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Modeling and Model Simplification of Aeroelastic Vehicles

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Abstract

The rigid-body degrees of freedom and elastic degrees of freedom of aeroelastic vehicles are typically treated separately in dynamic analysis. Such a decoupling, however, is not always justified and modeling assumptions that imply decoupling must be used with caution. The frequency separation between the rigid-body and elastic degrees of freedom for advanced aircraft may no longer be sufficient to permit the typical treatment of the vehicle dynamics. Integrated, elastic vehicle models must be developed initially and simplified in a manner appropriate to and consistent with the intended application. This paper summarizes key results from our research aimed at developing and implementing integrated aeroelastic vehicle models for flight controls analysis and design. Three major areas will be addressed; 1) the accurate representation of the dynamics of aeroelastic vehicles, 2) properties of several model simplification methods and 3) the importance of understanding the physics of the system being modeled and of having a model which exposes the underlying physical causes for critical dynamic characteristics.

Introduction

The means of obtaining the simplest valid mathematical model of an aeroelastic vehicle for dynamic analysis and control system design is a major issue in flight vehicle dynamics. The need to account for aeroelastic effects will make model formulation very important for flight vehicles of the future. Reduced structural weight, introduction of new materials, and application of high-authority feedback control systems will result in reduced frequency separation between the "rigid-body" modes and "elastic" modes. In addition, the potential for using control systems to influence the vehicle configuration, the so-called control configured vehicle concept [Schwanz (1977)], will require accurate aeroelastic models to be available very early in the design cycle.

Of particular importance is the potential for dynamic aeroelastic effects to influence "rigid-body" vehicle responses. Schmidt (1985), and Swaim and Poopaka (1982) have addressed the effects of aeroelastic/rigid-body modal coupling on flying qualities. A simulation study using the elastic vehicle model from Waszak and Schmidt (1988) and performed in the VMS moving-base simulator facility at NASA Langley Research Center addressed these effects [Waszak, Davidson, and Schmidt (1987)]. This study showed that increasing structural flexibility, even to moderate levels, can have a negative impact on vehicle handling qualities.

The view of the authors is that an integrated elastic vehicle model should be developed initially and simplified in a manner consistent with the intended application. This must be done in such a manner that the salient dynamic effects are retained in the model. This view must also be tempered with the need to have the simplest model possible to facilitate effective dynamic analysis and control synthesis, and to ease computational burden.

In this paper, several key aspects of formulating aeroelastic models for flight dynamics applications will be addressed. The equations of motion for any aeroelastic vehicle may be developed such that the structure is in many ways very similar to the equations of motion for a rigid vehicle. The similarities and differences will be discussed and the important features of the model structure will be addressed.

The method used for model simplification depends to a large degree on the ultimate use for the reduced-order model. However, one must never loose sight of the objective: obtain the simplest vehicle model that possesses the requisite accuracy. Techniques capable of delivering valid reduced-order models for control system design will be discussed.

Another key issue associated with the inherent complexity of aeroelastic vehicles is interpreting vehicle behavior. Understanding the sources of undesirable (or desirable) dynamic behavior is often required to design control systems or to design the airframe itself. The structure of the equations of motion and properties of the model simplification methods can aid the analyst/designer in developing this understanding. This is an area which is often overlooked in the development of modeling and model simplification methods and will be specifically addressed herein.

Equations of Motion

The development of the equations of motion of an elastic aircraft has been addressed many times in the literature. This subject has recently been revisited by the authors with emphasis on the need to develop accurate aeroelastic vehicle models and to clarify the assumptions associated with their development and assess their validity. The development of the equations of motion discussed here are intended for application to flight dynamics, simulation and control system

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design for elastic aircraft. This application places special requirements on the form and properties of the resulting equations. They must be able to describe large amplitude maneuvers in a body reference coordinate system while simultaneously describing the small amplitude structural deformations. This requires that the model be nonlinear in the large but allows the structural dynamics to be linear. The equations must also account for inertial and aerodynamic coupling which are normally neglected. An additional goal is to have the form of the equations be applicable over the entire design cycle, from the conceptual through detailed design phases.

Two recent studies [Waszak and Schmidt (1988) and Buttrill, Zeiler, and Arbuckle (1987)] have sought to develop the equations of motion for elastic aircraft to meet these goals. The emphasis of the first study (due to Waszak and Schmidt) was to assemble a mathematical model which integrated "rigid-body" and "elastic" degrees of freedom with particular emphasis on the assumptions made at the various stages in the development and on obtaining a set of equations that constitute an analytical (or symbolic) model. The second study (due to Buttrill, et al) focused on including effects of nonlinear inertial coupling between rigid-body angular rates and structural deformations and rates which are usually ignored in conventional aircraft modeling.

Both studies utilized Lagrange's method of deriving the equations of motion relative to a "mean axis" body reference coordinate system. The use of mean axes minimizes the degree of inertial coupling between the rigid-body and the elastic degrees of freedom. As a result, the equations of motion from the Waszak study have a structure similar to conventional rigid aircraft (with additional degrees of freedom associated with the elastic modes) and which demonstrate no inertial coupling between the elastic and the rigid degrees of freedom (all coupling occurs through the aerodynamic forces). The Buttrill study sacrificed some of the similarity to rigid aircraft demonstrated in the first study in the interest of more accurately modeling the inertial coupling effects. This results in the body axis rotational equations having additional terms not usually included.

Both studies also made use of the same basic assumptions in the derivation of the equations of motion;

a) the structure is treated as a collection of lumped masses with constant mass density,
b) the structural deformations are small (i.e. linear elastic theory is valid),
c) the structure exhibits synchronous elastic motion described by a complete set of normal modes, and
d) a local Earth-fixed inertial reference frame with constant uniform gravitational field was utilized.

The two studies differ slightly in that the Waszak study used the additional assumptions that;

e) each mass element is a point mass with no rotational inertia, and
f) elastic deformation and rate are sufficiently small or colinear that their cross product is negligible and the inertia tensor is constant.

The equations of motion from the Buttrill study can be simplified to a form identical to those in the Waszak study by adding these assumptions, therefore the two derivations are completely consistent.

Mean Axes

A short discussion of mean axes is warranted before addressing the specific issues associated with the two studies. As previously stated, the form of the equations of motion is facilitated by the use of a mean axes reference frame. The mean axis reference frame is not fixed to a material point in the body but floats so that its origin is at the instantaneous center of mass of the body. Furthermore, the mean axis frame is oriented in such a way that the reference frame motions are inertially decoupled from the structural deformations. An excellent survey of five types of floating reference frames is given in Canavin and Likins (1977). The practical mean axis conditions of Waszak and Schmidt (1988) are equivalent to the Buckins or linearized Tisserand frame of Canavin and Likins (1977).

Elastic modes of free vibration calculated from a structural model unconstrained in translation and rotation should satisfy the practical mean axis conditions [Canavin and Likins (1977)]. If the structural model has been restrained to create a nonsingular stiffness matrix, the resulting modes are inertially coupled with motion of the body frame and the equations of motion become more complex.

In the remainder of this section the important results from each of the two studies will be reviewed. Note that the equations of motion discussed herein are in a form which is applicable to a wide class of problems. Any airplane which has significant elastic dynamics and needs to be described dynamically in a body reference frame can be represented using these equations.

The Waszak Study

Tables 1 and 2 summarize the form of the equations of motion of an arbitrary elastic aircraft derived in Waszak and Schmidt (1988). Table 1 contains three translational equations and three rotational equations that describe the motion of the body reference coordinate system and a set of equations which describe the structural deformations relative to body axes.

Notice that the translational and rotational equations are identical in form to conventional rigid-body equations of motion. The differences lie in the representations of the aerodynamic forcing functions. These forces completely describe the coupling which exists between the rigid-body and elastic degrees of freedom. Also note that the moments of inertia, $I_{(.)}$, actually correspond to the deformed structure and change slightly as the vehicle deforms. It is assumed here that the variations in the moments of inertia due to elastic deformation are small and hence neglected. Thus, the moments of inertia used in the equations of motion correspond to the undeformed vehicle.

In addition to the body axis translational and rotational equations there are a set of equations that describe the elastic deformations of the body. The elastic mode equations are typical of a second-order oscillator with equivalent modal damping proportional to modal velocity, $\dot{\eta}$. Again, the only coupling with the rigid-body degrees of freedom is through the aerodynamic forces.

Table 2 summarizes expressions for the aerodynamic forces for an elastic aircraft. These expressions are presented in a stability derivative form. The difference between these expressions and those used for rigid aircraft is in the addition of elastic stability derivatives. These terms are similar to their rigid aircraft counterparts and serve to couple the elastic degrees of freedom with the rigid-body degrees of freedom. The relationships between the structural parameters (i.e. mode shapes) and the elastic stability derivatives are developed in Waszak and Schmidt (1988) by using strip theory aerodynamics. The validity of this approximation will not be discussed further other than to say that it is a reasonable approach for high aspect ratio configurations if numerical values are sought. However, the real importance is that the expressions for the elastic stability derivatives obtained using strip theory provide a conceptual link between physical parameters and their effect on the equations of motion independent of their numerical values.

The use of stability derivatives has the added advantage that the same form of the equations of motion can be used throughout the design cycle. Early on, before detailed analyses have been performed, strip theory or other first-order methods can be used to obtain a preliminary elastic vehicle model. Later, when more detailed analyses have been performed (e.g. computational fluids dynamics analyses, wind tunnel tests, finite element analyses), the data can be converted to a stability derivative form and substituted directly into the equations of motion.

The equations of motion summarized in Tables 1 and 2 also constitute a "literal" model for an elastic airplane. The literal model can be used to develop insight into the effects of various physical parameters on the dynamic characteristics of the vehicle. This is difficult to obtain from purely numerical models. It is important to note that even if a "numerical" model is available, it can be put in a form consistent with the "literal" model which allows the analyst to develop this understanding.

An example of the importance of modeling aeroelastic dynamics was demonstrated in Waszak and Schmidt (1988). The equations of motion discussed above were applied to a high speed transport aircraft with a moderate increase in structural flexibility over that of the baseline vehicle. The results of the study showed that neglecting aeroelastic dynamics during model development resulted in a model which incorrectly indicated the vehicle to have a stable phugoid mode and which had errors in short period frequency and damping of approximately 55% and 14%, respectively.

The Buttrill Study

The equations of motion developed in Buttrill, Zeiler, and Arbuckle (1987), and Zeiler and Buttrill (1988) are summarized in Table 3. These three sets of equations are presented in vector form and correspond to the scalar equations presented in Table 1 except for the addition of terms representing nonlinear inertial coupling. In Table 3, an open dot over a quantity indicates the time rate of change of that quantity expressed in body frame components.

The translational equation is completely analogous to the previous study. The use of mean axes has eliminated any inertial coupling between the rigid-body degrees of freedom and the elastic degrees of freedom.

The rotational equation on the other hand has additional terms not included in the Waszak study. These additional terms account for variations in the inertia terms with structural deformation (modal induced changes in mass distribution) and second-order coupling associated with the fact that the cross product between structural displacement and rate are nonzero (since all modes do not act in the same plane). When these terms are neglected the rotational equations are completely analogous to those in the first study.

The elastic mode equations also have additional terms. These are associated with angular acceleration of the body reference frame, Coriolis acceleration, and centrifugal loading. Neglecting these effects simplifies the elastic mode equations to those from the Waszak study.

The objective of the Buttrill study was to generate a high fidelity model of an elastic airplane with special attention to the effects of inertial coupling. Consequently, the representation of the aerodynamic forces was not explicitly addressed other than to discuss the numerical methods by which aerodynamic forces were computed for an example problem. It should be noted, however, that coupling between the rigid-body and elastic degrees of freedom also occurs through the aerodynamic forces.

In summary, it was found that in general, nonlinear inertial coupling can become a first-order effect if at least one of the following characteristics are reflected in the vehicle:

a) aerodynamic loads are small,
b) expected rotational rates are of the order of the elastic frequencies,
c) the model geometry is sufficiently complex that transverse deflections result in changes in mass distribution.

Summary

Both of these studies indicate that the structure of the equations of motion of elastic aircraft are quite complex, even when they are developed with the intent of minimizing the apparent modal coupling. Some of the complexities associated with the dynamics of aeroelastic vehicles are listed below.

a) The models which result are of high dynamic order, two additional states for each elastic mode.
b) The relationships between the various model parameters that are fairly well understood for rigid aircraft, (e.g. $\omega_{sp}^2 \approx Z_w M_q - M_\alpha$) are more difficult to identify for elastic aircraft.
c) The parameters that appear in the model, such as the generalized modal stability derivatives, are less well understood than classical parameters such as L_α, and accurate numerical values are more difficult to obtain.
d) There are significant uncertainties associated with the elastic parameters.

This complexity negatively impacts many aspects of aeroelastic aircraft dynamics and control. A natural next step is to address how the important effects of elastic dynamics can be retained in the model while at the same time simplifying the model structure. This is the role of model simplification.

Model Simplification

The design of effective and practical control systems requires that the designer understand which aspects of the vehicle dynamics are important, the uncertainties associated with the model, and a knowledge of the parameters which have a significant impact on critical vehicle responses. The size and complexity of the models which result from describing the aeroelastic interactions (via the equations of motion discussed previously) make this an extremely difficult task. It is likely, however, that once the key interactions have been accounted for, many fewer physical parameters need to be retained to capture the prominent aspects of the vehicle responses. It is therefore important to be able to simplify the models but still retain enough information to capture the salient features of the aeroelastic interactions. Note that the parameters which turn out to be key are not usually discernable before the detailed model is obtained and requires that such a model be obtained first followed by appropriate simplification.

Model simplification is also important for aeroelastic systems from the perspective of controller complexity. Many control design methodologies require the system model to be linear and result in controllers which are of dynamic order equal to or greater than the design model. The size and complexity of aeroelastic models therefore dictates large, complex controllers. A reduced-order linear model which retains the salient aspects of the nonlinear system dynamics within a simplified form may allow effective controllers to be designed with significantly simpler structure.

Model simplification has as its goal to obtain a model which, while simpler than the full-order model, approximates some aspect of the true system. The first step in this process is to linearize the system dynamics about an equilibrium condition. Reduction of the linear model is then performed and the desired manner in which the reduced-order linear model approximates the full-order model depends to a large degree on the intended application. For example, in control system synthesis it is important to accurately represent the system frequency response in the frequency range where crossover of the loop transfer function is likely to occur [McRuer, Ashkenas, and Graham (1973)]. Note that this implies that there are frequencies both above and below the critical frequency range which may not need to be well modeled. The frequency range of interest is very important in applying model simplification and plays a key role.

There are many methods by which linear elastic aircraft models can be simplified. A few of these will be discussed here and are summarized below.

1. Truncation - deletes some of the modes (modal truncation) or states (state truncation) from the full-order model.
2. Residualization - accounts only for the static effects of some modes or states whose dynamics are not crucial [Kokotovich, O'Malley, and Sannuti (1976)].
3. Balanced reduction - minimizes frequency response error in a normed sense and has certain advantages associated with obtaining desired accuracy [Bacon and Schmidt (1989)].
4. Literal (or symbolic) simplification - addresses the impact of various physical parameters on the system responses and ignores those which have little impact [Schmidt and Newman (1988)].

Each of these methods have advantages and disadvantages as they apply to simplification of elastic aircraft models. In this section the four simplification methods will be discussed in these terms.

Truncation

Truncation is a common form of model reduction. In fact, it is the most common form of reduction since every finite dimensional linear model is a truncated model in the sense that there is always some aspect of the physical system that is neglected in the modeling process. While truncation is a well established model simplification technique, a slightly different view on the subject is presented here with some interesting implications [Schmidt and Newman (1988)].

The degree to which truncation can be utilized depends on the degree to which the truncated degrees of freedom directly influence the vehicle response <u>and</u> the degree to which they couple with the retained degrees of freedom. Consider a frequency domain representation of a linear system as shown below.

$$\begin{bmatrix} A(s) & c(s) \\ r(s) & m(s) \end{bmatrix} \begin{bmatrix} Z(s) \\ z_r(s) \end{bmatrix} = \begin{bmatrix} B(s) \\ b_r(s) \end{bmatrix} U(s) \qquad (1a)$$

$$Y(s) = M(s)\,Z(s) + m_r(s)\,z_r(s) \qquad (1b)$$

Here Y(s) is the vector of responses, U(s) is the vector of commands and $[Z^T(s), z_r(s)]^T$ is the vector of states or system degrees of freedom. Assume for ease of discussion that $z_r(s)$ is a scalar. In this case m(s) is a scalar, r(s) and $b_r(s)$ are row vectors, and c(s) and $m_r(s)$ are column vectors. By applying Cramer's Rule for the determinant of a matrix [Strang (1980)], and an identity for the determinant of a partitioned matrix [Brogan (1974)] the transfer functions of the system can be written -

$$\frac{Z_i(s)}{U_j(s)} = \frac{\det\left\{ (A_i \,|\, B_j)(s) - c(s)m^{-1}(s)(r_i \,|\, b_{rj})(s) \right\}}{\det\left\{ A(s) - c(s)m^{-1}(s)r(s) \right\}} \qquad (2a)$$

$$\frac{z_r(s)}{U_j(s)} = \frac{b_{rj}(s)\det\left\{ A(s) - B_j(s)b_{rj}^{-1}(s)r(s) \right\}}{m(s)\det\left\{ A(s) - c(s)m^{-1}(s)r(s) \right\}} \qquad (2b)$$

where the notation $A_i | B_j$ represents the operation of replacing the ith column of A with the jth column of B. These forms of the system transfer functions are very useful in identifying some important aspects of applying model reduction via truncation.

From inspection of the transfer function expressions above one can see that if $c_k(s)r_l(s) << m(s)$ and $c_k(s)(r_i | b_{rj})_l(s) << m(s)$; $k,l = 1,2,...,n$ where n is the number of states, then

$$\frac{Z_i(s)}{U_j(s)} \approx \frac{\hat{Z}_i(s)}{U_j(s)} = \frac{\det\{A_i | B_j\}}{\det\{A\}} \quad (3)$$

Also, if in addition $B_{jk}(s)\, r_l(s) << b_{rj}(s)$; $k,l = 1,2,...,n$ then

$$\frac{z_r(s)}{U_j(s)} \approx \frac{b_{rj}(s)}{m(s)} \quad (4)$$

The first expression is exactly what results if the degree of freedom z_r is truncated from the model (or not included in the model from the outset).

Examining the transfer function $\frac{z_r(s)}{U_j(s)}$ from the context of physical systems one finds that the polynomial m(s) is usually of higher order than $b_{rj}(s)$. Therefore, for frequencies above some value determined by m(s) and $b_{rj}(s)$ (as $s \to \infty$) the effect of z_r on the output Y(s) is negligible which can be approximated by

$$Y(s) = M(s)\hat{Z}(s) \quad (5)$$

where $\frac{\hat{Z}_i(s)}{U_j(s)}$ is described in Eqn. (3). The implication is that truncation should be used to simplify a model in which the degree of freedom to be removed is much <u>slower</u> than the dynamics of interest.

The conditions that must be satisfied to apply truncation to slow dynamics have some important implications. The condition that $c_k(s)r_l(s) << m(s)$ implies that the degree of coupling between the deleted degree of freedom and the retained degrees of freedom is small. Similarly, $c_k(s)(r_i | b_{rj})_l(s) << m(s)$ implies that the coupling effect between the deleted degree of freedom and the retained degrees of freedom (via c) combined with the ability of the jth control input to excite the deleted degree of freedom is small. Finally, $B_{jk}(s)\, r_l(s) << b_{rj}(s)$ implies that the combination of the coupling effect between the retained degrees of freedom and the deleted degree of freedom (via r) and the ability of the jth control input to excite the retained degrees of freedom is small. These conditions will be referred to as the "decoupling conditions" and can generally serve as a test to determine if a particular degree of freedom can be legitimately truncated.

Note that these conditions are identically satisfied if the system is in modal form. Therefore, modal truncation can be effectively applied whenever there is sufficient frequency separation between the deleted mode and the dynamics of interest, so that in the frequency range of interest $m_r(s)z_r(s)$ can be neglected from the output equation (Eqn. (1b)).

The truncation of slow modes is contrary to the typical use of truncation which is to remove higher order dynamics. The reason that truncation works for some higher order dynamics can be seen by addressing the problem by using a partial fraction expansion of a transfer function,

$$\frac{Y(s)}{U(s)} = \frac{R_1}{s+\lambda_1} + \frac{R_2}{s+\lambda_2} + \ldots + \frac{R_n}{s+\lambda_n} \quad (6)$$

where R_i are the residues and λ_i are the eigenvalues of the system [D'Azzo and Hoopis (1975)]. If the desired frequency range of accuracy of the simplified model is well below λ_n, then the last term in the expansion can be approximated by $\frac{R_n}{\lambda_n}$. Since this term is associated with a high frequency mode, the value of λ_n is most likely much greater than unity. In addition, high frequency modes are frequently difficult to excite which results in small residues. Clearly, if $R_n << \lambda_n$ then the last term in the partial fraction expansion can be neglected without much impact on the frequency response in the frequency range of interest. Thus, truncation can also be used to remove high frequency dynamics.

Residualization

Residualization is another common method of model simplification. This method makes use of the fact that a system may have some dynamics that are fast compared to the dynamics of interest [Kokotovic, et al (1976)]. However, even the fast dynamics can interact with the slower dynamics. Residualization allows one to take into account the interaction without including the dynamic effects of the fast dynamics.

The same model structure used previously in the discussion of truncation will be used again here. Consider the frequency domain representation of a linear system presented in Eqns. (1). The system transfer functions can again be approximated by the expressions in Eqns. (3) and (4) when the decoupling conditions are satisfied (i.e. $c_k(s)r_l(s) <<$ m(s), $c_k(s)(r_i | b_{rj})_l(s) << m(s)$, and $B_{jk}(s)\, r_l(s) << b_{rj}(s)$).

Residualization is typically accomplished by letting the degrees of freedom to be removed from the model reach their steady state values instantaneously by setting their derivatives zero. An analogous interpretation is to let $s \to 0$ in the transfer function $\frac{z_r(s)}{U_j(s)}$ (as opposed to letting $s \to \infty$ for truncation).

The simplified model using residualization takes on the following form.

$$Y(s) \approx M(s)\hat{Z}(s) + m_r(s)\frac{b_{rj}(0)}{m(0)}U(s) \quad (7)$$

The implication here is that residualization can be legitimately applied to degrees of freedom which are much faster than the retained modes. Therefore, only degrees of freedom whose frequencies are well above the expected crossover frequency range should be considered for residualization.

Notice again that the decoupling conditions are automatically satisfied when the system is in modal form. When the frequency range of interest is well below λ_n, the simplified model produced by modal residualization (Eqn. (7)) is identical to the partial fraction expansion of the transfer function (Eqn. (6)) with the last term approximated by $\frac{R_n}{\lambda_n}$.

While the validity of performing model simplification via truncation or residualization can be evaluated by the degree to which the decoupling conditions are satisfied and the degree of frequency separation between the deleted dynamics and the desired dynamics, there is no guarantee on the accuracy of the resulting simplified model. The current approach is a cut and try (i.e. iterative), graphical procedure. A plot of the candidate frequency response is compared to that of the full-order system and a decision is made as to its acceptability. This is a basic limitation of these approaches.

An advantage of these approaches, however, is that the form of the model which results after simplification is the same as the corresponding portion of the original. Therefore, if the model had a special structure before simplification then that structure is retained in the simplified form. This can be important in allowing the analyst to use his knowledge of the physics to interpret the accuracy of the resulting simplified model as well as the effect of various physical parameters on the system response.

A comparison of truncation and residualization applied to a high speed transport aircraft is presented in Waszak and Schmidt (1988). The results indicate the these methods are often quite acceptable. However, as structural flexibility increases these methods may not provide the required accuracy for the desired model order. In these cases other model reduction methods should be considered.

Balanced Reduction

Internally balanced reduction and frequency weighted internally balanced reduction are two more model simplification methods that have received considerable attention recently. A considerable body of literature has addressed the concept of balanced reduction and its variants [Enns (1984), Bacon and Schmidt (1989), Glover]. We will simply address some of the issues that should be kept in mind when considering this model simplification approach.

Briefly stated, the balanced realization approach to model reduction chooses an ordered combination of state directions which dominate the input/output behavior of the system in decreasing order. An advantage of this approach is that the accuracy of the reduced-order model can be measured in a normed sense. The frequency response error between the full-order system and the simplified system is bounded by twice the sum of the Hankel singular values of the deleted degrees of freedom [Bacon and Schmidt (1989)]. In addition, the measure of accuracy is a direct byproduct of the reduction process.

Unfortunately, the balanced reduction method results in a simplified model which matches the frequency response of the full-order model in regions where the magnitude is greatest. This may not be the region of crossover. As a result the model may not be acceptable for application to control design regardless of the "accuracy." An example of this limitation is presented in Schmidt and Newman (1989). This weakness of the method can be resolved by applying weighting functions to the basic approach to emphasize a desired frequency range (e.g. the crossover region) [Enns (1984), Bacon and Schmidt (1989)]. However, when this is done the measure of the accuracy of the simplified model is no longer valid. Research into resolving this issue is currently underway.

Yet another limitation of the balanced reduction methods is associated with the fact that the state space form of the simplified model which results is only related to the original model through their frequency responses. The states of the simplified model are entirely different from those of the full-order model. In fact, the simplified state space model looses all structure that appeared in the full-order model. The implication is that any insight that the analyst has concerning the physical nature of the system cannot be utilized in any subsequent analysis using the simplified model.

An application of frequency weighted internally balanced model reduction was presented in Schmidt and Newman (1989). This study demonstrated the importance of appropriate model reduction for application in control synthesis. If the reduced order model does not show good agreement with the full order model in the region of crossover, even where the transfer functions have relatively small magnitude, the resulting control law may not perform as expected when applied to the full order model (even to the point of destabilizing the system).

Literal Simplification

The last model reduction method which will be discussed here and one which is often overlooked is literal approximation. This method is based on first-order perturbation theory and can, in principle, be applied to high order models.

An advantage of this approach is that it allows one to identify the cause and effect relationships between physical parameters and dynamic behavior. A disadvantage of this approach is that it is tedious to apply to more than very simple systems. The recent advances in symbolic mathematics computer programs however have fostered a renewed interest in this approach [Schmidt and Newman (1988)].

In an earlier section the equations of motion for an elastic aircraft were described in a literal form using modal structural representations. This form of the equations of motion lends itself to literal (symbolic) formulation of system transfer functions. This can be accomplished by hand or with the aid of one of the many symbolic mathematics computer programs.

Consider literal representations of the numerator and denominator polynomials for a pitch-rate-to-elevator transfer function of an elastic airplane in which the short period approximation has been applied [McRuer, Ashkenas, and Graham (1973)] and with one structural mode included in the model. These polynomials are presented in Table 4.

The parameters that appear in the polynomials represent stability and control derivatives (Z, M, and F)[1] , structural parameters (ω, ζ, and ϕ')[2], and the flight speed V. Those terms with subscripts α and q are associated with the rigid-body degrees of freedom (angle of attack and pitch rate, respectively). Those terms with the subscript δ_e are associated with the elevator deflection. Those terms with subscripts η and $\dot{\eta}$ are associated with the modeled elastic degree of freedom and its derivative. Those terms with double subscripts are associated with coupling between the rigid-body and the elastic degrees of freedom.

The numerical form of the transfer function structure represented by the polynomials in Table 4 is presented in Eqn. (8) for a high speed transport aircraft example from Waszak and Schmidt (1988). The numerical form of the transfer function was obtained by truncating the surge degree of freedom and residualizing the second, third and fourth structural modes from the elastic equations of motion of the aircraft. Note that the numerator has three real roots and one root at the origin and the denominator has two pairs of complex conjugate roots and one root at the origin.

$$G_q^{\delta_e} = \frac{13.06s(s+0.231)(s-3.362)(s+3.959)}{s(s^2+0.874s+1.572)(s^2+0.993s+36.51)} \quad (8)$$

Once the numerator and denominator polynomials of the desired transfer function are obtained, the approximate terms are chosen so that the following two criteria are satisfied.

a) The literal expressions must factor into the same form as the original polynomials (e.g. order of polynomial, number of real and complex roots), and
b) numerical values based on the simplified terms should accurately approximate the values based on the original polynomials.

The underlined terms of the numerator and denominator polynomials in Table 4 involve the key model parameters (stability derivatives and structural parameters), which factor into the appropriate form, and result in approximate polynomials with the desired properties described above. These terms are used to obtain the approximate literal model presented in Table 5. The corresponding numerical values for this approximate model are also presented in Eqn. (9).

$$\tilde{G}_q^{\delta_e} = \frac{13.06s(s+0.416)(s-3.265)(s+4.177)}{s(s^2+1.246s+3.758)(s^2+0.621s+34.83)} \quad (9)$$

Note that the numerical values from the approximate model agree to varying degree with the "truth" model in Eqn. (8). Those terms which are deemed to be of insufficient accuracy can be modified by computing correction terms.

Corrections to the approximate factors can be obtained in literal form by applying perturbation theory. Expanding the true polynomial coefficient, p_i, in a Taylor series about the approximate value, $\tilde{p}_i$, allows one to compute literal corrections. This requires that the Taylor series be truncated after the first-order term,

$$p_i \approx \tilde{p}_i + \frac{\partial \tilde{p}_i}{\partial \chi}\Delta\chi \quad (10)$$

Here χ is the vector of model parameters that contribute to the value of the polynomial coefficient p_i. The correction $\Delta\chi$ clearly requires literal expressions for $p_i - \tilde{p}_i$ and $\frac{\partial \tilde{p}_i}{\partial \chi}$ to be available. The difference expression, $p_i - \tilde{p}_i$, is simply what remains after the approximate factor is extracted from the literal expression for p_i and corresponds to the non-underlined terms in Table 4. The partial derivative term can be obtained by direct symbolic differentiation with respect to the model parameters, χ.

The correction factors can be used either to enhance the accuracy of the approximate model or to identify the sensitivity of the simplified model to variations in various physical parameters.

A numerical form of the literally simplified model can be obtained by substituting the values of the various parameters directly into the literal expressions. The literally simplified model, unlike strictly numerical models, can be used to assess the reason behind mismatches with the original model. If an error occurs at a particular frequency, the model parameters which are dominant at that frequency and so contribute significantly to the error can be directly identified. In addition, the impact of potential variations or uncertainties in a particular model parameter can be quantified in terms of its effect on the vehicle response.

An example of literal model simplification is presented in Schmidt and Newman (1988). This approach was shown to yield excellent results when applied to a high speed transport aircraft. Furthermore, the closed-form analytical expressions for the key dynamic characteristics that result allow one to identify critical parameters affecting the vehicle dynamics. Some of the key features associated with the aircraft example included magnitudes and frequencies of lightly damped modes and "rigid-body" modal characteristics.

Summary

Each of the model reduction methods described here have clear advantages and disadvantages. As such, it is unlikely that any one method will be able to satisfy all model

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[1] Z is the force oriented along the body axis orthogonal to the plane of the wing (and is predominantly lift), M is the pitching moment, and F is the generalized force associated with the elastic degree of freedom.

[2] ω and ζ are the invacuo frequency and damping of the elastic mode, and ϕ' is the mode slope at the point where pitch rate is measured.

simplification needs. In fact, it would be to the analyst's advantage to recognize the strengths and weaknesses of each method and use one which suits her particular needs.

These methods are not necessarily mutually exclusive either. One method can be used to compliment another and enhance ones understanding of the vehicle's dynamic behavior. For example, truncation and residualization may be used initially to reduce the model to a tractable form. Then literal methods may be used to identify the sensitivity of the model to parameter variations and uncertainties. Finally, frequency weighted internally balanced reduction might be used to obtain a numerical form of the model or further simplify a numerical version of the literal model.

The most important recommendation, however, is to use caution whenever applying model simplification to aeroelastic systems. Blindly applying any simplification method will lead to a simpler model, but one which may not accurately convey the important dynamic characteristics which influence the vehicle behavior.

Concluding Remarks

The objective of this paper was to emphasize some of the key issues associated with modeling elastic aircraft for dynamic analysis and control law synthesis. Emphasis has been placed on the importance of initially developing high fidelity models which are subsequently simplified for particular applications. This approach assures that the salient features of the vehicle dynamics will be represented in the design model. In addition, this approach results in a model structure which is consistent and applicable over the entire development cycle, including preliminary design. This is especially important in allowing control technologies to play a role in shaping the vehicle configuration.

The development of two modeling approaches were specifically addressed with particular attention paid to the underlying assumptions. The first approach results in a model structure with which literal models can be developed. The second modeling approach addressed the issues associated with including additional inertial coupling terms in the model and provided guidelines for when inertial coupling should be included.

The importance of model simplification was also addressed by considering the advantages and disadvantages of four model simplification methods. The first two simplification methods, truncation and residualization, represent traditional approaches. These were viewed in a slightly different way which resulted in some guidelines for when they can be legitimately applied. The third method, internally balanced reduction, represents the newer model simplification approaches which provide added capabilities subject to special limitations. The last method, literal simplification, summarized an approach which, while currently often overlooked, will become more attractive as symbolic mathematics computer programs become more capable.

The results from the studies described herein and the perceived need for accurate models of elastic aircraft for control design applications indicate that more emphasis should be placed on the modeling process. The recommendation of the authors is that model development should involve both formulating equations of motion and model simplification. Each phase should be treated separately but with knowledge of the other. This approach makes more likely, the possibility that the salient aspects of the system dynamics will be accurately modeled.

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Table 1 - Elastic aircraft equations of motion

$$M\,[\,\dot{U} - rV + qW + g\sin\theta\,] = Q_X$$
$$M\,[\,\dot{V} - pW + rU - g\sin\phi\cos\theta\,] = Q_Y$$
$$M\,[\,\dot{W} - qU + pV - g\cos\phi\cos\theta\,] = Q_Z$$

$$I_{xx}\dot{p} - (I_{xy}\dot{q} + I_{xz}\dot{r}) + (I_{zz} - I_{yy})qr + (I_{xy}r - I_{xz}q)p + (r^2 - q^2)I_{yz} = Q_\phi$$
$$I_{yy}\dot{q} - (I_{xy}\dot{p} + I_{yz}\dot{r}) + (I_{xx} - I_{zz})pr + (I_{yz}p - I_{xy}r)q + (p^2 - r^2)I_{xz} = Q_\theta$$
$$I_{xx}\dot{r} - (I_{xz}\dot{p} + I_{yz}\dot{q}) + (I_{yy} - I_{xx})pq + (I_{xz}q - I_{yz}p)r + (q^2 - p^2)I_{xy} = Q_\psi$$

$$\mathcal{M}_j\,[\,\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta} + \omega_j^2\eta_j\,] = Q^j_\eta \;;\; j=1,2,3,\ldots$$

Table 2 - Elastic aircraft generalized forces

$$Q_X = \frac{\rho V_0^2 S}{2}\left(C_{X_0} + C_{X_\alpha}\alpha + C_{X_\delta}\delta + \sum_{i=1}^{\infty} C_{X\eta}^{i}\eta_i\right) + \frac{\rho V_0 S\bar{c}}{4}\left(C_{X_{\dot\alpha}}\dot\alpha + C_{X_q}q + \sum_{i=1}^{\infty} C_{X\dot\eta}^{i}\dot\eta_i\right) + T_X$$

$$Q_Y = \frac{\rho V_0^2 S}{2}\left(C_{Y_0} + C_{Y_\beta}\beta + C_{Y_\delta}\delta + \sum_{i=1}^{\infty} C_{Y\eta}^{i}\eta_i\right) + \frac{\rho V_0 Sb}{4}\sum_{i=1}^{\infty} C_{Y\dot\eta}^{i}\dot\eta_i + T_Y$$

$$Q_Z = \frac{\rho V_0^2 S}{2}\left(C_{Z_0} + C_{Z_\alpha}\alpha + C_{Z_\delta}\delta + \sum_{i=1}^{\infty} C_{Z\eta}^{i}\eta_i\right) + \frac{\rho V_0 S\bar{c}}{4}\left(C_{Z_{\dot\alpha}}\dot\alpha + C_{Z_p}p + C_{Z_q}q + \sum_{i=1}^{\infty} C_{Z\dot\eta}^{i}\dot\eta_i\right) + T_Z$$

$$Q_\phi = \frac{\rho V_0^2 Sb}{2}\left(C_{L_0} + C_{L_\beta}\beta + C_{L_\delta}\delta + \sum_{i=1}^{\infty} C_{L\eta}^{i}\eta_i\right) + \frac{\rho V_0 Sb^2}{4}\left(C_{L_p}p + C_{L_q}q + \sum_{i=1}^{\infty} C_{L\dot\eta}^{i}\dot\eta_i\right) + L_T$$

$$Q_\theta = \frac{\rho V_0^2 S\bar{c}}{2}\left(C_{M_0} + C_{M_\alpha}\alpha + C_{M_\delta}\delta + \sum_{i=1}^{\infty} C_{M\eta}^{i}\eta_i\right) + \frac{\rho V_0 S\bar{c}^2}{4}\left(C_{M_{\dot\alpha}}\dot\alpha + C_{M_q}q + \sum_{i=1}^{\infty} C_{M\dot\eta}^{i}\dot\eta_i\right) + M_T$$

$$Q_\psi = \frac{\rho V_0^2 Sb}{2}\left(C_{N_0} + C_{N_\beta}\beta + C_{N_\delta}\delta + \sum_{i=1}^{\infty} C_{N\eta}^{i}\eta_i\right) + \frac{\rho V_0 Sb^2}{4}\left(C_{N_p}p + C_{N_r}r + \sum_{i=1}^{\infty} C_{N\dot\eta}^{i}\dot\eta_i\right) + N_T$$

$$Q^j_\eta = \frac{\rho V_0^2 S\bar{c}}{2}\left(C^j_0 + C^j_\alpha\alpha + C^j_\beta\beta + C^j_\delta\delta + \sum_{i=1}^{\infty} C^{ji}_{\eta}\eta_i\right) + \frac{\rho V_0 S\bar{c}^2}{4}\left(C^j_{\dot\alpha}\dot\alpha + C^j_p p + C^j_q q + C^j_r r + \sum_{i=1}^{\infty} C^{ji}_{\dot\eta}\dot\eta_i\right)$$

Table 3 - Elastic aircraft equations of motion with inertial coupling

$$m\mathring{\underline{V}} = \underline{F} - m(\underline{\omega} \times \underline{V}) + m\underline{g}$$

$$[J]\mathring{\underline{\omega}} + \underline{h}_{jk}\eta^j\ddot{\eta}^k = \underline{L} + \underline{\omega} \times [J]\underline{\omega} - [\mathring{J}]\underline{\omega} - \underline{h}_{jk}\dot{\eta}^j\dot{\eta}^k - \underline{\omega} \times \underline{h}_{jk}\eta^j\dot{\eta}^k$$

$$M_{jk}\ddot{\eta}^k - \mathring{\underline{\omega}} \cdot \underline{h}_{jk}\eta^k = Q_{\eta_j} - M_{jj}\omega_j^2\eta_j + 2\underline{\omega} \cdot \underline{h}_{jk}\dot{\eta}^k + \frac{1}{2}\underline{\omega}^T\{[\Delta J]_j + [\Delta^2 J]_{jk}\eta^k\}\underline{\omega}$$

$$[J] = [J_o] + [\Delta J]_j\eta^j + [\Delta^2 J]_{jk}\eta^j\eta^k$$

Table 4 - Example of literal transfer function polynomials - $G_q^{\delta_e} = \frac{N(s)}{D(s)}$

N(s)	$\frac{Z_{\delta_e}}{V}\{s[M_\alpha(s^2 + (2\zeta\omega - F_{\dot\eta})s + (\omega^2 - F_\eta)) + F_\alpha(M_{\dot\eta}s + M_\eta)] - \phi' s[M_\alpha F_q s + F_\alpha(s^2 - M_q s)]\}$ + $M_{\delta_e}\{\underline{s[(s - \frac{Z_\alpha}{V})(s^2 + (2\zeta\omega - F_{\dot\eta})s + (\omega^2 - F_\eta))} - F_\alpha(\frac{Z_{\dot\eta}}{V}s + \frac{Z_\eta}{V})] - \phi' s[F_q s(s - \frac{Z_\alpha}{V}) + F_\alpha(1 + \frac{Z_q}{V})s]\}$ $+ F_{\delta_e}\{s[(s - \frac{Z_\alpha}{V})(M_{\dot\eta}s + M_\eta) + M_\alpha(\frac{Z_{\dot\eta}}{V}s + \frac{Z_\eta}{V})] - \phi' s[\underline{(s - \frac{Z_\alpha}{V})(s^2 - M_q s)} - M_\alpha(1 + \frac{Z_q}{V})s]\}$
D(s)	$-(\frac{Z_{\dot\eta}}{V}s + \frac{Z_\eta}{V})[M_\alpha F_q s + F_\alpha(s^2 - M_q s)] - (M_{\dot\eta}s + M_\eta)[F_q s(s - \frac{Z_\alpha}{V}) + F_\alpha(1 + \frac{Z_q}{V})s]$ + $\underline{(s^2 + (2\zeta\omega - F_{\dot\eta})s + (\omega^2 - F_\eta))[(s - \frac{Z_\alpha}{V})(s^2 - M_q s) - M_\alpha(1 + \frac{Z_q}{V})s]}$

Table 5 - Approximate literal transfer function polynomials - $\tilde{G}_q^{\delta_e} = \frac{\tilde{N}(s)}{\tilde{D}(s)}$

Ñ(s)	$(M_{\delta_e} - \phi' F_{\delta_e})s[s + (-\frac{Z_\alpha}{V})][s + (\frac{b - (b^2 - 4c)^{1/2}}{2})][s + (\frac{b + (b^2 - 4c)^{1/2}}{2})]$
D̃(s)	$s[s^2 + (-\frac{Z_\alpha}{V} - M_q)s + (\frac{Z_\alpha}{V}M_q - (1 + \frac{Z_q}{V})M_\alpha)]\ (s^2 + (2\zeta\omega - F_{\dot\eta})s + (\omega^2 - F_\eta))$

$b = [M_{\delta_e}(2\zeta\omega - F_{\dot\eta}) + \phi' F_{\delta_e} M_q] / (M_{\delta_e} - \phi' F_{\delta_e})$ $c = M_{\delta_e}(\omega^2 - F_\eta) / (M_{\delta_e} - \phi' F_{\delta_e})$