AXISYMMETRICAL RESPONSE BY A PENNY-SHAPED INTERFACE CRACK IN MULTI-LAYERED COMPOSITES

Ma Xingrui, Zou Zhenzhu, Huang Wenhu and Shao Chengxun

Dept. of Appl. Mech., Harbin Institute of Technology
Harbin, P. R. China

Abstract

A rigorous theory of the scattering of normally incident longitudinal wave by a penny-shaped interface crack in multi-layered composites is presented. Made use of Hankel integral transform, a transfer matrix has been obtained, and the problem is reduced to a set of dual integral equations in matrix form, which are then reduced to a set of singular integral equations treated numerically by Jacobi polynomials. As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been investigated in more detail. The scattered field is derived in far field case by means of the contour integral technique and stationary phase method. The theoretical results have been shown that at large distance from the crack, the scattered displacements in the layer are composed of Rayleigh-Like-Mode waves predominantly, and the P wave and SV wave are predominant in the half space. The scattered amplitudes of displacements for the first two modes are plotted versus the incident wave frequency, and it is observed that the multi-resonances occur at some frequency.

I. Introdution

Recently, the theoretical problem of the scattering of elastic waves by crack has received considerable attention. Authors, working in many different areas, such as applied mathematics, applied mechanics, geophysics, seismical engineering and quantitative non-destructive evaluation (QNDE), have contributed to the aknowledge of this subject.

This paper is concerned with the scattering of time harmonic, normally incident longitudinal wave by a penny-shaped interface crack in multi-layered compositeds. The scattering of elastic wave by a penny-shaped crack located in an infinite isotropic medium has been considered by Mal[1], Matin[2], Matin and Wickham[3]. Srivastava et al.[4] have investigated the interaction of longitudinal wave with a penny-shaped crack at the interface of two bonded dissimilar elastic solids. The papers related to the problems are refered to those of Neerhoff[5], Yang and Bogy[6], Angel[7]. The problem of this paper is more complicated and difficult one.

The formulation of the problem is presented in section 2. The total field in the cracked layered media is analyzed as the superposition of the incident field and scattered field. The incident field has been given by Eeing et al.[8], and scattered field can be changed into mixed boundary value problems.

Hankel transform is used in section 3 to obtain a suitable general solution of the wave equations. The transfer matrices are obtained, and the problem is reduced to a set of dual integral equations in matrix form.

As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been considered in section 4. The discussion of numerical results are presented in section 5.

II. Formulation

Consider n layers and a half space are bonded together perfectly, except in the regin $0\leq r<1$, $z=0$, where is a penny-shaped crack, as shown in Fig.1.

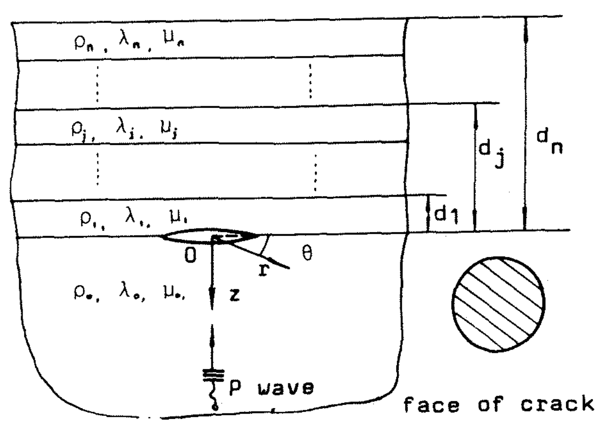

Fig.1 Incidence of a longitudinal wave in multi-layered composites on a penny-shaped interface crack

Suppose the layers and half space are occupied by homogineous isotropic materials with different properties and a incident wave is harmonic longitudinal wave which is impinging at the crack normally. The face of the crack is free of tractions. Thus, the problem is reduced to an axisymmetric elastodynamic problem.

The total field can be divided into the sum of the incident field and scattered field according to:

$$\{ \underset{\sim}{u}^{(t)}, \underset{\sim}{\tau}^{(t)} \} = \{ \underset{\sim}{u}^{(i)}, \underset{\sim}{\tau}^{(i)} \} + \{ \underset{\sim}{u}^{(s)}, \underset{\sim}{\tau}^{(s)} \} \quad (1)$$

in which, $\{ \underset{\sim}{u}^{(t)}, \underset{\sim}{\tau}^{(t)} \}$ represents the total field, $\{ \underset{\sim}{u}^{(i)}, \underset{\sim}{\tau}^{(i)} \}$ the incident field which is assumed to be known, and $\{ \underset{\sim}{u}^{(s)}, \underset{\sim}{\tau}^{(s)} \}$ the scattered field, i. e., the modification to the incident field due to the presence of crack. For the scattered field one should solve the following boundary value problem (superscription s and the time factor exp(-iωt) have been suppressed):

$$\nabla^2 \varnothing_j + K_{Lj}^2 \varnothing_j = 0$$
$$\nabla^2 \psi_j + K_{Tj}^2 \psi_j = 0 \quad (2)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1\,\partial}{r\partial r} + \frac{\partial^2}{\partial z^2}$

$$K_{Lj}^2 = \omega^2 \rho_j / (\lambda_j + 2\mu_j)$$
$$K_{Tj}^2 = \omega^2 \rho_j / \mu_j \quad (3)$$
$$j = 0, 1, 2, \dots n$$

ρ_j—the density, λ_j, μ_j—the lame constants, K_{Lj} and K_{Tj} are the P wave and SV wave numbers, respectively. The subscription j (j=0, 1, 2, ···, n) represents the half space and the layers.

The displacements and stresses can be represented in the $\varnothing_j$, ψ_j

$$u_{rj} = \frac{\partial \varnothing_j}{\partial r} + \frac{\partial^2 \psi_j}{\partial z \partial r}$$
$$u_{zj} = \frac{\partial \varnothing_j}{\partial z} - \frac{\partial}{r\partial r}\left(r \frac{\partial \psi_j}{\partial r}\right)$$
$$\tau_{zzj} = \lambda_j \nabla^2 \varnothing_j + 2\mu_j \left[\frac{\partial^2 \varnothing_j}{\partial z^2} - \left(\frac{\partial}{\partial z}\nabla^2 \psi_j - \frac{\partial^3 \psi_j}{\partial z^3}\right)\right]$$
$$\tau_{rrj} = \lambda_j \nabla^2 \varnothing_j + 2\mu_j \left[\frac{\partial^2 \varnothing_j}{\partial r^2} + \frac{\partial^3 \psi_j}{\partial z \partial r^2}\right]$$

$$\tau_{rzj} = \mu_j \left\{ 2\frac{\partial^2 \varnothing_j}{\partial r \partial z} + \left[2\frac{\partial^3 \psi}{\partial r \partial z^2} - \frac{\partial}{\partial r}(\nabla^2 \psi_j)\right]\right\} \quad (4)$$
$$j = 0, 1, \dots n$$

The boundary and continuous conditions are

$$\tau_{zzn} = \tau_{zrn} = 0 \qquad z=-d_n$$
$$\tau_{zzj} = \tau_{zzj-1}, \quad \tau_{zrj} = \tau_{zrj-1}$$
$$u_{rj} = u_{rj-1}, \qquad u_{zj} = u_{zj-1}$$
$$j=2,3,4\dots \qquad z=-d_{j-1} \quad (5)$$

$$\tau_{zr1} = \tau_{zr0}, \quad \tau_{zz1} = \tau_{zz0}$$
$$u_{r1} = u_{r0}, \quad u_{z1} = u_{z0} \qquad z = 0, \; 1 < r < \infty$$

$$\tau_{zr1} = \tau_{zr0} = -\tau_{zr}^{(i)} = 0$$
$$\tau_{zz1} = \tau_{zz0} = -\tau_{zz}^{(i)}, \quad z = 0, \; 0 \le r < 1 \quad (6)$$

where $\tau_{zz}^{(i)}$ is known.

In addition, the scattered field must satisfy the following radiation condition:

$$limit\; R \left\{ \frac{\partial u_0}{\partial R} - i\,|K|\, u_0 \right\} = 0 \quad (7)$$

where $R=(r^2+z^2)^{\frac{1}{2}}$, K is wave number vector modulus.

III. Integral Equations

Applying Hankel integral transform to wave motion equations (2), and taking into account of radiation condition (7), we obtain formulae as following:

$$\varnothing_j(r,z) = \int_0^\infty \xi i \left[A_j(\xi) e^{i\gamma_{Lj} z} + B_j(\xi) e^{-i\gamma_{Lj} z}\right] J_0(\xi r) d\xi$$
$$\psi_j(r,z) = \int_0^\infty \xi \left[\frac{C_j(\xi)}{\xi} e^{i\gamma_{Tj} z} + \frac{D_j(\xi)}{\xi} e^{-i\gamma_{Tj} z}\right] J_0(\xi r) d\xi$$
$$j = 1, 2, \dots, n \quad (8)$$

$$\varnothing_0(r,z)=\int_0^{\infty}\xi i A_0(\xi)e^{i\gamma_{L0}z}J_0(\xi r)d\xi$$

$$\psi_0(r,z)=\int_0^{\infty}\xi\left[\frac{1}{\xi}C_0(\xi)e^{i\gamma_{T0}z}J_0(\xi r)d\xi\right.$$

(9)

in which, $J_0(\xi r)$ is zero order Bessel function, $A_j(\xi)$, $B_j(\xi)$, $C_j(\xi)$ and $D_j(\xi)$ are unknown functions.

$$\gamma_{Lj}=(K_{Lj}^2-\xi^2)^{\frac{1}{2}},\quad \gamma_{Tj}=(K_{Tj}^2-\xi^2)^{\frac{1}{2}}$$

$$j=0,1,2,\cdots n \qquad (10)$$

The following two basic unknown functions $\overline{R}_1(\xi)$ and $\overline{S}_1(\xi)$ which are defined as:

$$\int_0^{\infty}[\overline{R}_1(\xi)/\xi]\xi J_1(\xi r)d\xi=\begin{cases}u_{r0}-u_{r1} & 0\le r<1\\ 0 & r>1\end{cases}$$

$$\int_0^{\infty}[\overline{S}_1(\xi)/\xi]\xi J_0(\xi r)d\xi=\begin{cases}u_{z0}-u_{z1} & 0\le r<1\\ 0 & r>1\end{cases} \qquad (11)$$

Taking the Hankel integral transform for the boundary and continuous conditions (5) (6), one obtain transfer matrices of $A_j(\xi)$, $B_j(\xi)$, $C_j(\xi)$ and $D_j(\xi)$:

$$[A_{j-1}(\xi),B_{j-1}(\xi),C_{j-1}(\xi),D_{j-1}(\xi)]^T=$$
$$=T_j[A_j(\xi),B_j(\xi),C_j(\xi),D_j(\xi)]^T$$

$$j=2,\cdots,n \qquad (12)$$

$$\begin{Bmatrix}A_0(\xi)\\C_0(\xi)\end{Bmatrix}=Q_0\begin{Bmatrix}A_1(\xi)\\B_1(\xi)\\C_1(\xi)\\D_1(\xi)\end{Bmatrix}$$

$$Q_1\begin{Bmatrix}A_1(\xi)\\B_1(\xi)\\C_1(\xi)\\D_1(\xi)\end{Bmatrix}=\begin{Bmatrix}\dfrac{-i\overline{R}_1(\xi)}{\xi}\\[2mm]\dfrac{\overline{S}_1(\xi)}{\xi}\end{Bmatrix} \qquad (13)$$

$$E\begin{Bmatrix}A_n(\xi)\\B_n(\xi)\end{Bmatrix}+F\begin{Bmatrix}C_n(\xi)\\D_n(\xi)\end{Bmatrix}=\begin{Bmatrix}-i\overline{R}_1(\xi)/\xi\\\overline{S}_1(\xi)/\xi\end{Bmatrix}$$

$$\begin{Bmatrix}C_n(\xi)\\D_n(\xi)\end{Bmatrix}=Q_n\begin{Bmatrix}A_n(\xi)\\B_n(\xi)\end{Bmatrix} \qquad (14)$$

where T_j, Q_0, Q_1, E, F and Q_n are known matrices, which can be refered to Ma[9].

Then, made use of the surface conditions of the crack, the integral equations in matrix are obtained as following:

$$\int_0^{\infty}\begin{bmatrix}\xi J_0(\xi r) & 0\\ 0 & \xi J_1(\xi r)\end{bmatrix}WQ\begin{bmatrix}[E+FQ_n]^{-1}\\Q_n[E+FQ_n]^{-1}\end{bmatrix}\begin{Bmatrix}-i\overline{R}_1(\xi)/\xi\\\overline{S}_1(\xi)/\xi\end{Bmatrix}d\xi=\begin{Bmatrix}-\tau_{zz}^{(1)}\\0\end{Bmatrix}$$

$$0\le r<1,\ z=0 \qquad (15)$$

$$\int_0^{\infty}\begin{bmatrix}J_1(\xi r) & 0\\ 0 & J_0(\xi r)\end{bmatrix}\begin{Bmatrix}\overline{R}(\xi)\\\overline{S}(\xi)\end{Bmatrix}d\xi=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$r>1,\ z=0$$

The equations (15) are a set of dual integral equations for the scattering of elastic wave by a penny-shaped interface crack in multi-layered composites. The derivation in more detail could be refered to Ma[9].

IV. Example

As an example, the scattering of elastic wave by a penny-shaped interface crack in one layered half space has been considered, i.e., $d_1=d$, $d_k=o$ (k=2,3,...,n). The set of dual integral equations (15) can be reduced to a set of singular integral equations by means of Abel integral transforms and the results of Lowengrub and Sneddon[10]:

$$\begin{bmatrix}\beta & 0\\0 & -\beta\end{bmatrix}\begin{Bmatrix}R(x)\\S(x)\end{Bmatrix}-\frac{\alpha}{\pi}\int_{-1}^{1}\begin{bmatrix}0 & \dfrac{1}{u-x}\\\dfrac{1}{u-x} & 0\end{bmatrix}\begin{Bmatrix}R(u)\\S(u)\end{Bmatrix}du$$

$$+\frac{1}{\pi}\int_{-1}^{1}\begin{bmatrix}K_1(u,x) & -iK_2(u,x)\\iK_3(u,x) & -K_4(u,x)\end{bmatrix}\begin{Bmatrix}R(u)\\S(u)\end{Bmatrix}du$$

$$=\begin{Bmatrix}C_0\\\dfrac{f(x)}{\mu_1}\end{Bmatrix}\qquad |x|<1 \qquad (16)$$

where α, β, C_0, $f(x)$ and $K_j(u,x)$ (j=1,2,3,

4) are given in Ma[9], R(x) and S(x) are the unknown functions. Following the method of Erdogan[11], and expanding the unknown functions as an infinite series in Jacobi polynimals, the singular integral equations (16) can be reduced to a set of algebraic equations which are refered to Ma[9].

The expandation can be expressed:

$$\begin{Bmatrix} R(\bar{x}) \\ S(\bar{x}) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} W_1(\bar{x}) P_n^{(a_1, b_1)}(\bar{x}) & 0 \\ 0 & W_2(\bar{x}) P_n^{(a_2, b_2)}(\bar{x}) \end{bmatrix} \begin{Bmatrix} C_{1n} \\ C_{2n} \end{Bmatrix} \qquad (17)$$

where c_{1n} and c_{2n} (n=0,1,2,...) are under-mined constants which can be solvee by the algebraic equations, and P_n (n=0,1,2,...) is Jacobi polynimals.

$$W_1(x) = (1-x)^{a_1}(1+x)^{b_1}$$

$$W_2(x) = (1-x)^{a_2}(1+x)^{b_2}$$

$$a_1 = -\frac{i}{2\pi} \ln\left(\frac{1+\gamma}{1-\gamma}\right)$$

$$b_1 = \frac{i}{2\pi} \ln\left(\frac{1+\gamma}{1-\gamma}\right)$$

$$a_2 = b_1 \qquad b_2 = a_1$$

$$\gamma = \frac{\alpha}{\beta} \qquad (18)$$

Upon the constants c_{1n} and c_{2n} (n=0,1,2, ...) are obtained , using the equations (8)-(14),(4), displacements can be expressed in integral form. making use of contour integral technique and the some asymptotic analysis methods, the surface scattered displacements in the layer at large distance from the crack are presented

$$u_r^{(sf)}(r,t) = \frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{rK_j}\right)^{\frac{1}{2}} A_j^{(sf)} \cdot e^{i(K_j r - wt - \frac{\pi}{4})} + 0\left(\frac{1}{r^{3/2}}\right)$$

$$u_z^{(sf)}(r,t) = \frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{rK_j}\right)^{\frac{1}{2}} B_j^{(sf)} \cdot e^{i(K_j r - wt - \frac{\pi}{4})} + 0\left(\frac{1}{r^{3/2}}\right) \qquad (19)$$

in which,

$$A_j^{(sf)} = \int_{-1}^{1} R(\bar{x}) \cos K_j \bar{x}\, d\bar{x}\, \frac{T_{11}^{(sf)}(K_j)}{\Delta'(K_j)} - i\int_{-1}^{1} S(\bar{x}) \sin K_j \bar{x}\, d\bar{x}\, \frac{T_{12}^{(sf)}(k_j)}{\Delta'(K_j)}$$

$$B_j^{(sf)} = \int_{-1}^{1} R(\bar{x}) \cos K_j \bar{x}\, d\bar{x}\, \frac{T_{21}^{(sf)}(K_j)}{\Delta'(K_j)} - i\int_{-1}^{1} S(\bar{x}) \sin K_j \bar{x}\, d\bar{x}\, \frac{T_{22}^{(sf)}(K_j)}{\Delta'(K_j)}$$

$$j = 1, \cdots, N, \qquad (20)$$

where $T_{mn}^{(sf)}(K_j)$ (m=1,2,n=1,2) and $\Delta'(K_j)$ are given in Ma[9]. K_j(j=1,2,...,N) is root from the generalized Rayleigh function $\Delta(K)$.

The scattered displacements for the half space at large distance form the crack are obtained:

$$u_R^{(sp)}(R,\varnothing,t) = G_L(\varnothing,w)\frac{e^{i(K_{L0}R - wt)}}{(RK_{L0})} + 0\left(\frac{1}{R}\right)^{3/2}$$

$$u_\varnothing^{(sp)}(R,\varnothing,t) = G_T(\varnothing,w)\frac{e^{i(K_{T0}R - wt)}}{(RK_{T0})} + 0\left(\frac{1}{R}\right)^{3/2} \qquad (21)$$

where $R = (r^2+z^2)^{\frac{1}{2}}$, $\varnothing = -tg^{-1}\left(\frac{r}{z}\right)$,

$G_L(\varnothing,w)$ and $G_T(\varnothing,w)$ are given in Ma[9].

From the formulae (19) and (21), it can be seen that the scattered displacements in the layer are composed of Rayleigh-Like Mode waves predominantly, and the P wave and SV wave are predominant in the half space.

V. Discussion of Numerical Results

Numerical results are presented in Figs. 2-5. These results were computed for two groups parameters.

1) the layer is AL, and the half space is NI:

$d=0.3$ or $d=0.6$, $\mu_1=26.5\times10^9 (N/m^2)$,

$\rho_1=2.7\times10^3 (kg/m^3)$, $\lambda_1=56.3\times10^9 (N/m^2)$

$\mu_0=66.5\times10^9 (N/m^2)$, $\rho_0=8.8\times10^3 (kg/m^3)$

$\lambda_0=108.5\times10^9 (N/m^2)$

2) the layer is AU, and the half space is AL:

$d=0.8$, $\mu_1=28.0\times10^9 (N/m^2)$

$\rho_1=19.3\times10^3 (kg/m^3)$, $\lambda_1=147.0\times10^9 (N/m^2)$

$\mu_0=26.5\times10^9 (N/m^2)$, $\rho_0=2.7\times10^3 (kg/m^3)$

$\lambda_0=56.3\times10^9 (N/m^2)$

For the layer -half space combination—the first groups parameters which are said to " stiffen" case ($V_{t1}>V_{to}$), there is only one scattered Rayleigh-Like-Mode wave in the layer. Figure 2 and Figure 3 show the modulus of the ratios of coefficient A_1 and B_1 to incident P wave amplitude A_0 versus the frequency, respectively. For $w=0$, there is no scattered field in the solid. The several resonance peaks can occur in some range of frequency. The resonance amplitude decreases as d increases.

However, for the other layer-half space combination——the second groups parameters which are said to " loading " case ($V_{t1}<V_{to}$), there are multi-scattered Rayleigh-Like -Mode waves in the layer. The first two mode Rayleigh - waves occur with the range of the frequency of this paper. Fig. 4 and Fig.5 represent the first mode and the second mode of Rayleigh wave, respectively. The resonance peaks in Fig.4-5 appear also. The numerical results show that the first mode amplitude is larger than that of the second mode.

The computations have been performed on the SIEMENS 7570c computer of the computing centre of Harbin Institute of Technology.

Finally, it is pointed out that the method used in this paper can be applied to the scattering of elastic waves by multi-cracks in multi-layered composites.

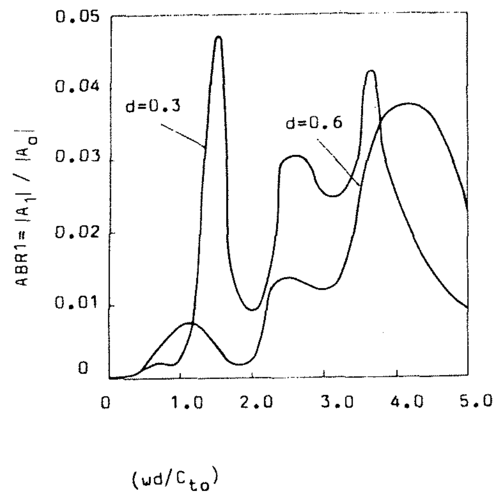

Fig.2 Modulus of the ratio of coefficient A_1 to incident P wave amplitude A_0 for the first layer-half space combination

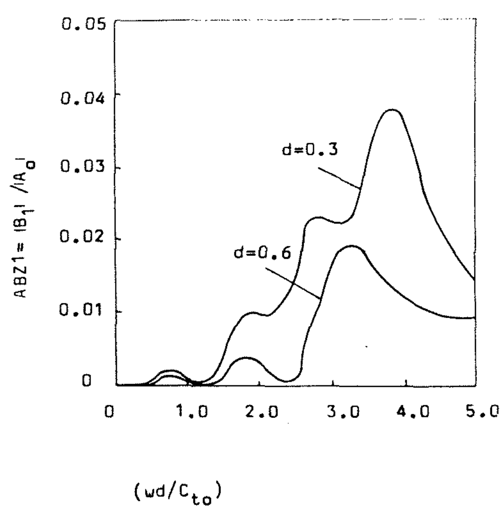

Fig.3 Modulus of the ratio of coefficient B_1 to incident P wave amplitude A_0 for the first layer-half space combination

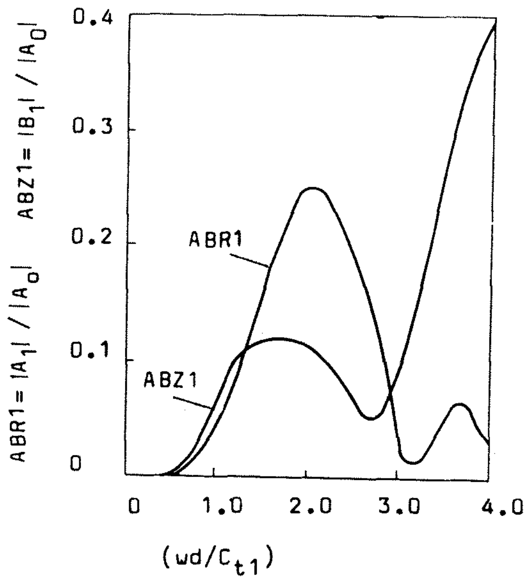

Fig.4 Modulus of the ratio of coefficients A_1 and B_1 to incident P wave amplitude A_0 for the second layer-half space combination

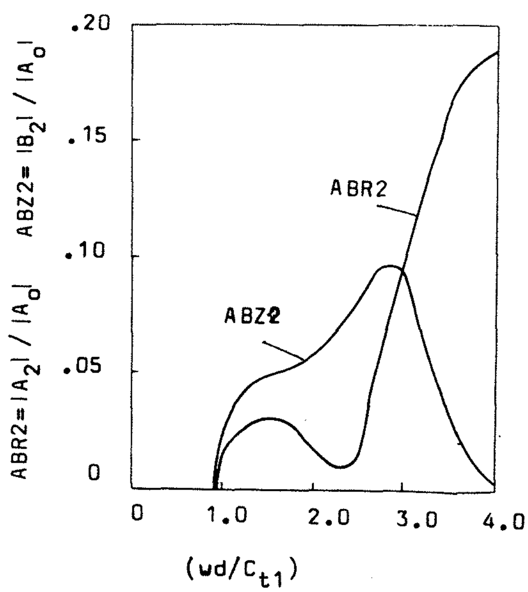

Fig.5 Modulus of the ratio of coefficients A_2 and B_2 to incident P wave amplitude A_0 for the second layer-half space combination

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