VIBRATION ANALYSIS OF STIFFENED CIRCULAR CYLINDRICAL THIN SHELLS

Z. Mecitoğlu* and M.C. Dökmeci**
Istanbul Technical University,
Maslak, Istanbul 80626, Turkey

Abstract

This paper is addressed to the free vibrations of stiffened circular cylindrical thin shells within the frame of Love's first approximation theory of elastic shells. The circular cylindrical thin shells with shear diaphragm ends is reinforced at uniform intervals by elastic stringers and rings. The effects of stiffeners are taken into account by the orthotropic material approach of stiffened shells. The governing equations are deduced from the three-dimensional equations of elastodynamics by means of Hamilton's principle together with the usual kinematic hypothesis of circular cylindrical thin shells. The governing equations are simplified for various special cases involving the material and geometrical properties of stiffened cylindrical thin shells. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells. Moreover, the stiffened shell is discretized by Semiloof shell elements and a matrix equation is obtained by means of the variational equation to determine the vibration characteristics of the stiffened shells. The influence of geometrical parameters and material properties of stiffened circular cylindrical shells is investigated on the vibration characteristics. Numerical results are plotted with respect to the parameters of stiffened cylindrical shells, and they are compared with certain experimental results.

1. Introduction

It is required that an aerospace vehicle must be light weight. In order to overcome this problem, the designer usually resorts to the use of thin shells reinforced by the stiffeners, because of their high structural efficiency. Particularly, circular cylindrical shells reinforced by stringers and rings are used in the structure of various aerospace vehicles.

The stiffener members, representing a relatively small part of the total weight of a structure, substantially influence its dynamics behavior, stability, stiffeness, and strength. Hence, the prediction of dynamic characteristics of stiffened circular cylindrical shells is very important for determination of inflight behavior, fatigue life, noise generation of the aerospace vehicle and vibration isolation of the sensitive electronic instrumentation and on-board computers.

Numerous studies examining free vibration characteristics have been conducted on elastic stiffened circular cylindrical thin shells. Eggle and Sewall[1] studied the vibration of orthogonally stiffened cylindrical shells with discrete axial stiffeners by the Ritz method. Bushnell[2] evaluated various analytical models for vibrations of stiffened shells. The free vibrations of a thin cylindrical shell have been investigated for discrete axial and circumferential stiffeners by Mead and Bardell[3]. Mustafa and Ali determined the frequencies of ring stiffened, stringer stiffened and orthogonally stiffened shell using super shell finite elements[4] and a formulation of energy functional[5]. The free vibration characteristics of stiffened circular cylindrical shells have been studied numerically using the finite element method by Mecitoğlu[6]. Although the uniqueness in solutions of the elastodynamic problems is very important, only a few work is found in literature examining with this subject[7].

The purpose of this paper is (i) to derive all the governing equations for vibrations of cylindrical thin shells reinforced by stringers and rings, (ii) to examine the uniqueness in solutions of the governing shell equations, and (iii) to study numerically the vibration characteristics of some special cases.

The dynamic governing equations of the stiffened circular cylindrical shells are derived within the frame of Love's first approximation theory of elastic thin shells. By means of Hamilton's principle together with the usual kinematic hypothesis of cylindrical thin shells, all the governing equations are deduced in a systematic manner from the three-dimensional equations of elastodynamics under the well-known assumptions of regularity and smoothness of field variables. The stiffeners are taken to be along the usual cylindrical coordinates and their dimensions to be small compared to the radius of cylindrical shell. The effects of stiffeners are taken into account by the orthotropic material approach of stiffened shells. The governing equations are simplified for various special cases involving material and geometrical properties of stiffened circular cylindrical thin shells. The uniqueness is examined in solutions of the dynamic governing equations of stiffened shells, and a theorem of uniqueness is given which enumerates the initial and boundary conditions sufficient for the uniqueness.

The free vibrations of a stiffened circular cylindrical thin shell with shear diaphragm ends are numerically studied by the finite element method. The stiffened shell is discretized by Semiloof shell elements. Then, by means of the variational equation, the matrix equation is obtained to determine the vibration characteristics of the stiffened shell. The natural frequencies and mode shapes of the stiffened circular cylindrical shell are determined, and their accuracy is tested with earlier experimental results[8]. The influence of geometrical parameters and material properties of the stiffened cylindrical shell is investigated on the vibration characteristics. In addition, the effect of the spacings and dimensions of stiffeners is studied. The numerical results are plotted with respect to the parameters of the stiffened cylindrical shell.

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* Assistant Professor, Aeronautics and Astronautics Faculty.
** İstanbul Teknik Üniversitesi, P.K.9, Taksim, 80191 İstanbul, Turkey.

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3. Governing Equations

The membran strains in the middle surface of the thin circular cylindrical shell can be written as

$$e_x = \frac{\partial u_x}{\partial x} \quad e_y = \frac{1}{R}\left(\frac{\partial u_\theta}{\partial \theta} + w\right) \quad e_{x\theta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x}\right) \quad (1a)$$

where u_x, u_θ, and w denote the displacements of a point on the shell middle surface in the coordinate system x, θ, and z; R the radius of cylindrical shell.

It is assumed by Donnell[9] that the bending strains of a thin cylindrical shell are negligibly affected by the strectching displacement u_θ. Then the bending strains of a thin circular cylidrical shell can be expressed

$$k_x = -\frac{\partial^2 w}{\partial x^2} \quad k_\theta = -\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} \quad k_{x\theta} = \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \theta} \quad (1b)$$

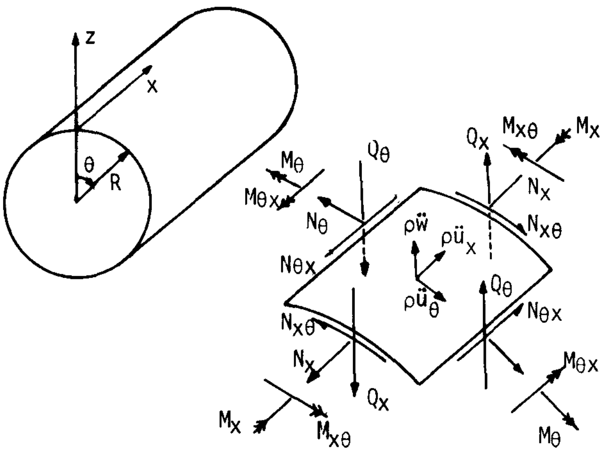

Figure 1 Circular cylindrical shell.

The force and moment resultants are related to the membran and bending strains by

$$N_x = C(e_x + \nu e_\theta) \quad N_\theta = C(e_\theta + \nu e_x) \quad N_{x\theta} = N_{\theta x} = 2Ghe_{x\theta}$$
$$M_x = D(k_x + \nu k_\theta) \quad M_\theta = D(k_\theta + \nu k_x) \quad M_{x\theta} = M_{\theta x} = \frac{Gh^3}{6}k_{x\theta} \quad (2)$$

where $C = Eh/(1-\nu^2)$ is the streching rigidity of shell, $D = Eh^3/12(1-\nu^2)$ is the bending rigidity of shell and h is the thickness of shell. Here E and ν denote the modulus of elasticity and Poisson's ratio, respectively.

The specific strain energy of a thin circular cylindrical shell can be expressed in terms of the force and moment resultants and the strains as follows

$$W_c = \frac{1}{2}(N_x e_x + N_\theta e_\theta + 2N_{x\theta}e_{x\theta} + M_x k_x + M_\theta k_\theta + 2M_{x\theta}k_{x\theta}) \quad (3a)$$

The specific strain energy of the shell is related only to the strains by using the equations (2).

$$W_c = \frac{1}{2}C[e_x^2 + 2\nu e_x e_\theta + e_\theta^2 + 2(1-\nu)e_{x\theta}^2] + \frac{1}{2}D[k_x^2 + 2\nu k_x k_\theta + k_\theta^2 + 2(1-\nu)k_{x\theta}^2] \quad (3b)$$

The strain energy of a single stringer per unit length of the stringer centroidal axis can be approximated by

$$U_s \approx \frac{1}{2}E_s\iint_{A_s}[e_x - \zeta k_x]^2 dA + \frac{1}{2}(GJ)_s k_{x\theta}^2 \quad (4a)$$

where E_s, A_s, and G_sJ_s are the modulus of elasticity, cross-sectional area, and torsional stiffness of a stringer, respectively; ζ is the distance measured along the normal to the shell middle surface (Fig.2)

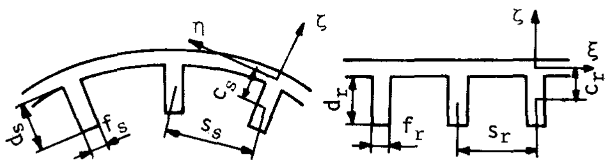

Figure 2 Stringers and rings.

On carrying out the integrations and dividing U_s by the stringer spacing s_s we obtain the specific strain energy of a stringer per unit area of the shell middle surface:

$$W_s \approx \frac{1}{2}\left(\frac{EA}{s}\right)_s e_x^2 - \left(\frac{EAc}{s}\right)_s e_x k_x + \frac{1}{2}\left(\frac{EI_\eta}{s}\right)_s k_x^2 + \frac{1}{2}\left(\frac{GJ}{s}\right)_s k_{x\theta}^2 \quad (4b)$$

where I_η is the moment of inertia of a stringer about the η axis and c_s is distance to the centroid from the shell middle surface (eccentricity)(Fig.2).

A similar expression can be written for the rings

$$W_r \approx \frac{1}{2}\left(\frac{EA}{s}\right)_r e_\theta^2 - \left(\frac{EAc}{s}\right)_r e_\theta k_\theta + \frac{1}{2}\left(\frac{EI_\xi}{s}\right)_r k_\theta^2 + \frac{1}{2}\left(\frac{GJ}{s}\right)_r k_{x\theta}^2 \quad (5)$$

By adding the specific strain energy of the stringer and ring to that of the circular cylindrical shell, we obtain the total specific strain energy of the stiffened circular cylindrical shell.

$$W \approx W_c + W_s + W_r = E_1 e_x^2 + E_{12}e_x e_\theta + E_2 e_\theta^2 + G_1 e_{x\theta}^2 + F_1 k_x^2 + F_{12}k_x k_\theta + F_2 k_\theta^2 + G_2 k_{x\theta}^2 - H_1 e_x k_x - H_2 e_\theta k_\theta \quad (6)$$

where the denotations by

$$\{E_1, E_2, E_{12}\} = \frac{Eh}{2(1-\nu^2)}\{1,1,2\nu\} + \frac{1}{2}\left\{\left(\frac{EA}{s}\right)_s, \left(\frac{EA}{s}\right)_r, 0\right\}$$

$$\{F_1, F_2, F_{12}\} = \frac{Eh^3}{24(1-\nu^2)}\{1,2,1\} + \frac{1}{2}\left\{\left(\frac{EI_\eta}{s}\right)_s, \left(\frac{EI_\xi}{s}\right)_r, 0\right\}$$

$$G_1 = 2Gh, \quad G_2 = \frac{Gh^3}{6} + \frac{1}{2}\left(\frac{GJ}{s}\right)_s + \frac{1}{2}\left(\frac{GJ}{s}\right)_r$$

$$H_1 = \left(\frac{EAc}{s}\right)_s, \quad H_2 = \left(\frac{EAc}{s}\right)_r$$

Substituting the equations (1a) and (1b) into the equation (6) and integrating over the middle surface of the circular cylindrical shell, we obtain the strain energy of stiffened shell in the form

$$U \approx \iint_A [E_1(\frac{\partial u_x}{\partial x})^2 + \frac{E_{12}}{R}\frac{\partial u_x}{\partial x}(\frac{\partial u_\theta}{\partial \theta} + w) + \frac{E_2}{R^2}(\frac{\partial u_\theta}{\partial \theta} + w)^2$$
$$+ \frac{G_1}{4}(\frac{\partial u_\theta}{\partial x} + \frac{1}{R}\frac{\partial u_x}{\partial \theta})^2 + F_1(\frac{\partial^2 w}{\partial x^2})^2 + \frac{F_{12}}{R^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2}$$
$$+ \frac{F_2}{R^4}(\frac{\partial^2 w}{\partial \theta^2}) + \frac{G_2}{R^2}(\frac{\partial^2 w}{\partial x \partial \theta})^2 + H_1\frac{\partial u_x}{\partial x}\frac{\partial^2 w}{\partial x^2}$$
$$+ \frac{H_2}{R^3}(\frac{\partial u_\theta}{\partial \theta} + w)\frac{\partial^2 w}{\partial \theta^2}\}dA \qquad (6a)$$

The kinetic energy of stiffened cylindrical shell may be expressed as

$$K = \frac{1}{2}m\iint_A(\dot{u}_x^2 + \dot{u}_\theta^2 + \dot{w}^2)dA \qquad (7)$$

Here, $m = \rho h_e$ is the average smeared-out mass per unit area of the stiffened circular cylindrical shell; ρ is the material densty and h_e is the equivalent thickness of stiffened shell.

Now, the governing equations and boundary conditions of the stiffened circular cylindrical shell are derived from Hamilton's principle

$$\delta\int_T(U - K)dt = 0 \qquad (8)$$

by allowing the variations of the three displacements u_x, u_θ, and w to be arbitrary. Here $T = [t_0, t_1)$. Hence, we obtain

$$\tau_x = 2E_1\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{R}(E_{12} + \frac{G_1}{2})\frac{\partial^2 u_\theta}{\partial x \partial \theta} + \frac{G_1}{2R^2}\frac{\partial^2 u_x}{\partial \theta^2}$$
$$+ H_1\frac{\partial^3 w}{\partial x^3} + \frac{E_{12}}{R}\frac{\partial w}{\partial x} - ma_x = 0 \qquad (9a)$$

$$\tau_\theta = \frac{2E_2}{R^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{1}{R}(E_{12} + \frac{G_1}{2})\frac{\partial^2 u_x}{\partial x \partial \theta} + \frac{G_1}{2}\frac{\partial^2 u_\theta}{\partial x^2}$$
$$+ \frac{H_2}{R^3}\frac{\partial^3 w}{\partial \theta^3} + \frac{2E_2}{R^2}\frac{\partial w}{\partial \theta} - ma_\theta = 0 \qquad (9b)$$

$$\tau_w = 2F_1\frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}(F_{12} + G_2)\frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{2F_2}{R^4}\frac{\partial^4 w}{\partial \theta^4}$$
$$+ \frac{2H_2}{R^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{2E_2}{R^2}w + H_1\frac{\partial^3 u_x}{\partial x^3} + \frac{E_{12}}{R}\frac{\partial u_x}{\partial x}$$
$$+ \frac{H_2}{R^3}\frac{\partial^3 u_\theta}{\partial \theta^3} + \frac{2E_2}{R^2}\frac{\partial u_\theta}{\partial \theta} - ma = 0 \qquad (9c)$$

For a cylindrical shell with shear diaphragm ends, the solutions must satisfy the boundary conditions

$$u_\theta = w = M_x = N_x = 0 \qquad \text{at } x = 0, L \qquad (10)$$

and the initial contitions[10]

$$u_x = u_\theta = w = 0$$
$$\dot{u}_x = \dot{u}_\theta = \dot{w} = 0 \qquad \text{at } t = 0 \qquad (11)$$

4. Uniqueness of Solutions

In the previous section, a set of two-dimensional, differential, approximate equations of Donnel's type is derived for the dynamic response of a cylindrical elastic shell with stringers and rings. The two-dimensional governing equations of stiffened cylindrical shell are constructed by use of Hamilton's principle within the limits of the well-known Kirchhoff-Love hypothes of thin shells. Now, the boundary and initial conditions are obtained which are sufficient to ensure the uniqueness in solutions of the dynamical governing equations. Of the several arguments to be used to establish the uniqueness of solutions in elasticity[8], the clasical energy argument is used. The energy argument relies upon the positive-definiteness of strain and kinetic energies. Kirchhoff[11] used the energy argument at establishing uniqueness in elastostatics, so did Neumann[12] in elastodynamics and Weiner[13] in thermoelasticity. A uniqueness theorem of Neumann's type is proved for solutions of the initial mixed-boundary value problems defined by the two-dimensional governing equations of stiffened cylindrical shell (cf.[14] for elastic shells).

To begin with, consider two possible sets of solutions to the governing equations of stiffened cylindrical shell, namely,

$$\Lambda = (u_x, u_\theta, w;\ e_x, e_\theta, e_{x\theta}, k_x, k_\theta, k_{x\theta};\ N_x, N_\theta, N_{x\theta}, M_x, M_\theta, M_{x\theta}), \qquad \alpha = 1,2 \qquad (12)$$

Let the difference set of two solutions be denoted by

$$\Lambda = \Lambda_2 - \Lambda_1.$$

The difference set of solutions apparently satisfies all the governing equations of stiffened cylindrical shell due to their linearity. It will be shown that the homogeneous linear governing equations possess only the zero solution, that is, the two sets of solutions (12) are equivalent under the pertinent boundary and initial conditions. In so doing, we introduce a relation of the form

$$\Gamma = \int_T(\Gamma_x + \Gamma_\theta + \Gamma_w)dt = 0 \qquad (13)$$

with

$$\Gamma_x = \iint_A \tau_x \dot{u}_x dA \qquad \Gamma_\theta = \iint_A \tau_\theta \dot{u}_\theta dA \qquad \Gamma_w = \iint_A \tau_w \dot{w} dA$$

where τ_x, τ_θ, and τ_w are defined by the equations (9a-c).

Now, let us calculate the rates of the kinetic and strain energies of the stiffened cylindrical shell in terms of the displacement components. The rate of the kinetic energy is expressed with respect to the difference set of solutions in the form

$$\dot{K} = \iint_A m(a_x\dot{u}_x + a_\theta\dot{u}_\theta + a\dot{w})dA \qquad (14)$$

In this equation, $a_x(=\ddot{u}_x)$, a_θ and a are the components of acceleration, and the equation (7) are used.

Likewise, the rate of the total strain energy is obtained by use of the equation (6a) as follows

$$\dot{U} = \iint_A \{[2E_1 \frac{\partial u_x}{\partial x} + \frac{E_{12}}{R}(\frac{\partial u_\theta}{\partial \theta} + w) + H_1 \frac{\partial^2 w}{\partial x^2}] \frac{\partial \dot{u}_x}{\partial x}$$
$$+ \frac{G_1}{2R}(\frac{\partial u_\theta}{\partial x} + \frac{1}{R}\frac{\partial u_x}{\partial \theta}) \frac{\partial \dot{u}_x}{\partial \theta} + \frac{1}{R}[E_{12}\frac{\partial u_x}{\partial x} + \frac{2E_2}{R}(\frac{\partial u_\theta}{\partial \theta} + w)$$
$$+ \frac{H_2}{R^2}\frac{\partial^2 w}{\partial \theta^2}] \frac{\partial \dot{u}_\theta}{\partial \theta} + \frac{G_1}{2}(\frac{\partial u_\theta}{\partial x} + \frac{1}{R}\frac{\partial u_x}{\partial \theta}) \frac{\partial \dot{u}_\theta}{\partial x}$$
$$+ \frac{1}{R}[E_{12}\frac{\partial u_x}{\partial x} + \frac{2E_2}{R}(\frac{\partial u_\theta}{\partial \theta} + w) + \frac{H_2}{R^2}\frac{\partial^2 w}{\partial \theta^2}] \dot{w}$$
$$+ (2F_1 \frac{\partial^2 w}{\partial x^2} + \frac{F_{12}}{R^2}\frac{\partial^2 w}{\partial \theta^2} + H_1 \frac{\partial u_x}{\partial x}) \frac{\partial^2 \dot{w}}{\partial x^2}$$
$$+ \frac{1}{R^2}[F_{12}\frac{\partial^2 w}{\partial x^2} + \frac{2F_2}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{H_2}{R}(\frac{\partial u_\theta}{\partial \theta} + w)] \frac{\partial^2 \dot{w}}{\partial \theta^2}$$
$$+ (\frac{2G_2}{R^2}\frac{\partial^2 w}{\partial x \partial \theta}) \frac{\partial^2 \dot{w}}{\partial x \partial \theta} \}dA \qquad (15)$$

in terms of the difference set of solutions. Integrating this equation by parts, one arrives at the rate of the form

$$\dot{U} = -\iint_A \{[2E_1 \frac{\partial^2 u_x}{\partial x^2} + \frac{E_{12}}{R}(\frac{\partial^2 u_\theta}{\partial x \partial \theta} + \frac{\partial w}{\partial x})$$
$$+ \frac{G_1}{2R}(\frac{\partial^2 u_\theta}{\partial x \partial \theta} + \frac{1}{R}\frac{\partial^2 u_x}{\partial \theta^2}) + H_1 \frac{\partial^3 w}{\partial x^3}] \dot{u}_x$$
$$+ \frac{1}{R}[\frac{2E_2}{R}(\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial w}{\partial \theta}) + E_{12}\frac{\partial^2 u_x}{\partial x \partial \theta} + \frac{H_2}{R^2}\frac{\partial^3 w}{\partial \theta^3}$$
$$+ \frac{G_1}{2}(\frac{\partial^2 u_\theta}{\partial x^2} + \frac{1}{R}\frac{\partial^2 u_x}{\partial x \partial \theta})] \dot{u}_\theta$$
$$+ [2F_1 \frac{\partial^4 w}{\partial x^4} + \frac{2}{R^2}(F_{12} + G_2)\frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{2F_2}{R^4}\frac{\partial^4 w}{\partial \theta^4}$$
$$+ \frac{H_2}{R^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{2E_2}{R^2}(\frac{\partial u_\theta}{\partial \theta} + w) + H_1 \frac{\partial^3 u_x}{\partial x^3}$$
$$+ \frac{E_{12}}{R}\frac{\partial u_x}{\partial x} + \frac{H_2}{R^3}(\frac{\partial^3 u_\theta}{\partial \theta^3} + \frac{\partial^2 w}{\partial \theta^2})] \dot{w} \}R d\theta dx$$
$$+ \oint_C \Psi_\theta R d\theta + \int_L \Psi_x dx \qquad (16)$$

In this equation, the quantities of the form

$$\Psi_\theta = \{[2E_1 \frac{\partial u_x}{\partial x} + \frac{E_{12}}{R}(\frac{\partial u_\theta}{\partial \theta} + w) + H_1 \frac{\partial^2 w}{\partial x^2}] \dot{u}_x$$
$$+ \frac{G_1}{2}(\frac{\partial u_\theta}{\partial x} + \frac{1}{R}\frac{\partial u_x}{\partial \theta}) \dot{u}_\theta$$
$$- [(\frac{2G_2}{R} + \frac{F_{12}}{R^2})\frac{\partial^3 w}{\partial x \partial \theta^2} + 2F_1 \frac{\partial^3 w}{\partial x^3} + H_1 \frac{\partial^2 u_x}{\partial x^2}] \dot{w}$$
$$+ (2F_1 \frac{\partial^2 w}{\partial x^2} + \frac{F_{12}}{R^2}\frac{\partial^2 w}{\partial \theta^2} + H_1 \frac{\partial u_x}{\partial x}) \frac{\partial \dot{w}}{\partial x} \} \Big|_{x=0}^{L}$$

and

$$\Psi_x = \{\frac{G_1}{2}(\frac{\partial u_\theta}{\partial x} + \frac{1}{R}\frac{\partial u_x}{\partial \theta}) \dot{u}_x$$
$$+ [E_{12}\frac{\partial u_x}{\partial x} + \frac{2E_2}{R}(\frac{\partial u_\theta}{\partial \theta} + w) + \frac{H_2}{R^2}\frac{\partial^2 w}{\partial \theta^2}] \dot{u}_\theta$$
$$- \frac{1}{R}[F_{12}\frac{\partial^3 w}{\partial x^2 \partial \theta} + \frac{2F_2}{R^2}\frac{\partial^3 w}{\partial \theta^3} + \frac{H_2}{R}(\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial w}{\partial \theta})] \dot{w}$$
$$+ \frac{1}{R}[F_{12}\frac{\partial^2 w}{\partial x^2} + \frac{2F_2}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{H_2}{R}(\frac{\partial u_\theta}{\partial \theta} + w)]\frac{\partial \dot{w}}{\partial \theta}$$
$$+ \frac{2G_2}{R}\frac{\partial^2 w}{\partial x \partial \theta}\frac{\partial \dot{w}}{\partial x} \} \Big|_{\theta=0}^{2\pi} \qquad (17)$$

are introduced.

In view of the energy rates (14) and (16), the relation (13) is expressed as

$$\Gamma = -\int_T (\dot{K} + \dot{U})dt + \int_T dt(\oint_C \Psi_\theta R d\theta + \int_L \Psi_x dx) = 0 \qquad (18)$$

An integration of this equation with respect to time yields

$$K(t_1) + U(t_1) = K(t_0) + U(t_0) + \int_T \Psi dt \qquad (19)$$

with

$$\Psi = \oint_C \Psi_\theta R d\theta + \int_L \Psi_x dx \qquad (20)$$

The kinetic and strain energy densities are positive-definite, by definition, and initially zero; so that the total kinetic energy and strain energy, K and U, calculated by integration from the difference set of solutions for the stiffened cylindrical elastic shell have the same properties. Thus, it follows from the equation (19) that

$$K(t_1) = U(t_1) = K(t_0) = U(t_0) = 0 \qquad (21)$$

This implies a trivial solution for the difference set of solutions, Λ, since the remaining term Ψ of the equation (19) vanishes in view of the equation (11). Hence, the uniqueness is ensured in solutions of the governing equations of stiffened cylindrical elastic shell. The following theorem of uniqueness is then concluded.

Theorem - Given the regular region of a stiffened cylindrical elastic shell in the Eucliden three-dimensional space, then there exists at most one set of single-valued solutions, Λ_c, namely,

$$\Lambda_c = [u_x, u_\theta \text{ and } w \in C_{12};\ e_x, e_\theta, e_{x\theta} \text{ and } k_x, k_\theta, k_{x\theta} \in C_{00};$$
$$N_x, N_\theta, N_{x\theta}, M_x, M_\theta, M_{x\theta} \in C_{10}]$$

which satisfies all the governing equations of the stiffened shell, provided that the kinetic and strain energies are positive-definite, and the boundary (10) and initial conditions (11) are prescribed. C_{mn} refers to the function with derivatives of order up to and including (m) and (n) with respect to space coordinates (x,θ) and time t.

5. Method of Solution

In this section some properties of the Semiloof element developed by Irons[15] are briefly reviewed. Figure 3 shows the Semiloof cylindrical shell element where the global coordinate system (x, θ, z), isoparametric curvilinear

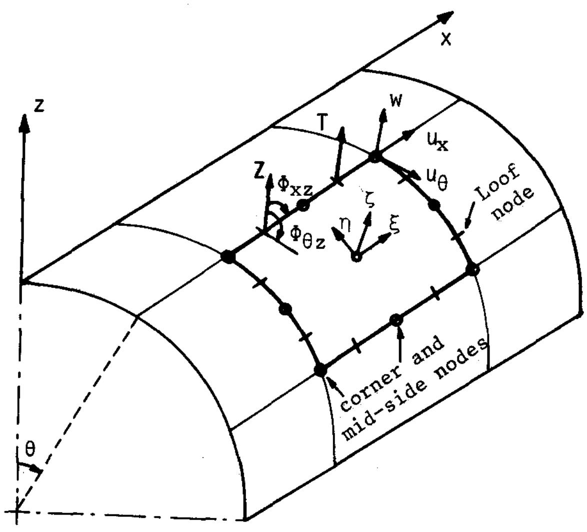

Figure 3 Semiloof cylindrical shell element

coordinate system (ξ, η, ζ) are depicted. Semiloof element has three types of nodes: the corner and midside nodes, Loof nodes, and central node. The displacements u_x, u_θ, and w are defined as variables at the corner and midside nodes, and central node. In addition, the rotations Φ_{xz} and $\Phi_{\theta z}$ are defined at the Loof nodes and the central node. Hence, an element includes 45 degrees of freedom (d.o.f.). Two d.o.f. are eliminated from the element combining the displacements at the central node to obtain only a normal deflection. Then, applying the Kirchhoff-Love hypothesis of thin shells in discrete manner[15], the number of freedoms are reduced to 32 d.o.f.

Semiloof element is an isoparametric element which adopts 8-noded parabolic model. The shape functions are defined in terms of the curvilinear coordinate systems (ξ,η) as

$$N_i = \frac{1}{4}(1+\xi_0)(1+\eta_0)(\xi_0+\eta_0-1) \quad \text{at corner nodes}$$

$$N_i = (1-\xi^2)(1+\eta_0), \quad \xi_i = 0$$
$$N_i = (1-\eta^2)(1+\xi_0), \quad \eta_i = 0 \quad \text{at mid-side nodes}$$

$$L_i = \frac{3}{32}(3\xi^2-\eta^2)+\frac{1}{8}[3\xi_0(1-\eta^2) + 3\eta_0\{3\xi^2+\xi_0-1+\frac{3}{2}\xi_0(\xi^2-\eta^2)\}] \quad \xi_i = \pm 1 \tag{22}$$
$$L_i = \frac{3}{32}(3\eta^2-\xi^2)+\frac{1}{8}[3\eta_0(1-\xi^2) + 3\xi_0\{3\eta^2+\eta_0-1+\frac{3}{2}\eta_0(\eta^2-\xi^2)\}] \quad \eta_i = \pm 1$$
$$N_c = (1-\xi^2)(1-\eta^2) \quad \text{at the central node}$$

where L_i stand for the shape functions at the Loof nodes.

The global displacements **q** of a point P are related to the nodal displacements $\mathbf{Q}^e$ by means of the shape functions:

$$\mathbf{q} = \{u_x, u_\theta, w\}^T = \mathbf{N}\mathbf{Q}^e \tag{23}$$

The derivatives in the middle surface are obtained easily as

$$\frac{\partial \mathbf{q}}{\partial x} = \frac{\partial \mathbf{N}}{\partial x}\mathbf{Q}^e, \quad \frac{\partial \mathbf{q}}{\partial \theta} = \frac{\partial \mathbf{N}}{\partial \theta}\mathbf{Q}^e \tag{24a}$$

It is used to the following equation to get the x, θ derivatives of the shape functions

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial \theta}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial \theta}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \tag{25}$$

The derivative $\partial u_x/\partial z$ are given by

$$\frac{\partial u_x}{\partial z} = (\frac{\partial u_x}{\partial z})^N + (\frac{\partial u_x}{\partial z})^L = \frac{1}{h}\{H_x\frac{\partial \mathbf{N}}{\partial x} + H_\theta\frac{\partial \mathbf{N}}{\partial \theta}\}\mathbf{Q}^e + \mathbf{L}\{\Phi_{xz}\}^e \tag{24b}$$

where h represents the shell thickness at the point P and H_x and H_θ are the components of a thickness vector **H** defined by the following formula

$$\mathbf{H} = \sum_{j=1}^{9} h_j \mathbf{Z}_j \tag{26}$$

Here, $\mathbf{Z}_j$ is the unit normal vector of the midle surface at the Loof nodes, and the central node. It can be computed from the vector product at any point, say P, as

$$\mathbf{Z} = \frac{\partial \mathbf{P}}{\partial \xi} \times \frac{\partial \mathbf{P}}{\partial \eta} \tag{27}$$

Similar expression can be obtained for $\partial u_\theta/\partial z$. Since it is assumed that the shears γ_{xz} and $\gamma_{\theta z}$ are zero, it may be written

$$\frac{\partial w}{\partial x} = \frac{\partial u_x}{\partial z}, \quad \frac{\partial w}{\partial \theta} = R\frac{\partial u_\theta}{\partial z} \tag{28}$$

The derivatives governing the bending behavior of the shell are given by

$$\frac{\partial^2 u_x}{\partial \theta \partial z} = \frac{1}{h}\{-\frac{\partial H_x}{\partial \theta}\frac{\partial \mathbf{N}}{\partial x} - \frac{\partial H_\theta}{\partial \theta}\frac{\partial \mathbf{N}}{\partial \theta} + \frac{\partial H_z}{\partial x}\frac{\partial \mathbf{N}}{\partial \theta}\}\mathbf{Q}^e + \frac{\partial \mathbf{L}}{\partial \theta}\{\Phi_{xz}\}^e \tag{29}$$

Similarly, the derivatives $\partial^2 u_x/\partial x \partial z$, $\partial^2 u_\theta/\partial x \partial z$, and $\partial^2 u_\theta/\partial \theta \partial z$ can be obtained.

The local displacement components and their derivatives are used to determine the stiffeness and mass matrices after an application of the discrete Kirchhoff-Love hypothesis[15].

Discretizing the stiffened cylindrical shell with Semiloof elements, we obtain the matrix equation

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = \mathbf{0} \tag{30}$$

by means of Hamilton's principle (8) for the dynamic behavior of the stiffened shell. Here, **M** and **K** are the mass and stiffeness matrices of the stiffened cylidrical shell, respectively. **Q** is the vector of nodal displacements and rotations.

Assuming harmonic motion, $\mathbf{Q} = \bar{\mathbf{Q}}^{e i\Omega t}$, the equation (30) reduces a linear eigenvalue problem as

$$[\mathbf{K} - \Omega^2\mathbf{M}]\{\bar{\mathbf{Q}}\} = \mathbf{0} \tag{31}$$

This equation is solved by using the EISPACK routines.

6. Numerical Results

The vibration characteristics are determined for stiffened circular cylindrical shells having shear diaphragm ends, which have integral stringers and/or rings. Two types of stiffened circular shell model are used. C model is reinforced by the stringers with rectangular cross-sections and F model is reinforced by the stringers and/or rings with profile shape cross-section. Geometrical and material properties of the models are listed in Table 1.

Table 1 Geometrical and material properties.

Properties	Model C	Model F
L(m)	0.39446	18.00
R(m)	0.04976	1.95
h(m)	0.001651	0.0012
$E(N/m^2)$	68.95×10^9	68.95×10^9
$\rho(kg/m^3)$	2760	2760
ν	0.3	0.3

The natural frequencies and the mode shapes are listed in Table 2 for a stringer stiffened shell, (model C). Natural frequencies obtained by using the super shell finite element[4] and measured values obtained by Hoppmann[9] are also included for the purpose of comparison. Dimensions of a stringer are d_s = 0.005334 m, f_s = 0.003175 m and the number of external stringers is N_s = 16. The present results are in good agreement with earlier experimental and numerical results.

Table 2 Natural frequencies of a stringer stiffened shell with shear diaphragm ends (Model C).

mode number m,n	experiment[9]	predicted natural frequencies (Hz): present d.o.f.=272	present d.o.f.=376	nine-noded element[4] d.o.f. = 592
2,1	-	4216	3964	4264
2,2	-	3228	2134	2011
2,3	1830	2686	1827	1729
2,4	2600	2330	2480	2450
2,5	4080	-	3977	3680

The variation of the natural frequency ratio Ω/Ω_0 of circular cylindrical shell (model C) with the depth ratio of a stringer is shown in Fig. 4 for different width ratio of stringers. Here, Ω_0 is the natural frequency of the unstiffened circular cylindrical shell. Number of stringers is N_s = 16.

The influence of the number of the stringers and rings on the vibration characteristics is investigated for orthogonally stiffened circular cylindrical shells (model F). The cross-section of a stringer and a ring is depicted in Fig. 5.

The effects of the number of stringers on the vibration characteristics are investigated for an orthogonally stiffened circular cylindrical shell. The number of rings is taken to be 36. The variation of the lowest three natural frequencies is plotted in Fig. 6.

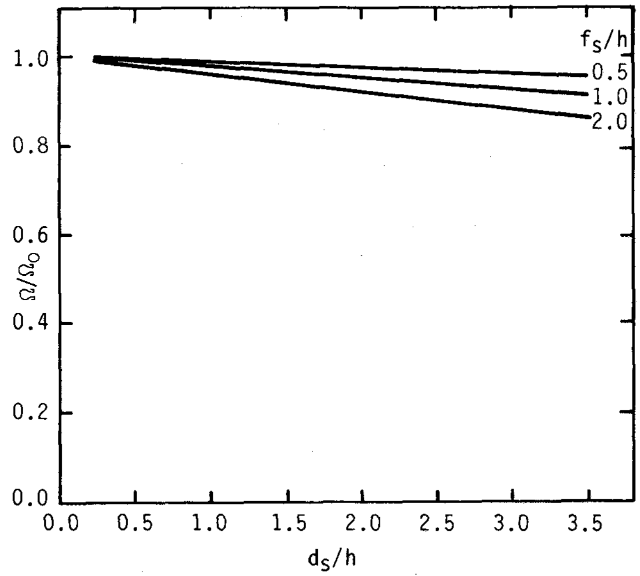

Fig. 4 The depth ratio effects on the natural frequency ratio of a stiffened cylindrical shell.

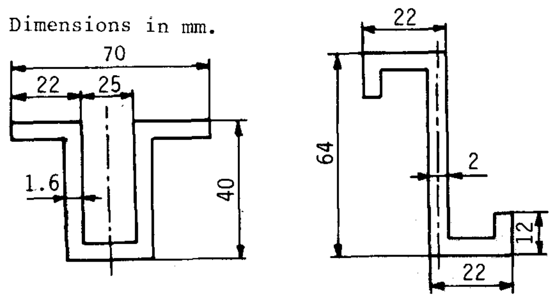

Fig. 5 Dimensions of profiles.

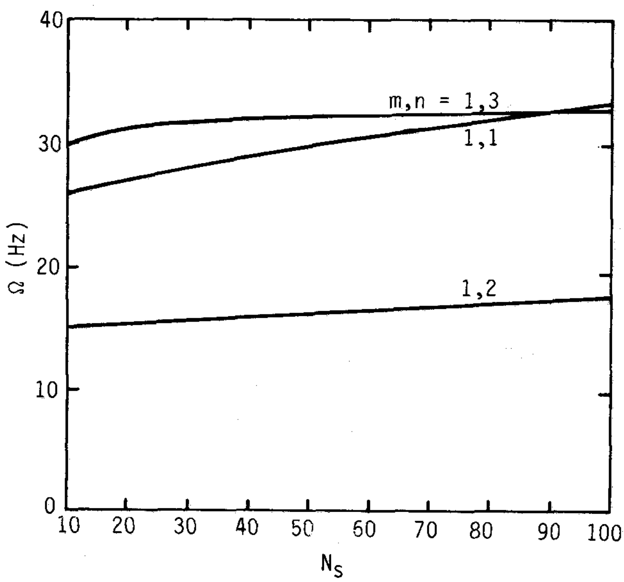

Fig. 6 Natural frequencies vs the number of stringers. N_r = 36.

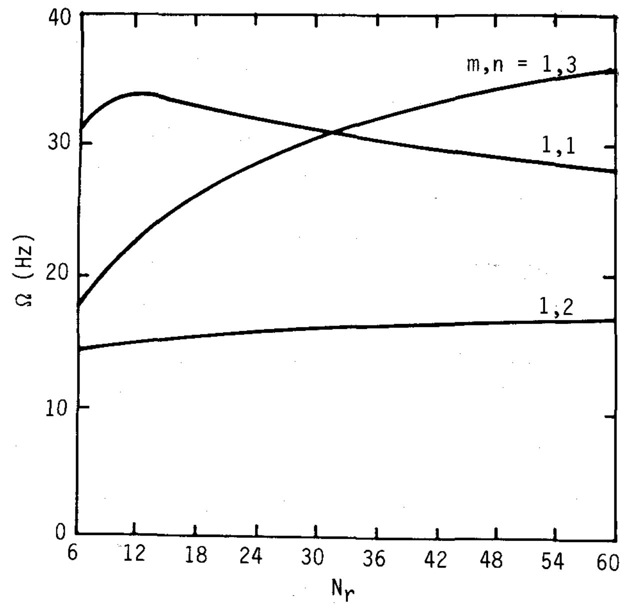

Fig. 7 Natural frequencies vs the number of rings. N_s = 60.

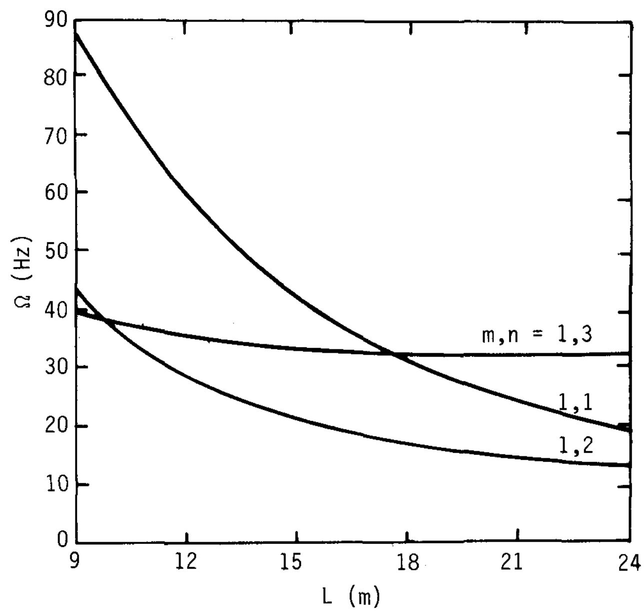

Fig. 8 Natural frequencies vs the length of the cylindrical shell. R=1.95 m.

Similarly, the effects of the number of rings on the vibration characteristics are examined and the numerical results is ploted in Fig. 7. The number of stringers is selected to be 60.

Finally, the effects of length of an orthogonally stiffened cylindrical shell on the vibration characteristics are investigated using the model F. The spacings of the stringers and rings are s_s = 0.2042 m and s_r = 0.5 m, respectively. The numerical results are ploted in Fig. 8.

7. Discussion

The free vibrations of stiffened circular cylindrical thin shells with shear diaphragm ends are studied numerically within the frame of Love's first approximation theory of elastic shells. The governing equations are obtained from the three-dimensional equations of elastodynamics by means of a generalized variational principle together with the usual kinematic hypothesis of cylindrical thin shells. The effects of stiffeners are taken into account by the orthotropic material approach. The uniqueness is tested in solutions of the dynamic governing equations of stiffened shell. The vibration characteristics are obtained numerically by means of the finite element method. A comparison of the present results with earlier results shows good agreement.

In the uniqueness theorem, the boundary and initial conditions which render Ψ to zero are shown to be sufficient for the uniqueness in solutions of the governing equations of stiffened cylindrical elastic shell. The conditions (10) and (11), and also, to specify one member of each of the products in Ψ of (20) ensure a unique solution for the governing equations. Besides, the sufficient conditions can be expressed in terms of the stress resultants as well as the displacement components[16], and they can obtained by logarithmic convexity arguments with no restrictions on the positive-definiteness of energies[17].

For a circular cylindrical shell reinforced by stringers, the natural frequency slightly decreases with the depth ratio of the stringer. The influence of the width ratio intends to decrease the natural frequency. Because an increment at the depth ratio and the width ratio of a stiffener contributes to the stiffeness and mass of the cylindrical shell, but the contribution to the mass of the shell is significant for the considered examples.

The number of the stringers slightly effects the dynamic behavior of the stiffened cylindrical shell. But the number of rings considerably increases the natural frequency corresponding to the mode shape m,n = 1,3, and decreases the natural frequency of mode shape m,n = 1,1. It should be noticed that the mode shape corresponding to the lowest natural frequency is not the simplest one, and this mode shape may change with the geometrical properties of the stiffened cylindrical shell and the conditions of reinforcing. Sometimes although the variation of the lowest natural frequency with a parameter is negligible, other frequencies may show considerable increasing with the parameter.

The natural frequencies corresponding some mode shapes strongly decreases with the increasing in the length of orthogonally stiffened circular cylindrical shells.

The present method can be successfully used so as to investigate the effect of rectangular cutouts on the free vibration characteristics of stiffened cylindrical shell. Also, by the method, the free vibration analysis can be carried out for the stiffened cylindrical shell with discrete stiffeners. The method can be extended to the forced-response dynamic behavoir and linear buckling analysis of stiffened cylindrical shell; this is will be reported else where.

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