partial y}\right)_{R0} F_{f3} + \left(\frac{\partial V}{\partial z}\right)_{R0} F_{f4}\Bigg] = 0 \tag{60}$$

where the various velocities in Eqs. (59)(60) are defined by Eqs. (24a)(24c)(32)(34c)(43b)(43c)(43d)(50b)(50c). Eqs. (59)(60) are integro-differential equations for the solution of perturbations y_n, z_n for every n and $Q_1(x_R, \theta_{R0}, t)$ as functions of the basic-state solution and the surface perturbation $R_1(x_R, t)$.The solution of these equations has to use the "cutoff-distance" model (Eqs. (25)(26)) for the calculation of $\bar{v}^V_{nn}, \bar{w}^V_{nn}$. Eqs. (59)(60) admit a solution of an exponential form (Fourier integrals) and can be transformed by a Fourier transformation into a system of equations for the solution of the stability of the basic state vortex dynamics.

3.5 Fourier transformation of perturbation equations

Each of the perturbations is assumed to be given by a Fourier integral,

$$y_n(x_n, t) = \int_{-\infty}^{+\infty} \hat{y}_n(k, t)e^{ikx_n}dk,$$

$$z_n(x_n, t) = \int_{-\infty}^{+\infty} \hat{z}_n(k, t)e^{ikx_n}dk$$

$$Q_1(x_R, \theta_{R0}, t) = \int_{-\infty}^{+\infty} \hat{Q}_1(\theta_{R0}, k, t)e^{ikx_R}dk,$$

$$R_1(x_R, t) = \int_{-\infty}^{+\infty} \hat{R}_1(k, t)e^{ikx_R}dk \tag{61}$$

where k is the wave number of perturbations in x direction (k is a real number), $i = \sqrt{-1}$, and $\hat{y}_n, \hat{z}_n, \hat{Q}_1, \hat{R}_1$ are the Fourier transforms of y_n, z_n, Q_1, R_1 respectively. No generality is lost by considering such a solution. Also, $k = 0$ describes a two-dimensional perturbation.

Substituting Eqs. (61) in Eqs. (59)(60) and changing the order of the integrations in the various equations of the velocities in these equations, changes (details in Rusak[14]) the perturbation equations into a linear system of equations for $\hat{y}_n, \hat{z}_n, \hat{Q}_1, \hat{R}_1$ for every dimensionless wave number $\beta \equiv k\ell$ of the perturbations. For every (n, t)

$$\frac{\partial \hat{y}_n}{\partial t} + i\frac{\beta}{\ell}U\hat{y}_n = \hat{y}_n\Bigg[\left(\frac{\partial V}{\partial y}\right)_{0n} + \sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{\pi}G_{mn}H_{mn} - \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0(\theta'_{R0}, t)}{2\pi}(H'^2_{nR0} - G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0}\Bigg] +$$

$$+\hat{z}_n\Bigg[\left(\frac{\partial V}{\partial z}\right)_{0n} + \frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2\omega(\delta_n) + \sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}(G^2_{mn} - H^2_{mn}) - \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0(\theta'_{R0}, t)}{\pi}G'_{nR0}H'_{nR0}S(\theta'_{R0})d\theta'_{R0}\Bigg] +$$

$$+ \sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}\Big\{\hat{z}_m[H^2_{mn}\psi(\beta\rho_{mn}) - G^2_{mn}\chi(\beta\rho_{mn})] - \hat{y}_m G_{mn}H_{mn}[\psi(\beta\rho_{mn}) + \chi(\beta\rho_{mn})]\Big\}$$

$$+\hat{R}_1\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0(\theta'_{R0}, t)}{2\pi f^2_{R0}}\Big\{H'_{nR0}[H'_{nR0}f'_{Ry0} + G'_{nR0}f'_{Rz0}]\psi(\beta\rho'_{nR0}) + G'_{nR0}[H'_{nR0}f'_{Rz0} - G'_{nR0}f'_{Ry0}]\chi(\beta\rho'_{nR0})\Big\}S(\theta'_{R0})d\theta'_{R0}$$

$$+ \int_{\theta_{RL}}^{\theta_{RU}} \frac{\hat{Q}'_1(\theta'_{R0}, \beta, t)}{2\pi f^2_{R0}}H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \tag{62a}$$

$$\frac{\partial \hat{z}_n}{\partial t}+i\frac{\beta}{\ell}U\hat{z}_n=\hat{z}_n\Bigg[\left(\frac{\partial W}{\partial z}\right)_{0n}-\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{\pi}G_{mn}H_{mn}$$

$$+\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{2\pi}(H'^2_{nR0}-G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0}\Bigg]$$

$$+\hat{y}_n\Bigg[\left(\frac{\partial W}{\partial y}\right)_{0n}-\frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2\omega(\delta_n)$$

$$+\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}(G^2_{mn}-H^2_{mn})$$

$$-\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{\pi}G'_{nR0}H'_{nR0}S(\theta'_{R0})d\theta'_{R0}\Bigg]$$

$$-\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}\Big\{\hat{y}_m[G^2_{mn}\psi(\beta\rho_{mn})-H^2_{mn}\chi(\beta\rho_{mn})]$$

$$-\hat{z}_mG_{mn}H_{mn}[\psi(\beta\rho_{mn})+\chi(\beta\rho_{mn})]\Big\}$$

$$+\hat{R}_1\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{2\pi f^2_{R0}}\Big\{H'_{nR0}[G'_{nR0}f'_{Ry0}-H'_{nR0}f'_{Rz0}]\chi(\beta\rho'_{nR0})$$

$$+G'_{nR0}[H'_{nR0}f'_{Ry0}+G'_{nR0}f'_{Rz0}]\psi(\beta\rho'_{nR0})\Big\}S(\theta'_{R0})d\theta'_{R0}$$

$$+\int_{\theta_{RL}}^{\theta_{RU}}\frac{\hat{Q}_1'(\theta'_{R0},\beta,t)}{2\pi f^2_{R0}}G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \tag{62b}$$

and for every (θ_{R0},t),

$$\hat{Q}_1(\theta_{R0},\beta,t)+\int_{\theta_{RL}}^{\theta_{RU}}\hat{Q}_1'(\theta'_{R0},\beta,t)(H'_{RR0}\hat{f}_{Ry0}+G'_{RR0}\hat{f}_{Rz0})\chi(\beta\rho'_{RR0})\frac{S(\theta'_{R0})}{\pi}d\theta'_{R0}=$$

$$=\sum_{m=1}^{N}\frac{\Gamma_m}{\pi}(\hat{y}_m\bar{a}_{ym}-\hat{z}_m\bar{a}_{zm})+\hat{R}_1\bar{a}_{R1}+\frac{\partial\hat{R}_1}{\partial t}\bar{a}_{R^\bullet 1}$$

where the various functions $\bar{a}_{ym},\bar{a}_{zm},\bar{a}_{R_1},\bar{a}_{R^\bullet 1}$ depend on (θ_{R0},β,t) and are defined by,

$$\bar{a}_{ym}=[G^2_{mR0}\psi(\beta\rho_{mR0})-H^2_{mR0}\chi(\beta\rho_{mR0})]\hat{f}_{Rz0}$$
$$+G_{mR0}H_{mR0}[\psi(\beta\rho_{mR0})+\chi(\beta\rho_{mR0})]\hat{f}_{Ry0} \tag{63a}$$

$$\bar{a}_{zm}=[H^2_{mR0}\psi(\beta\rho_{mR0})-G^2_{mR0}\chi(\beta\rho_{mR0})]\hat{f}_{Ry0}$$
$$+G_{mR0}H_{mR0}[\psi(\beta\rho_{mR0})+\chi(\beta\rho_{mR0})]\hat{f}_{Rz0} \tag{63b}$$

$$\bar{a}_{R1}=-\frac{2}{f_{R0}}\Big\{-i\frac{\beta}{\ell}U+\sum_{m=1}^{N}\frac{\Gamma_m}{\pi}[G_{mR0}H_{mR0}(\hat{f}^2_{Ry0}-\hat{f}^2_{Rz0})$$
$$+(G^2_{mR0}-H^2_{mR0})\hat{f}_{Ry0}\hat{f}_{Rz0}]$$

$$+\Big[V(\theta_{R0})+\sum_{m=1}^{N}\frac{\Gamma_m}{2\pi}G_{mR0}$$

$$+\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{2\pi}H'_{RR0}S(\theta'_{R0})d\theta'_{R0}\Big](\hat{f}_{Ryy0}\hat{f}_{Ry0}+\hat{f}_{Ryz0}\hat{f}_{Rz0})$$

$$+\Big[W(\theta_{R0})-\sum_{m=1}^{N}\frac{\Gamma_m}{2\pi}H_{mR0}$$

$$+\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{2\pi}G'_{RR0}S(\theta'_{R0})d\theta'_{R0}\Big](\hat{f}_{Rzz0}\hat{f}_{Rz0}+\hat{f}_{Ryz0}\hat{f}_{Ry0})$$

$$+\left(\frac{\partial V}{\partial y}\right)_{R0}(\hat{f}^2_{Ry0}-\hat{f}^2_{Rz0})+\left(\frac{\partial V}{\partial z}\right)_{R0}2\hat{f}_{Ry0}\hat{f}_{Rz0} \tag{63c}$$

$$+\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'(\theta'_{R0},t)}{2\pi}\Big[(H'_{RR0}\hat{f}'_{Ry0}+G'_{RR0}\hat{f}'_{Rz0})$$
$$(H'_{RR0}\hat{f}_{Ry0}+G'_{RR0}\hat{f}_{Rz0})\psi(\beta\rho'_{RR0})$$
$$+(H'_{RR0}\hat{f}'_{Rz0}-G'_{RR0}\hat{f}'_{Ry0})$$
$$(G'_{RR0}\hat{f}_{Ry0}-H'_{RR0}\hat{f}_{Rz0})\chi(\beta\rho'_{RR0})$$
$$+(H'^2_{RR0}-G'^2_{RR0})(\hat{f}^2_{Ry0}-\hat{f}^2_{Rz0})$$
$$+4G'_{RR0}H'_{RR0}\hat{f}_{Ry0}\hat{f}_{Rz0}\Big]S(\theta'_{R0})d\theta'_{R0}\Big\}$$

$$\bar{a}_{R^\bullet 1}=\frac{2}{f_{R0}} \tag{63d}$$

In Eqs. (62), $\delta_n\equiv kd_n$ is the dimensionless cutoff distance, $\delta_n=C^*_n\beta$,

$$\rho_{mn}=\frac{\ell_{mn}}{\ell},\qquad \rho_{mR0}=\frac{\ell_{mR0}}{\ell},\qquad \rho'_{RR0}=\frac{\ell'_{RR0}}{\ell} \tag{64a}$$

and in deriving these equations the following equations were used,

$$\int_{-\infty}^{+\infty}\frac{dx}{(x^2+\ell_s^2)^{3/2}}=\frac{2}{\ell_s^2}\qquad (s=mn,RR0) \tag{64b}$$

$$\int_{-\infty}^{+\infty}\frac{e^{ikx}-ikxe^{ikx}}{(x^2+\ell_s^2)^{3/2}}dx=\frac{2\psi(\beta\rho_s)}{\ell_s^2},$$

$$\int_{-\infty}^{+\infty}\frac{e^{ikx}dx}{(x^2+\ell_s^2)^{3/2}}dx=\frac{2\chi(\beta\rho_s)}{\ell_s^2}\quad (s=mn,mR0,RR0) \tag{64c}$$

$$\int_{-\infty}^{+\infty}\frac{e^{ikx}-1-ikxe^{ikx}}{|x|^3}=2\frac{\beta^2}{\ell^2}\omega(\delta_n) \tag{64d}$$

where χ and ψ are Crow's[9] interaction functions and $\omega(\delta_n)$ is Crow's[9] self-induced function,

$$\chi(x)=xK_1(x)\qquad \psi(x)=x^2K_0(x)+xK_1(x)$$

$$\omega(\delta_n)=\frac{1}{2}\left[\frac{\cos\delta_n-1}{\delta_n^2}+\frac{\sin\delta_n}{\delta_n}-Ci(\delta_n)\right] \tag{65}$$

where $K_0(x),K_1(x)$ are modified Bessel functions of the second kind (Watson[26]) and $Ci(\delta_n)$ is the integral cosine (Jahnke and Emde[27]).

3.6 Model equations

Eq. (62c) is a Fredholm integral equation of the second kind for the solution of the Fourier-transformed source fluctuations $\hat{Q}_1(\theta_{R0},\beta,t)$. Assumption of a solution where $\hat{Q}_1$ is a linear combination of all the vortex and surface perturbations,

$$\hat{Q}_1(\theta_{R0},\beta,t)=\sum_{m=1}^{N}\frac{\Gamma_m}{\pi}(\hat{y}_m\bar{Q}_{ym}-\hat{z}_m\bar{Q}_{zm})+\hat{R}_1\bar{Q}_{R1}+\frac{\partial\hat{R}_1}{\partial t}\bar{Q}_{R^\bullet 1} \quad (66)$$

and substitution in Eq. (62c) results for every (β,t), in a set of $2N+2$ independent Fredholm integral equations of the second kind for the solution of the source distribution functions $\bar{Q}_s(s=ym,zm,R1,R^\bullet 1)$

$$\bar{Q}_s(\theta_{R0},\beta,t)+\int_{\theta_{RL}}^{\theta_{RU}}\bar{Q}'_s(\theta'_{R0},\beta,t)(H'_{RR0}\hat{f}_{Ry0}+G'_{RR0}\hat{f}_{Rz0})\chi(\beta\rho'_{RR0})\frac{S(\theta'_{R0})}{\pi}d\theta'_{R0}$$
$$=\bar{a}_s(\theta_{R0},\beta,t) \qquad (s=ym,zm,R1,R^\bullet 1) \quad (67)$$

where the various functions $\bar{a}_s$ are defined by Eqs. (63).

The solution of Eqs. (67) gives the distribution of $\bar{Q}_s$ over the surfaces $f_R=0$ for every wave number β and time t. From Eqs. (63)(67) it can be seen that for every β the solution of the functions $\bar{Q}_s$ depends only on the basic-state solution (of Eqs. (58)) and it is independent of the solution of the vortex perturbations development that is described by Eqs. (63a)(63b).

The solution of Eqs. (66) is quite difficult as the unknown function $\bar{Q}_s$ appears both outside and inside the integral component of this equation. An analytical solution can be found for specific cases where the surface $f_R=0$ is an infinite straight plane or a circular cylinder (Rusak[14]). For general surfaces $f_R=0$ a numerical solution is required, such as the well-known source panel method of Hess and Smith[21]. The solution scheme of Eqs. (67) is identical for each $s=ym,zm,R1,R^\bullet 1$ and only the forcing functions $\bar{a}_s$ have to be adjusted for each s.

The solution of the source distribution functions $\bar{Q}_s$ for every wave number β satisfy identically the linearized tangency boundary condition over the given solid surfaces. Since for every β none of the functions $\bar{a}_s$ vanish identically, the functions $\bar{Q}_s$ always exist and are nonlinear in β. This means that the inclusion of the source fluctuations Q_1 is a necessary part of the present model, for the satisfaction of the linearized tangency boundary condition over the given solid surfaces. The source fluctuations Q_1 represent the solid surfaces resistance to be deformed by the influence of the perturbed vortices.

Assuming that a solution of Eqs. (67) is known, substituting the expression of $\hat{Q}_1$ from Eq. (66) into Eqs. (62a)(62b) results in a system of $2N$ first-order linear differential equationsthat describe the time history of the Fourier- transformed vortex displacements of the vortices, for every (β,t) and n,

$$\frac{\partial\hat{y}_n}{\partial t}=\sum_{m=1}^{N}[(a_{yy})_{nm}\hat{y}_m+(a_{yz})_{nm}\hat{z}_m]+b_{ynR}\hat{R}_1+b_{ynR^\bullet}\frac{\partial\hat{R}_1}{\partial t}$$

$$\frac{\partial\hat{z}_n}{\partial t}=\sum_{m=1}^{N}[(a_{zy})_{nm}\hat{y}_m+(a_{zz})_{nm}\hat{z}_m]+b_{znR}\hat{R}_1+b_{znR^\bullet}\frac{\partial\hat{R}_1}{\partial t} \quad (68)$$

where for $m=n$,

$$(a_{yy})_{nn}=-i\frac{\beta}{\ell}U+\left(\frac{\partial V}{\partial y}\right)_{0n}+\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{\pi}G_{mn}H_{mn}-\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q'_0(\theta'_{R0},t)}{2\pi}(H'^2_{nR0}-G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0}$$
$$+\frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}}\bar{Q}'_{yn}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \quad (69a)$$

$$(a_{yz})_{nn}=\left(\frac{\partial V}{\partial z}\right)_{0n}+\frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2\omega(\delta_n)+\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}(G^2_{mn}-H^2_{mn})-\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q'_0(\theta'_{R0},t)}{\pi}G'_{nR0}H'_{nR0}S(\theta'_{R0})d\theta'_{R0}$$
$$-\frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}}\bar{Q}'_{zn}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \quad (69b)$$

$$(a_{zz})_{nn}=-i\frac{\beta}{\ell}U+\left(\frac{\partial W}{\partial z}\right)_{0n}-\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}G_{mn}H_{mn}+\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q'_0(\theta'_{R0},t)}{2\pi}(H'^2_{nR0}-G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0} \quad (69c)$$
$$-\frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}}\bar{Q}'_{zn}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}$$

$$(a_{zy})_{nn}=\left(\frac{\partial W}{\partial y}\right)_{0n}-\frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2\omega(\delta_n)+\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}(G^2_{mn}-H^2_{mn})-\int_{\theta_{RL}}^{\theta_{RU}}\frac{Q'_0(\theta'_{R0},t)}{\pi}G'_{nR0}H'_{nR0}S(\theta'_{R0})d\theta'_{R0} \quad (69d)$$
$$+\frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}}\bar{Q}'_{yn}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}$$

and for $m \neq n$,

$$(a_{yy})_{nm} = -\frac{\Gamma_m}{2\pi}\Big\{G_{mn}H_{mn}[\psi(\beta\rho_{mn}) + \chi(\beta\rho_{mn})] - \frac{1}{\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{ym}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\Big\} \quad (70a)$$

$$(a_{yz})_{nm} = -\frac{\Gamma_m}{2\pi}\Big\{[H^2_{mn}\psi(\beta\rho_{mn}) - G^2_{mn}\chi(\beta\rho_{mn})] - \frac{1}{\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{zm}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\Big\} \quad (70b)$$

$$(a_{zz})_{nm} = \frac{\Gamma_m}{2\pi}\Big\{G_{mn}H_{mn}[\psi(\beta\rho_{mn}) + \chi(\beta\rho_{mn})] - \frac{1}{\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{zm}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\Big\} \quad (70c)$$

$$(a_{zy})_{nm} = -\frac{\Gamma_m}{2\pi}\Big\{[G^2_{mn}\psi(\beta\rho_{mn}) - H^2_{mn}\chi(\beta\rho_{mn})] - \frac{1}{\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{ym}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\Big\} \quad (70d)$$

$$b_{ynR} = \int_{\theta_{RL}}^{\theta_{RU}}\Big\{\frac{Q'_0(\theta'_{R0}t)}{2\pi f^2_{R0}}\Big[H'_{nR0}(H'_{nR0}f'_{Ry0} + G'_{nR0}f'_{Rz0})\psi(\beta\rho'_{nR0}) + G'_{nR0}(H'_{nR0}f'_{Rz0} - G'_{nR0}f'_{Ry0})\chi(\beta\rho'_{nR0})\Big] + \frac{1}{2\pi}\bar{Q}'_{R1}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})\Big\}S(\theta'_{R0})d\theta'_{R0} \quad (71a)$$

$$b_{znR} = \int_{\theta_{RL}}^{\theta_{RU}}\Big\{\frac{Q'_0(\theta'_{R0}t)}{2\pi f^2_{R0}}\Big[H'_{nR0}(G'_{nR0}f'_{Ry0} - H'_{nR0}f'_{Rz0})\chi(\beta\rho'_{nR0}) + G'_{nR0}(H'_{nR0}f'_{Ry0} + G'_{nR0}f'_{Rz0})\psi(\beta\rho'_{nR0})\Big] + \frac{1}{2\pi}\bar{Q}'_{R1}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})\Big\}S(\theta'_{R0})d\theta'_{R0} \quad (71b)$$

$$b_{ynR^\bullet} = \frac{1}{2\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{R^\bullet 1}(\theta'_{R0},\beta,t)H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \quad (71c)$$

$$b_{znR^\bullet} = \frac{1}{2\pi}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{R^\bullet 1}(\theta'_{R0},\beta,t)G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \quad (71d)$$

The coefficients of Eqs. (68) depend on the basic-state solution of Eqs. (58) and on the source-fluctuation functions that are solved by Eqs. (67). In the general case when the basic-state motion of the vortices is unsteady these coefficients are also time-dependent.The solution of Eqs. (67) is subjected for every wave number β to $2N$ initial conditions that are given by Eqs. (11)(61) in the form,

$$\hat{y}_n(\beta,t_0) = \int_{-\infty}^{+\infty} y_n(x_n,t_0)\exp\left(-i\frac{\beta}{\ell}x_n\right)dx_n,$$
$$\hat{z}_n(\beta,t_0) = \int_{-\infty}^{+\infty} z_n(x_n,t_0)\exp\left(-i\frac{\beta}{\ell}x_n\right)dx_n \quad (72)$$

Eqs. (68) can be written in a matrix formulation for every wave number β of the imposed perturbations as,

$$\partial\hat{\mathbf{Y}}/\partial t = A(\beta,t)\hat{\mathbf{Y}}(\beta,t) + B(\beta,t)\hat{\mathbf{r}}(\beta,t) + \hat{\mathbf{d}}(\beta,t) \quad (73)$$

where $\hat{\mathbf{Y}}(\beta,t)$ is a $2N$-dimensional column vector of the $2N$ Fourier-transformed vortex perturbations, $\hat{\mathbf{Y}}(\beta,t) = \{\cdots,\hat{y}_n(\beta,t),\hat{z}_n(\beta,t),\cdots\}$, $A(\beta,t)$ is a $2N\times 2N$ influence matrix the components of which are given by,

$$a_{2n-1,2m-1} = (a_{yy})_{nm}, \qquad a_{2n-1,2m} = (a_{yz})_{nm}$$
$$a_{2n,2m-1} = (a_{zy})_{nm}, \qquad a_{2n,2m} = (a_{zz})_{nm} \quad (74)$$

$\hat{\mathbf{r}}$ is a 2×1-dimensional vector, $\hat{\mathbf{r}}^T = (\hat{R}_1, \partial\hat{R}_1/\partial t)$, and $B(\beta,t)$ is its $2N\times 2$-dimensional influence matrix the components of which are given by,

$$b_{2n-1,1} = b_{ynR}, \qquad b_{2n-1,2} = b_{ynR^\bullet}$$
$$b_{2n,1} = b_{znR}, \qquad b_{2n,2} = b_{znR^\bullet} \qquad (1 \le n \le N) \quad (75)$$

The matrices A, B are written in Eq. (73) as functions of (β,t) to stress the major parameters that are needed for further analysis, but they are also functions of the basic-state solution, the source functions and the "core diameter" parameters.

The vector $\hat{\mathbf{d}}$ in Eqs. (73) is added simply to represent any other disturbances that may influence the vortex perturbations, but are not connected to any of the vortex-displacements or the solid-surface perturbations.

In the general case where the matrices A, B are time-dependent, the solution of Eqs. (73) can be given for each wave number β by the transfer-matrix $\Phi(\beta,t,t_0)$ that is a $2N\times 2N$-dimensional matrix (Kwakernaak and Sivan[28]),

$$\hat{\mathbf{Y}}(\beta,t) = \Phi(\beta,t,t_0)\hat{\mathbf{Y}}_0(\beta,t_0) + \int_{t_0}^{t}\Phi(\beta,t,\tau)[B(\beta,\tau)\hat{\mathbf{r}}(\beta,\tau) + \hat{\mathbf{d}}(\beta,\tau)]d\tau \quad (76)$$

where the initial-conditions vector $\hat{\mathbf{Y}}_0$ is given by, $\hat{\mathbf{Y}}_0(\beta,t_0) = \{\cdots,\hat{y}_n(\beta,t_0)\hat{z}_n(\beta,t_0),\cdots\}$ and the transfer -matrix $\Phi(\beta,t,t_0)$ is solved for every (β,t) by,

$$\frac{d}{dt}\Phi = A\Phi, \qquad \Phi(\beta, t_0, t_0) = I \tag{77}$$

The solution of Eqs. (73) or (77) in the general case is quite difficult, and can be obtained in most cases only numerically. On the other hand, when the basic state is steady or quasi-steady (where the vortex system and the surface $f_R = 0$ create a self-preserving configuration) the matrices A, B are time invariant and Eq. (73) can be solved analytically, where the homogeneous solution is constructed of $2N$ modes that depend on the $2N$ eigenvalues of the matrix A.

3.7 Summary

A general theory has been developed for the calculation of the three-dimensional long-wave linear stability of a system of straight parallel vortices immersed in an external incompressible and inviscid, potential flow and in the vicinity of solid surfaces. Model equations are composed of:

(i) basic-state equations (Eqs. (58)) for the solution of the basic-state dynamics of the vortices $y_{0n}(t), z_{0n}(t)$ for every (n, t), and the basic-state source distribution $Q_0(\theta_{R0}, t)$ over the surfaces $f_R = 0$,

(ii) source-fluctuations equations (Eqs. (67)) for the solution of the source functions $\bar{Q}_s(s = ym, zm, R_1, R_1^*)$,

(iii) vortex-perturbations equations (Eqs. (68)) for the calculation of the stability of the vortex system.

The basic state equations are independent of the perturbation equations. The source equations depend only on the basic state solution, whereas the vortex perturbations equations depend both on the basic state solution and on the solution of the source functions. In Eqs. (68) the core-diameter parameters C_n^* (Eqs. (26)) are used as free parameters, the influence of which has to be investigated in any specific problem.

The model can deal with a general system of straight parallel vortices, in the vicinity of general two-dimensional surfaces. Also, the model equations were reduced into a simple set of equations that can be used for the calculation of the three-dimensional stability of the vortex system. Model equations are also applicable to the specific cases of free vortex systems without any surfaces, where the sources Q_0 and $\bar{Q}_S$ and the perturbation R_1 are identically zero, and to the two-dimensional linear stability of a system of vortex points near surfaces, where $\beta = 0$ and $\chi(0) = \psi(0) = 1$ and $[\beta^2\omega(\delta_n)]_{\beta=0}=0$. The model equations were validated by Rusak[14] by complete agreement with the equations and results of all the known analyses of References 5, 6, 9-13.

The applicability of the present model is restricted to cases where a finite characteristic length (ℓ) can be defined from the basic-state solution by Eq. (13), to small-amplitude vortex and surface perturbations, to long-wave disturbances ($\beta C_n^* \ll 1$), and to slender-core vortices ($C_n^* \ll 1$). Cases where the characteristic length (ℓ) cannot be defined for all time, or where the perturbations are not small or the vortices have thick cores, cannot be represented by the present model. However, despite its simplicity and limitations, the present model is capable of representing the basic inviscid and long-wave mutual interations between the slender vortices and between them and solid surfaces, and of introducing new ideas about the stability of vortex systems and its control (see Section 4, 5 and 6 below and also Rusak and Seginer[15,16]).

4. Mathematical and physical analysis of vortex-perturbations equations

4.1 Analysis of the influence matrix

Using Eqs. (69)(70)(73)(74) the influence matrix A can be written generally in the form,

$$A = A_1 - i\frac{\beta}{\ell}UI \tag{78}$$

where A_1 is a $2N \times 2N$-dimensional reduced influence matrix and I is a $2N \times 2N$-dimensional unit matrix. It will be show that matrix A_1 has a vanishing trace, trace $(A_1) = 0$. From Eqs. (22a)(43a)(58) the velocities induced in the basic state on each of the vortices are,

$$\begin{aligned}(\mathbf{u}_{n0})_y &\equiv V(y_{0n}, z_{0n}) + v_{n0}^V + v_{n0}^S, \\ (\mathbf{u}_{n0})_z &\equiv W(y_{0n}, z_{0n}) - w_{n0}^V + w_{n0}^S\end{aligned} \tag{79}$$

where the velocities (v_{n0}^V, w_{n0}^V) and (v_{n0}^S, w_{n0}^S) are given by Eqs. (22a) and (43a) respectively. Since,

$$\begin{aligned}\frac{\partial G_{mn}}{\partial y_{0n}} &= \frac{\partial H_{mn}}{\partial z_{0n}} = 2G_{mn}H_{mn}, \\ \frac{\partial H_{nR0}}{\partial y_{0n}} &= -\frac{\partial G_{nR0}}{\partial z_{0n}} = H'^2_{nR0} - G'^2_{nR0}\end{aligned} \tag{80}$$

and since for a periodic perturbation with a wave number k, $\partial y_{0n} = \hat{y}_n e^{ikx_n}$, $\partial z_{0n} = \hat{z}_n e^{ikx_n}, \partial Q_0 = \hat{Q}_1 e^{ikx_R}$, Eq. (66) results in,

$$\begin{aligned}\left.\frac{\partial Q_0}{\partial y_{0n}}\right|_{\substack{\hat{y}_m = \hat{z}_m = 0, \hat{R}_1 = 0 \\ m \neq n}} &= \frac{\Gamma_n}{\pi}\bar{Q}_{yn} e^{ik(x_R - x_n)} \\ \left.\frac{\partial Q_0}{\partial z_{0n}}\right|_{\substack{\hat{y}_m = \hat{z}_m = 0, \hat{R}_1 = 0 \\ m \neq n}} &= -\frac{\Gamma_n}{\pi}\bar{Q}_{zn} e^{ik(x_R - x_n)}\end{aligned} \tag{81}$$

Using Eqs. (64b)(64c)(80)(81) it is found that for every (β, t),

$$\begin{aligned}\frac{\partial v_{n0}^S}{\partial y_{0n}} &= \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q_0'}{2\pi}(G'^2_{nR0} - H'^2_{nR0})S(\theta'_{R0})d\theta'_{R0} \\ &\quad + \frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{yn} H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0} \\ \frac{\partial w_{n0}^S}{\partial z_{0n}} &= \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q_0'}{2\pi}(H'^2_{nR0} - G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0} \\ &\quad - \frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{zn} G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\end{aligned} \tag{82}$$

Eqs. (79)(80)(82) together with Eqs. (69a)(69c)(74)(78) result in,

$$\frac{\partial(\mathbf{u}_{n0})_y}{\partial y_{0n}} = (a_1)_{2n-1,2n-1}, \qquad \frac{\partial(\mathbf{u}_{n0})_z}{\partial z_{0n}} = (a_1)_{2n,2n} \tag{83}$$

where $(a_1)_{rq}$ are the components of matrix (A_1). Therefore,

$$(a_1)_{2n,2n} = -(a_1)_{2n-1,2n-1} \tag{84}$$

since the basic-state problem (Eqs. (58)) describes a two-dimensional inviscid and incompressible, potential flow field where,

$$\frac{\partial(\mathbf{u}_{n0})y}{\partial y_{0n}} + \frac{\partial(\mathbf{u}_{n0})_z}{\partial z_{0n}} = 0 \tag{85}$$

Also from Eqs. (78)(84) for every (β, t),

$$\text{trace } (A_1) = 0, \qquad \text{trace } (A) = -2Ni\frac{\beta}{\ell}U \tag{86}$$

Eqs. (86) hold in the general case and reflect the fact that the basic state presents an incompressible two-dimensional flow. A similar result was found by Levitas and Seginer[29] for the specific case of a free-vortex system without any boundary surfaces. Eqs. (86) hold also for any "core-diameter" parameter C_n^* and doesn't depend on it at all, since all the C_n^* appear only in off-diagonal components of matrix A (or A_1).

4.2 Stability of a vortex system

Defining the Euclidian norm of vortex-perturbations vector $\mathbf{Y} \equiv \{\cdots, y_n, z_n, \cdots\}$ as $\|\mathbf{Y}\| \equiv [\sum_{n=1}^{N}(y_n^2 + z_n^2)]^{1/2}$, and vector $\mathbf{Y}_0$ as vector $\mathbf{Y}$ at time t_0, the stability of the vortex system can be defined according to Willems[30] by,

(i) The basic state dynamics of the vortex system are definedas stable according to Liapunov, if for every initial time t_0 and $\epsilon > 0$ there exists$\nu(\epsilon, t_0) > 0$, such that if $\|\mathbf{Y}_0\| < \nu$ then for every time $t \geq t_0$ the norm of vector $\mathbf{Y}$ is limited by, $\|\mathbf{Y}\| < \epsilon$.

(ii) The basic state dynamics of the vortex system are defined asasymptotically stable, if theystable, and if for every initial time t_0 there exists$\nu_0(t_0) > 0$, such that if $\|\mathbf{Y}_0\| < \nu_0$ then $\lim_{t\to\infty} \|\mathbf{Y}\| = 0$.

(iii) When condition (i) is not satisfied the basic state dynamics of the vortex system are defined as unstable.

In the present model both ν and ϵ have to satisfy $\nu^2 \ll \ell^2, \epsilon^2 \ll \ell^2$ to fit linearization requirements (Eqs. (14)). The stability of the vortex system at a specific wave number β can be defined analogously by replacing vector $\mathbf{Y}$ with vector $\hat{\mathbf{Y}}$, and ϵ, ν, ν_0, with $\hat{\epsilon}(\beta), \hat{\nu}(\hat{\epsilon}, \beta, t_0), \hat{\nu}_0(\beta, t_0)$ where the vectors $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Y}}_0$ are defined in Section 3.6.

Application of Willems[30] stability theorems to Eqs. (73) results in,

(i) For every wave number β,

$$\det \Phi(\beta, t, t_0) = \exp\left[\int_{t_0}^{t} \text{trace } (A(\beta, \tau))d\tau\right] \tag{87}$$

(ii) The homogeneous solution of Eqs. (73) is asymptotically stable at a specific wave number β if and only if the norm of the transfer matrix Φ, $\|\Phi(\beta, t, t_0\|$ is bounded at any time t, and when $t \to \infty, \lim_{t\to\infty} \|\Phi(\beta, t, t_0)\| = 0$ (where the norm of matrix Φ is defined by its components Φ_{rq} as $\|\Phi\| = [\sum \Phi_{rq}^2]^{1/2}$).

Using Eqs. (86)(87) for every wave number β results in,

$$\det \Phi(\beta, t, t_0) = \exp[-2Ni\frac{\beta}{\ell}U(t - t_0)] \tag{88}$$

Therefore, $\lim_{t\to\infty} \|\Phi(\beta, t, t_0)\| \neq 0$ for every β, which means that the basic-state vortex dynamics can never be asymptotically stable in the linear approximation of small perturbations. The vortex-system basic-state dynamics can be either stable (bounded oscillations) or unstable at a specific wave number. This general conclusion holds for every "core-diameter" parameter C_n^*, and results directly from the fact that the basic-state problem represents a two-dimensional incompressible and inviscid, potential flow. A similar result was found by Levitas and Seginer[29] for the specific case of a free-vortex system. Also, all the known two-dimensional or three-dimensional long-wave stability analyses[4-13] are compatible with this general conclusion.

4.3 Basic interactions of the vortex system

Eqs. (68)(69) can be rewritten for every (n, β, t) in the following special form,

$$\begin{aligned}
\frac{\partial \hat{y}_n}{\partial t} &= \left[-i\frac{\beta}{\ell}U\hat{y}_n\right] + \left[\hat{y}_n\left(\frac{\partial V}{\partial y}\right)_{0n} + \hat{z}_n\left(\frac{\partial V}{\partial z}\right)_{0n}\right] \\
&+ \left[\hat{y}_n\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{\pi}G_{mn}H_{mn} + \hat{z}_n\sum_{\substack{m=1\\m\neq n}}^{N}\frac{\Gamma_m}{2\pi}(G_{mn}^2 - H_{mn}^2)\right] + \\
&+ \int_{\theta_{RL}}^{\theta_{RU}}\frac{Q_0'}{2\pi}[\hat{y}_n(G_{nR0}'^2 - H_{nR0}'^2) - 2\hat{z}_n G_{nR0}' H_{nR0}']S(\theta_{R0}')d\theta_{R0}' \\
&+ \hat{z}_n\frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2\omega(\delta_n) + \\
&+ \frac{\Gamma_n}{2\pi^2}\int_{\theta_{RL}}^{\theta_{RU}}H_{nR0}'\chi(\beta\rho_{nR0}')[\hat{y}_n\bar{Q}_{yn}' - \hat{z}_n\bar{Q}_{zn}']S(\theta_{R0}')d\theta_{R0}' \\
&+ \sum_{\substack{m=1\\m\neq n}}^{N}\left[(a_{yy})_{nm}\hat{y}_m + (a_{yz})_{nm}\hat{z}_m\right] + \left[b_{ynR}\hat{R}_1 + b_{ynR\bullet}\frac{\partial \hat{R}_1}{\partial t}\right]
\end{aligned} \tag{89a}$$

$$\frac{\partial \hat{z}_n}{\partial t} = \left[-i\frac{\beta}{\ell}U\hat{z}_n\right] + \left[\hat{y}_n\left(\frac{\partial W}{\partial y}\right)_{0n} + \hat{z}_n\left(\frac{\partial W}{\partial z}\right)_{0n}\right]$$

$$+ \left[\hat{y}_n \sum_{\substack{m=1\\m\neq n}}^{N} \frac{\Gamma_m}{2\pi}(G^2_{mn} - H^2_{mn}) - \hat{z}_n \sum_{\substack{m=1\\m\neq n}}^{N} \frac{\Gamma_m}{\pi} G_{mn}H_{mn}\right]$$

$$+ \int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi}[\hat{z}_n(H'^2_{nR0} - G'^2_{nR0}) - 2\hat{y}_n G'_{nR0}H'_{nR0}]S(\theta'_{R0})d\theta'_{R0}$$

$$- \hat{y}_n \frac{\Gamma_n}{2\pi}\left(\frac{\beta}{\ell}\right)^2 \omega(\delta_n) +$$

$$+ \frac{\Gamma_n}{2\pi^2} \int_{\theta_{RL}}^{\theta_{RU}} G'_{nR0}\chi(\beta\rho'_{nR0})[\hat{z}_n\bar{Q}'_{zn} - \hat{y}_n\bar{Q}'_{yn}]S(\theta'_{R0})d\theta'_{R0}$$

$$+ \sum_{\substack{m=1\\m\neq n}}^{N} \left[(a_{zy})_{nm}\hat{y}_m + (a_{zz})_{nm}\hat{z}_m\right] + \left[b_{znR}\hat{R}_1 + b_{znR^\bullet}\frac{\partial \hat{R}_1}{\partial t}\right] \quad (89b)$$

As was mentioned, when the basic state is steady (or quasi-steady) the homogeneous solution of Eqs. (89) depends on the $2N$ eigenvalues $\lambda_j (j = 1, \cdots, 2N)$ of the matrix A.

The terms in Eqs. (89) have been arranged in nine columns, each of which has a distinct physical meaning. The left hand column represents the rate of change of the vortex displacement amplitudes with time. The first column on the right-hand side represents the effect of the uniform axial flow (U) on the development of the perturbations. Were only this column effective, then $\lambda_j = -i\frac{\beta}{\ell}U$, which means that the disturbances would be swept downstream with the uniform flow. The second column on the right represents the influence of gradients of the external cross-flow velocity on vortex-perturbations development. Were only this column effective, then $\lambda_{n,n+1} = \pm[(\frac{\partial V}{\partial y})^2_{0n} + (\frac{\partial V}{\partial z})^2_{0n}]^{1/2}$ for every $n (n = 1, \cdots, N)$, where Eqs. (7)(17) were used. It means that each vortex displacement would tend to grow exponentially in time under external cross-flow gradients, independent of the other vortex displacements and with a growth rate (λ_n), that is equal to the rate of deformation of the basic-state cross-flow field at the vortex basic-state position.

The third column on the right-hand side of Eqs. (89) represents the influence of the vortices' basic-state positionson the perturbations of a specific vortex. Were only this influence effective, then when the vortex basic state motion is steady,

$$\lambda_{n,n+1} = \left\{\left[\sum_{m=1}^{N} \frac{\Gamma_m}{2\pi}(G^2_{mn} - H^2_{mn})\right]^2 + \left[\sum_{m=1}^{N} \frac{\Gamma_m}{\pi}(G_{mn}H_{mn})\right]^2\right\}^{1/2} \quad (90)$$

for every $n = 1, \cdots, N$. It means that each vortex displacement would tend to grow exponentially in time under the influence of the vortex basic-state flow, independent of the other vortex displacements and with a growth rate (λ_n), that is equal to the rate of deformation that the other vortices induce on the basic-state flow at a specific vortex basic-state position.

The fourth column on the right-hand side of Eqs. (89) represents the influence of the basic-state sources Q_0 distributed over the surfaces $f_R = 0$ on the perturbations of a specific vortex. Were only this influence effective in a steady basic-state, then,

$$\lambda_{n,n+1} = \left\{\left[\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{2\pi}(H'^2_{nR0} - G'^2_{nR0})S(\theta'_{R0})d\theta'_{R0}\right]^2 + \left[\int_{\theta_{RL}}^{\theta_{RU}} \frac{Q'_0}{\pi}G'_{nR0}H'_{nR0}S(\theta'_{R0})d\theta'_{R0}\right]^2\right\}^{1/2} \quad (91)$$

for every $n = 1, \cdots, N$. It means that each vortex displacement would tend to grow exponentially under the influence of basic-state sources, independent of the other vortex displacements and with a growth rate (λ_n), that is equal to the rate of deformation that the sources Q_0 induce on a specific basic state position of a vortex.

The fifth column on the right-hand side of Eqs. (89) represents the self-induction effect on the perturbation. Were only this column effective then (λ) would be $\lambda_{n,n\mp 1}$ $\pm i\frac{\Gamma_n}{2\pi}(\frac{\beta}{\ell})^2|\omega(\delta_n)|$ for every $n = 1, \cdots, N$, and the disturbances of each vortex would develop into independently stable and bounded oscillations. The nature of the oscillations would depend on the initial phase difference between $\hat{y}_n$ and $\hat{z}_n$. Two modes, a helical vortex line or a rotating sinusoidal curve, could be developed. For slender vortices ($C^{*2}_n \ll 1$) the second column must dominate the other columns, since $\omega(\delta_n)$ (Eqs. (65)) grows like $\ell n(1/\delta_n)$ when δ_n becomes small.

The sixth column on the right-hand side of Eqs. (89) represents the interaction of a perturbed vortex with the surface $f_R = 0$. Were only this interaction effective in a steady basic-state, then,

$$\lambda_{n,n+1} = \pm\frac{\Gamma_n}{2\pi^2}\left\{\left[\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{yn}H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\right]^2 - \left[\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{zn}H'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\right] \left[\int_{\theta_{RL}}^{\theta_{RU}} \bar{Q}'_{yn}G'_{nR0}\chi(\beta\rho'_{nR0})S(\theta'_{R0})d\theta'_{R0}\right]\right\}^{1/2} \quad (92)$$

for every $n = 1, \cdots, N$. The growth rates $\lambda_{n,n+1}$ of Eq. (92) can be either real, where perturbations tend to grow exponentially, or purely imaginary, where the disturbances would develop into stable and bounded oscillations.

The seventh column on the right-hand side of Eqs(89) represents the influence that all the perturbed vortices in the flow field exert on the basic-state position of a specific vortex. This interaction depends on the wave number β of the perturbations, the various circulations (Γ_m), surface shape of $f_R = 0$, and the solution of the source functions $\bar{Q}_{ym}, \bar{Q}_{zm}$. This interaction can result in either stable bounded oscillations or growing instabilities.

The eight column on the right-hand side of Eqs. (89) represents the influence of the imposed perturbations along the solid surfaces and their rate of change with time, on the development of vortex perturbations. This influence dependson the function $\hat{R}_1(\beta, t)$. Under certain conditions it may result in a static instability (see Section 5 and Rusak and Seginer[15]). Also, using the time-dependent perturbations to the solid surfaces as a controller results in a new approach to the control of vortex systems stability (See Section 6 and Rusak and Seginer[16]).

To summarize, the stability of the vortex system is influenced by the stabilizing effects of the constant streamwise component of the external flow and of the self-induced velocity, and by the destabilizing effects of the external cross-flow field and the interactions between the perturbed vortices and the basic-state flow.

5. Dynamic and Static Stability of Vortex Systems

5.1 Dynamic and Static Stability

Using the stability definition in Section 4.2, the basic-state dynamics of a vortex system are defined as dynamically stable (or unstable)if theystable (or unstable) according to Liapunov under effects that depend on the change of the vortex perturbations with time (dynamic effects). It is defined as statically unstable if the instability is not caused by dynamic effects.

This section deals only with flows that have a steady or quasi-steady (self-preserving configuration) basic-state, and $\hat{R}_1$ is a general oscillatory perturbation to the solid surfaces. In these cases the matrices A and B are invariant in time and Eq. (73) can be solved analytically, and the stability of the basic state can be determined from the solution for the development of the perturbations along the vortex filaments.

The solution of Eq. (73) is composed of a characteristic solution of the free (unforced, i.e. $\hat{R}_1 = 0, \hat{\mathbf{d}} = 0$) problem, and of a particular solution of the forced problem. The characteristic solution comprises $2N$ modes of the development in time of the perturbations. The growth rate of these perturbations depends on the characters of the $2N$ eigenvalues, $\lambda_j(\beta)(j = 1, \cdots, 2N)$, of the matrix $A(\beta)$. Using Eq. (78), the eigenvalues $\lambda_j(\beta)$ are given by $\lambda_j(\beta) = -i\frac{\beta}{\ell}U + \lambda_{1j}(\beta)$, where $\lambda_{1j}(\beta)$ are the $2N$ eigenvalues of the matrix $A_1(\beta)$.

Dynamic instability develops at a certain wave number β, when the real part of any one of the eigenvalues $\lambda_{1j}(\beta)$ is positive at this wave number. On the other hand, when the real parts of all $2N$ eigenvalues $\lambda_{1j}(\beta)$ are non-positive, the vortex system is dynamically stable.

The particular solution of Eq. (73) that is forced by a general oscillating perturbation $\hat{R}_1(\beta, t) = \tilde{R}_1(\beta)e^{i\alpha t}$ to the solid surfaces (where $\tilde{R}_1(\beta)$ and α are the amplitude and frequency of the perturbation) is given for every β and α by,

$$\hat{\mathbf{Y}}_p = -\bar{A}^{-1}(\beta, \alpha)B(\beta)\mathbf{r}^*(\alpha)\tilde{R}_1(\beta)e^{i\alpha t} \quad (93a)$$

when $\det[\bar{A}(\beta, \alpha)] \neq 0$, or by

$$\hat{\mathbf{Y}}_p = \left[B_1(\beta, \alpha)t + B_0(\beta, \alpha)\right]B(\beta)\mathbf{r}^*(\alpha)\tilde{R}_1(\beta)e^{i\alpha t} \quad (93b)$$

when $\det[\bar{A}(\beta, \alpha)] = 0$.

In Eqs. (93) the matrix $\bar{A}(\beta, \alpha)$ and vector $\mathbf{r}^*(\alpha)$ are given by,

$$\bar{A}(\beta, \alpha) = A(\beta) - i\alpha I = A_1(\beta) - i\frac{\beta}{\ell}\bar{U}I, \quad \mathbf{r}^{*T} = (1, i\alpha) \quad (94)$$

where $\bar{U} = U + U_\alpha$, and U_α is the phase speed of the traveling perturbation R_1 at wave number $\beta, U_\alpha = \alpha\ell/\beta$.

The matrices $B_1(\beta, \alpha), B_0(\beta, \alpha)$ in Eq. (93b) are given by,

$$B_1(\beta, \alpha) = \frac{1}{\bar{a}_1}[\bar{A}^{2N-1} + \bar{a}_{2N-1}\bar{A}^{2N-2} + \cdots + \bar{a}_2\bar{A} + \bar{a}_1 I] \quad (95)$$

$$B_0(\beta, \alpha) = \frac{1}{\bar{a}_1}[\bar{A}^{2N-2} + \bar{a}_{2N-1}\bar{A}^{2N-3} + \cdots + \bar{a}_2\bar{A} + \bar{a}_2 I]$$

where $\bar{a}_j$ are the coefficients of the characteristic equation of the matrix $\bar{A}(\beta)$, $\det[\lambda I - \bar{A}(\beta)] = \lambda^{2N} + \bar{a}_{2N-1}\lambda^{2N-1} + \cdots + \bar{a}_1\lambda + \bar{a}_0 = 0$, where $\bar{a}_0 = \det[\bar{A}(\beta)]$. Eqs. (95) are obtained by using the Cayley-Hamilton theorem (Bellman[31]). A similar particular solution is found when the disturbance vector $\hat{\mathbf{d}}(\beta, t)$ has a general oscillatory form, $\hat{\mathbf{d}} = \hat{\mathbf{d}}(\beta)e^{i\alpha t}$.

Equation (93b) proves that when $\det[\bar{A}(\beta_d, \alpha)]$ vanishes at a specific wave number β_d, and when the oscillatory forcing vectors $B(\beta_d)\mathbf{r}^*(\alpha)\tilde{R}_1(\beta_d)$ or $\hat{\mathbf{d}}(\beta_d)$ exist a static instability can occur at this wave-number. This phenomenon can be defined as the "Divergence of the vortex system". To the best of the author's knowledge, such a phenomenon was never identified before, in any of the known stability analyses of vortex systems.

The divergence condition $\det[\bar{A}(\beta, \alpha)] = 0$ and Eq(94) show that divergence can happen only at wave numbers β_d for which the eigenvalues of the matrix $A_1(\beta)$ are purely imaginary. However, having the freedom to choose any frequency α of the forcing oscillation, it is concluded that a static instability can occur at every wave number β for which the matrix A_1 has purely imaginary eigenvalues, $Re[\lambda_{1j}(\beta)] = 0$, where

$$\alpha = -\frac{\beta}{\ell}U + Im[\lambda_{1j}(\beta)] \qquad (96a)$$

Significantly, the vortex system tends to diverge statically under the action of oscillating forcing perturbations, at every wave number where it is dynamically stable!

Of specific interest is the case where the perturbations $\hat{R}_1$ or $\hat{\mathbf{d}}$ are steady, $\alpha = 0$ (or $U_\alpha = 0$). Then $\bar{A} = A$, $\bar{U} = U$ and divergence can occur when $\det[A(\beta)] = 0$ (Rusak and Seginer[15]). It means that a vortex system tendsto diverge statically under a steady perturbation at specific wave numbers β_d for which,

$$Re[\lambda_{1j}(\beta_d)] = 0 \quad \text{and} \quad \beta_d\frac{U}{\ell} = Im[\lambda_{1j}(\beta_d)] \qquad (96b)$$

5.3 Physical Model

When Eq. (73) can be reduced to a two-rank problem, the time-invariant matrices $A(\beta)$ and $B(\beta)$ are 2×2-dimensional. The terms in Eq. (73) can in this case be written in the following form,

$$A = A_1 - i\frac{\beta}{\ell}UI, \; A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\hat{\mathbf{Y}} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix}, \qquad \hat{\mathbf{r}} = \begin{pmatrix} \hat{R}_1 \\ \partial\hat{R}_1/\partial t \end{pmatrix}, \qquad \hat{\mathbf{d}} = \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \end{pmatrix} \qquad (97)$$

where the components of matrices A_1 and B are not functions of the time t. It has to be emphasized at this point, that the analyses of most of the above-mentioned two-dimensional stability problems,[3-8] and of all the above-mentioned three-dimensional problems,[9-13] can be reduced to solutions of two-rank problems described by Eqs. (73) and (97).

It can be shown, after performing Laplace transformations, that Eqs. (73) and (97) are in this case equivalent to the following second-order, linear differential equations for each of the components of the displacements vector $\hat{\mathbf{Y}}$ (for any β), $(j = 1, 2)$

$$\partial^2\hat{y}_j/\partial t^2 + 2i(\beta/\ell)U\partial\hat{y}_j/\partial t + \det[A(\beta)]\hat{y}_j = \hat{\epsilon}_j(\beta, t), \qquad (98)$$

where $\hat{\epsilon}_j(\beta, t)$ are the forcing functions that depend on the components of matrices $A(\beta)$, $B(\beta)$, on $\hat{R}_1$, on vector $\hat{\mathbf{d}}$ and on the initial conditions of $\hat{y}_j$.

The solution of Eq. (98) is composed of a characteristic solution of the free (unforced) problem ($\hat{\epsilon}_j \equiv 0$), and of a particular solution of the forced problem. The characteristic solution comprises two modes. The development in time of the perturbations in these modes depends on the characters of the eigenvalues $(\lambda_{1,2})$ of the matrix $A(\beta)$,

$$\lambda_{1,2} = -i(\beta/\ell)U \pm \sqrt{a_{11}^2 + a_{12}a_{21}} \qquad (99)$$

The particular solution of Eq. (98) depends on the functional form of $\hat{\epsilon}_j(\beta, t)$. When $\hat{\epsilon}_j(\beta, t)$ has an oscillatory form, $\hat{\epsilon}_j = \tilde{\epsilon}_j(\beta)e^{i\alpha t}$, the respective solution $\hat{y}_{jp} = \tilde{y}_{jp}(\beta, t)e^{i\alpha t}$ results from the following equation,

$$\partial^2\tilde{y}_{jp}/\partial t^2 + 2\left(i\frac{\beta}{\ell}\bar{U}\right)\partial\tilde{y}_{jp}/\partial t + \det[\bar{A}(\beta,\alpha)]\tilde{y}_{jp} = \tilde{\epsilon}_j(\beta) \qquad (100a)$$

and is,

$$\hat{y}_{jp} = \frac{\tilde{\epsilon}_j(\beta)}{\det\bar{A}(\beta)}e^{i\alpha t} \quad \text{when } \det[\bar{A}(\beta,\alpha)] \neq 0$$

or (100b)

$$\hat{y}_{jp} = \left[\frac{\tilde{\epsilon}_j(\beta)}{2i(\beta/\ell)\bar{U}}t + b_{j0}\right]e^{i\alpha t} \quad \text{when } \det[\bar{A}(\beta,\alpha)] = 0 \text{ and } \beta\bar{U} \neq 0$$

or

$$\hat{y}_{jp} = \left[\frac{1}{2}\tilde{\epsilon}_j(\beta)t^2 + b_{j1}t + b_{j0}\right]e^{i\alpha t} \quad \text{when } \det[\bar{A}(\beta,\alpha)] = 0 \text{ and } \beta\bar{U} = 0$$

where $\bar{A}(\beta, \alpha)$ and $\bar{U}$ are given by Eqs. (94) and b_{j0}, b_{j1} are determined by the initial conditions. It is again obvious from Eqs. (100b), that when $\det[\bar{A}(\beta, \alpha] = 0$ and when the forcing functions $\hat{\epsilon}_j$ are oscillatory, a static instability can occur.

Equations (99)(100) give a new physical meaning to the stability, or to the development of perturbations of a vortex system. For each value of β and α, let us define the following parameters (that are new in the context of vortex-stability analysis):

a. the "Rigidity of the vortex system"

$K_v(\beta, \alpha) \equiv \det[\bar{A}(\beta, \alpha)]$

b. the "Damping of the vortex system" $C_v(\beta, \alpha) \equiv i(\beta/\ell)\bar{U}$

c. the "Generalized damping of the vortex system" $\bar{C}_v^2(\beta) \equiv (C_v^2 - K_v)$.

Using an analogy to the solid-mechanics model of a unit point mass on a spring with a damper, a similar physical model can be proposed for a vortex system. The development of the perturbations to the vortex lines relative to the basic state (for each value of β and α), resembles the kinematics of a unit point mass on a spring of rigidity $K_v(\beta, \alpha)$ with a damper $C_v(\beta, \alpha)$ (Fig. 1). Of specific interest is the case $\alpha = 0$ or ($U_\alpha = 0$) where $\bar{A} = A, \bar{U} = U$ and $K_v \equiv \det A(\beta), C_v = i\frac{\beta}{\ell}U$ (see Eq. (98)).

The stability of the vortex system can now be easily deduced from this model by the well known and understood classic rules of the kinematics of the equivalent mechanical system:

I. The dynamic stability of a given basic state of a vortex system is governed, according to Eq. (99), by the generalized damping $\bar{C}_v^2$ of the vortex system (which from its definition (c) above is $\bar{C}_v^2 = a_{11}^2 + a_{12}a_{21}$).

I.a) At every value of β for which the generalized damping is positive ($\bar{C}_v^2 > 0$), one of the eigenvalues of $A_1(\beta)$

has a positive real component. This indicates that the oscillatory development of the perturbation at these wave numbers with frequency $\frac{\beta}{\ell}U$, has a tendency to grow exponentially in time, which also means dynamic linear instability (Fig. 2.1).

I.b) If, for a certain wave number (β), the generalized damping vanishes, ($\bar{C}_v^2 = 0$), the two eigenvalues of $A_1(\beta)$ coincide. The oscillating perturbations with the frequency $\frac{\beta}{\ell}U$ in this case usually have a tendency to grow linearly in time, which is defined here as the "Threshold of Stability" (Fig. 2.2).

I.c) At all wave numbers β, for which the generalized damping is negative, ($\bar{C}_v^2 < 0$), the two eigenvalues of $A_1(\beta)$ are imaginary and, $\lambda_{1,2} = -i\frac{\beta}{\ell}U \pm |\bar{C}_v|$. The perturbations are then bounded and develop as two displacement waves along the vortex lines, and the vortex system is neutrally stable (Fig. 2.3).

II. The static stability of the basic state of a vortex system is governed, according to Eqs. (100), by the rigidity K_v of the system. When, for a certain wave number β_d and frequency α the rigidity vanishes, ($K_v = 0$), a static instability can occur. As mentioned in Section 5.1 this phenomenon can be defined as the "Divergence of the Vortex System", because it resembles the divergence of the analogous mechanical system (e.g. the aeroelastic divergence of an aircraft wing). The condition for the divergence of the vortex system is, according to Eqs. (97)

$$\left(\frac{\beta}{\ell}\bar{U}\right)^2 + a_{11}^2 + a_{12}a_{21} = 0 \qquad (101a)$$

Eq. (101a) shows that the divergence is strongly affected by the $\bar{U}$ component. Divergence can occur at every β for which $\bar{C}_v^2 < 0$, and the oscillating perturbations have a frequency $\alpha = \frac{\beta}{\ell}\bar{U} \pm |\bar{C}_v^2(\beta)|$ and tend to grow linearly in time, similarity to the "threshold of stability" case (Fig. 2.2). However on the specific case when $\alpha = 0$ (a steady forcing perturbation) divergence develops as a standing wave at a specific β_d that is given by,

$$\left(\frac{\beta_d}{\ell}U\right)^2 + \bar{C}_v^2(\beta_d) = 0 \qquad (101b)$$

and the amplitude of which tends to grow linearly in time (see Eqs. (100b) and Fig. 2.4). Eq. (101b) shows that divergence under a steady forcing is strongly affected by the U component of the external flow.

Eqs. (99)(101a) *also show that the characteristic* solution under divergence conditions is always dynamically stable, and is composed of the two waves that travel with group velocities of U_α and $2U+U_\alpha$. Eqs. (100) indicate that divergence is not just a local phenomenon at a specific β_d, but also that the disturbances tend to grow statically to infinite values in the vicinity of β_d, when the wave number β approaches this value.

It is interesting to note that in two-dimensional stability problems (for which $\beta \equiv 0$), or in three-dimensional problems when $\bar{U} \equiv 0$, the conditions for the stability threshold and for divergence coincide. This may explain why static instability was never before identified in any of the known two-dimensional or three-dimensional stability analyses.

In summary, when the basic state is steady, and when the vortex perturbation equations can be reduced to two-rank problems, the vortex system can suffer either a dynamic instability when the generalized damping is positive, $\bar{C}_v^2 > 0$, or a static divergence under an oscillating perturbation when the generalized damping is negative, $\bar{C}_v^2 < 0$ and the rigidity vanishes, $K_v = 0$!

The application of these new model and concepts to several basic vortex flows, (Rusak[14] and Rusak and Seginer[15,32,33]) demonstrates that a vortex system can tend to diverge statically, as well as to develop dynamic instabilities.

This work is, to the best of the author's knowledge, the first in which the divergence of a vortex system was identified as part of its stability characteristics. The divergence phenomenon in vortex systems has to be considered in all future vortex stability analyses.

6. The Control of Vortex Systems Stability

6.1 Control Theory Approach

When an equation for the output $\hat{\mathbf{Y}}_c$ of the measurement of $\hat{\mathbf{Y}}$ at every β:

$$\hat{\mathbf{Y}}_c(\beta,t) = C(\beta,t)\hat{\mathbf{Y}}(\beta,t) \qquad (102)$$

is added, Eqs. (73) and (102) have the general form of the basic equations of the classic linear control theory (Kwakernaak and Sivan[28]), where $\hat{\mathbf{Y}}$ is the state-variable vector, $A(\beta,t)$ is the system matrix, $\hat{\mathbf{r}}(\beta,t)$ is the input or control vector, $B(\beta,t)$ is the control matrix, $\hat{\mathbf{Y}}_c$ is the measurement vector and $C(\beta,t)$ is the measurement matrix. Following Kwakernaak and Sivan[28], the concepts of "Controllability" and "Observability" can now be defined:

a) A vortex system is defined as "controllable" at a certain wave number β, if the disturbances state vector $\hat{\mathbf{Y}}(\beta,t)$ can be transferred from its initial state at any time t_0, to any terminal state at time t_1, within a finite time interval $t_1 - t_0$.

b) A vortex system is defined as "observable" (or "reconstructible") at a certain wave number β, if for every time t_1, there exists a time t_0 within the interval $-\infty < t_0 < t_1$, such that for all $\hat{\mathbf{r}}(\beta,t)$ and $t_0 \le t \le t_1$, the condition $\hat{\mathbf{Y}}_c(t,t_0,\hat{Y}_0(\beta,t_0),\hat{\mathbf{r}}) = \hat{\mathbf{Y}}_c(t,t_0,\hat{\mathbf{Y}}_0'(\beta,t_0),\hat{\mathbf{r}})$ implies that $\hat{\mathbf{Y}}_0 = \hat{\mathbf{Y}}_0'$ (where $\mathbf{Y}_0(\beta,t_0)$ is the initial condition at time t_0).

Controllability means that the perturbations along the vortex lines can, for a given β, be steered from one given state to any other state. It can be shown[14] that a vortex system that is steady in its basic state, is controllable at a certain wave number β if, and only if, the rank $[P(\beta)]$ of the controllability matrix $P(\beta)$ equals $2N$, where

$$P(\beta) = [B, AB, \cdots, A^{2N-1}B] \qquad (103)$$

In this case, the unstable modes of the disturbances to the vortex filaments, can theoretically be stabilized. Following Kwakernaak and Sivan[28] a controllability criterion can be found also in the general case where the basic state is unsteady (Rusak[14]).

Observability means that the behaviour of all the disturbances state vector $\hat{\mathbf{Y}}$ can be determined by the behaviour of the output-measurement vector $\hat{\mathbf{Y}}_c$. Observability also includes the determination of the minimum number of parameters that have to be measured for the reconstruction of the behaviour of the perturbations. It can be shown[14] that a vortex system that is steady in its basic state, is observable if, and only if, the rank $[Q(\beta)]$ of the observability matrix $Q(\beta)$ equals $2N$, where

$$Q(\beta) = [C^T, A^T C^T, \cdots (A^{2N-1})^T C^T] \qquad (104)$$

Following Kwakernaak and Sivan[28] an observability criterion can be derived also in the general case where the basic state is unsteady (Rusak[14]).

When the problem described by Eq. (73) can be reduced to a two-rank problem, the time-invariant matrices $A(\beta)$ and $B(\beta)$ are 2×2-dimensional and the matrix $C(\beta)$ is 1×2-dimensional. The terms in Eq. (73) are described by Eqs. (97) and those in Eq. (102) are given by,

$$\hat{\mathbf{Y}}_c = [\hat{y}_{c1}], \qquad C = [C_1, C_2] \qquad (105)$$

Sufficient conditions for the controllability of this vortex system at a certain wave number β are,

$$\det B(\beta) = b_{11}b_{22} - b_{12}b_{21} \neq 0$$

or

$$a_{21}b_{11}^2 - 2a_{11}b_{11}b_{21} - a_{12}b_{21}^2 \neq 0 \qquad (106)$$

and the condition for observability of this vortex system at a certain wave number β, by measuring one parameter $\hat{y}_{c1}$ only, is:

$$\det Q(\beta) = a_{12}C_1^2 - 2a_{11}C_1C_2 - a_{21}C_2^2 \neq 0 \qquad (107)$$

6.2 Active Control of Vortex Stability

The new control-theory approach to vortex-system stability that has presented above, introduces the possibility of actively controlling vortex stability by two basic classic methods.

The first method of controlling a vortex-system stability is in an open-loop mode. This method is derived directly from Eq. (73), and uses an apriori known surface oscillation $R_1(x_R, t)$ to control the development of the disturbances along the vortex system. When the basic vortex state is steady, or quasi-steady, a Laplace transform of Eq. (73) gives the following transfer function for each β:

$$\hat{\mathbf{Y}}(\beta, S) = [SI - A(\beta)]^{-1} \left\{ B(\beta) \begin{bmatrix} 1 \\ S \end{bmatrix} \hat{R}_1(\beta, S) + \hat{\mathbf{Y}}_0(\beta) + \hat{\mathbf{d}}(\beta, S) \right\} \qquad (108)$$

where S is the Laplace-transform variable. A description of the open-loop scheme in this case is shown in Fig. 3 Such a control method can be applied only at wave numbers β when the system is controllable according to the criteria defined above.

The second control method is in a closed-loop feedback mode (Fig. 4), that can be derived from Eqs. (73) and (102), together with an appropriate control-law in the general form of:

$$\hat{R}_1(\beta, t) = H\{\hat{\mathbf{Y}}_c(\beta, t)\} \qquad (109)$$

where H is a control operator that acts on the output vector $\hat{\mathbf{Y}}_c(\beta, t)$. In this method the vortex disturbances $\hat{\mathbf{Y}}(\beta, t)$ are measured (Eq. (102)) at each wave number β. The resulting measured data $\hat{\mathbf{Y}}_c(\beta, t)$ are used by the control operator in a feed-back loop to activate surface oscillations $\hat{R}_1(\beta, t)$ in order to control the vortex disturbances vector $\hat{\mathbf{Y}}(\beta, t)$.

When the basic vortex state is steady, or quasi-steady, and when H is a time-invariant linear control operator, a Laplace transform of Eqs. (73), (102) and (109) results, at each β, in:

$$\begin{aligned} S\hat{\mathbf{Y}}(\beta, S) &= A(\beta)\hat{\mathbf{Y}}(\beta, S) + B(\beta)\hat{\mathbf{r}}(\beta, S) + \hat{\mathbf{Y}}_0(\beta) + \hat{\mathbf{d}}(\beta, S) \\ \hat{\mathbf{Y}}_c(\beta, S) &= C(\beta)\hat{\mathbf{Y}}(\beta, S) \\ \hat{R}_1(\beta, S) &= H(\beta)\hat{\mathbf{Y}}_c(\beta, S) \\ \hat{\mathbf{r}}(\beta, S) &= \begin{bmatrix} 1 \\ S \end{bmatrix} \hat{R}_1(\beta, S) \end{aligned} \qquad (110)$$

The controlled vortex disturbances are characterized by the eigenvalue of the controlled matrix,

$$A_c(\beta) = [I - \mathbf{b}_2(\beta)H(\beta)C(\beta)]^{-1}[A(\beta) + \mathbf{b}_1(\beta)H(\beta)C(\beta)] \qquad (111)$$

where $\mathbf{b}_1(\beta), \mathbf{b}_2(\beta)$ are the column vectors of matrix $B(\beta) : B(\beta) = [\mathbf{b}_1(\beta) | \mathbf{b}_2(\beta)]$.

The closed-loop feed-back control can be applied only at wave numbers β when the system is controllable and observable according to the criteria defined above.

6.3 Summary

The linear equations, that describe the linear three-dimensional stability of vortex systems near a parallel surface have the general form of the classic linear control equations,with unsteady surface oscillations as the controller. "Controllability" and "Observability" of a vortex-system stability are defined by the classic criteria of the linear control theory. Criteria for the analysis of the control of vortex systems, result in a proposed method of actively controlling the vortex stability. The methods of open-loop control and of closed-loop with feed-back control are presented for vortex svstems. The examples shown in Rusak[14] and Rusak and Seginer[16,32,33] demonstrate the powerful potential of the new combined theories of vortex stability and control. Using active solid-surface oscillations, a control law is proposed for the closed-loop method, to suppress or to amplify at will the well-known Crow instability of a vortex moving near a straight plane (see Rusak and Seginer[16]).

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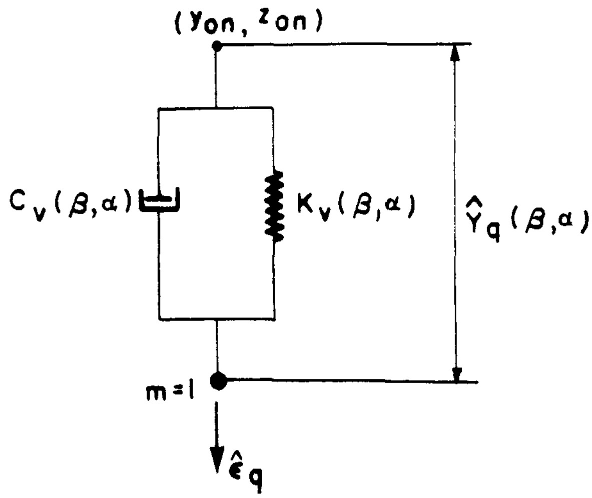

Fig. 1. The physical analogy for the perturbation development.

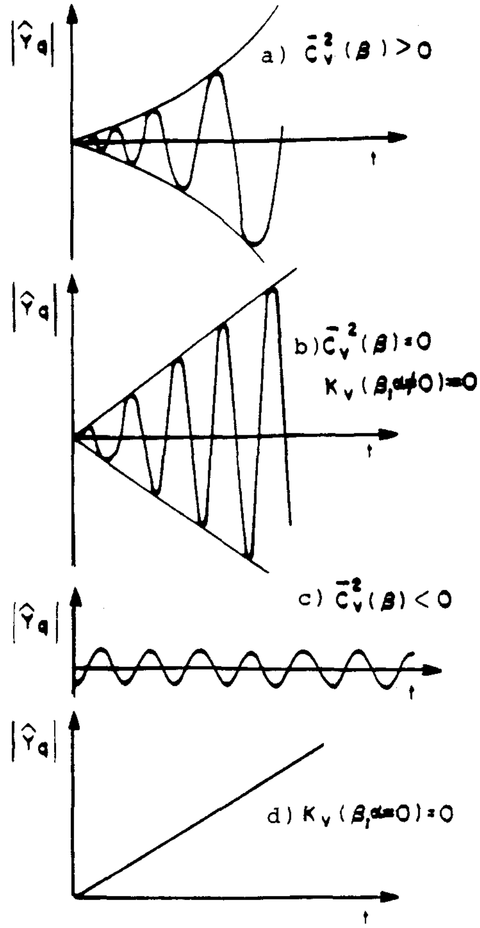

Fig. 2. Development of perturbations, several modes:

a) positive generalized damping

b) zero generalized damping

c) negative generalized damping

d) zero rigidity.

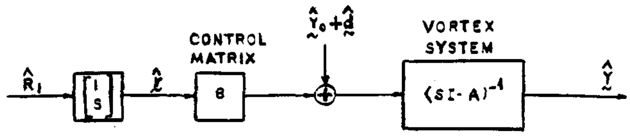

Fig. 3. Open-Loop Control.

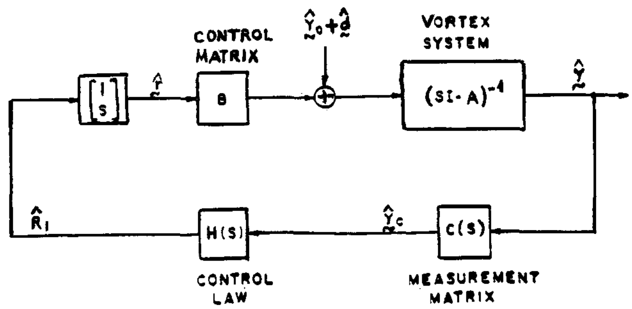

Fig. 4. Closed-Loop Feed-Back Control.