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THE CALCULATION OF ENERGY RELEASE RATE COMPONENTS USING THE COUPLED STRAIN ENERGY*

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ABSTRACT

An accurate estimate of energy release rate components (G_I, G_{II}) is crucial to understending as well as predicting delamination-related damage in composite laminates. The virtual crack extension method uses the difference between the strain energy for two crack positions to evaluate the total energy release rate (G_T). This method supplies one algebraic equation in two unknowns G_I and G_{II}, which does not provide enough information to find these components separately. This makes it impractical in mixed mode situations.

A new method the "Coupled Strain Energy" based upon the coupled strain energy of two superimposed equilibrium states is presented in this paper. It's main contribution consists in provinding an equation for G_T components in addition to the existing one supplied by the virtual crack extension method. This method utilizes several properties of the conservation J integral [1] and its equality to the energy release rate for linear elastic situations. This approach can utilize numerically provided results as well as analytical solutions .

Applications which utilize analytical results obtained by the shear deformation (SD) model [2] , as well as numerical results obtained by the FEM are presented. The results obtained using the "Coupled Strain Energy" method are compared with results obtained by different numerical based methods and by analytical models.

I. INTRODUCTION

Currently, a considerable amount of research activity is devoted to the study of failure mechanisms in laminated composite structures. A thorough knowledge of these mechanisms is necessary, not only to avoid catastrophic failures, but also to create efficient and durable structures.

The delamination is one of the most predominant modes of damage. It is characterized by a complex state of stress with steep gradients in the vicinity of its initiation or by an energy balance in the cracked structure in classical fracture mechanics.

It is assumed that energy is dissipated when a cracked surface is created in a stressed body. The rate of the strain energy dissipation per cracked surface is known as the energy release rate (G) and can be obtained from energy considerations or, by using the stress functions from the crack tip region. It can be defined as the sum of three particular modes of crack action which are mode I or the opening mode, mode II or the forward shearing mode, and mode III, the tearing mode. If the magnitude of G_T exceeds a critical value G_{cr} - a material property and fracture toughness - crack propagation is assumed to occur. Therefore, an accurate knowledge of G_T component values is needed to analyze composite structure against failure and damage growth.

Irwin [3] showed that the elastic strain energy released during an incremental crack extension can be equated to the work done in closing the incremental crack. Based upon this formulation various numerical methods have been developed and utilized for G predictions in laminated composite structures. All these methods assume an existing crack-like flaw in the structure [4],[5],[6],[7],[8].

An alternative method that uses analytical formulations as well as numerical results to obtain stress intensity factors is the J integral [1]. This approach uses the stress and displacement field distributions along a path around the crack to obtain the stress intensity factor and is based on the fact that the J integral is path independent [9],[10].Based on the J integral and the conservation laws of anisotropic elasticity a mixed mode crack analysis of rectilinear anisotropic solids was formulated in [11].

An additional method that uses the forces and the displacements obtained at two opposite nodes of an assumed crack extension to evaluate the stress intensity factors is presented in [12].

A new method based upon the coupled strain energy of two superimposed equilibrium states follows.

II. The Coupled Strain Energy Approach

The fundamental basis of the present approach exploits the superposition of an auxiliary equilibrium state to an unknown mixed mode situation under consideration. The analysis requires the evaluation of the coupled strain energy between the two states. Several properties related to the conservation integral J are recalled and used in the present formulation. Finally, unknown stress intensity factors or G_T components are obtained in terms of the auxiliary solutions and the coupled strain energy.

Consider two equilibrium states "1" and "2" which are denoted by the superscripts of the field variables. From superposition, the strain energy for the superimposed state "0" is:

$$U^{(0)} = U^{(1)} + U^{(2)} + U^{(1,2)} \qquad (1.1)$$

where the coupling term $U^{(1,2)}$ comes from the reciprocal theorem.

The identity between the J integral and Irwin's energy release rate G can be expressed for constant applied load as:

$$J = G_T = -\frac{dU}{da} \qquad (1.2)$$

For the superimposed state, the substitution of Equation (1.1) into Equation (1.2) yields

$$J^{(0)} = G_T^{(0)} = \frac{dU^{(0)}}{da} = \frac{dU^{(1)}}{da} + \frac{dU^{(2)}}{da} + \frac{dU^{(1,2)}}{da} \qquad (1.3)$$

The J integral relation for the superimposed state is defined in [9],[11] by:

$$J^{(0)} = J^{(1)} + J^{(2)} + M^{(1,2)} \qquad (1.4)$$

where the coupled term $M^{(1,2)}$ is defined by

$$M^{(1,2)} = 2\alpha_{11}K_I^{(1)}K_{II}^{(2)} + \alpha_{12}\left[K_I^{(1)}K_{II}^{(2)} + K_{II}^{(1)}K_I^{(2)}\right] + 2\alpha_{22}K_{II}^{(1)}K_{II}^{(2)} \qquad (1.5)$$

and by the integral equation:

$$M^{(1,2)} = \int \left[U^{(1,2)}dy - \left(T_i^{(1)} \frac{\partial u_i^{(2)}}{\partial x} + T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial x}\right) ds\right] \qquad (1.6)$$

The parameters α_{11}, α_{12}, and α_{22} are related to the roots of the fourth-order governing partial differential equation in plane anisotropic elasticity [13], and K_I and K_{II} are the stress intensity factors related to the appropriate superimposed states denoted by the superscripts 1 and 2. The u_i, ds, and the T_i are the displacement vector, an element of arc length and the traction vector components, respectively.

Equations (1.5) and (1.6) use the J integral approach to relate the stress intensity factors to the $M^{(1,2)}$ integral. This is done through a tedious evaluation of the $M^{(1,2)}$ integral where considerable difficulties accompany its determination [11].

A direct comparison between Equations (1.3) and (1.4) reveals

$$J^{(1)} = \frac{dU^{(1)}}{da} \qquad (a)$$

$$J^{(2)} = \frac{dU^{(2)}}{da} \qquad (b) \qquad (1.7)$$

$$M^{(1,2)} = \frac{dU^{(1,2)}}{da} \qquad (c)$$

Equation (1.7c) relates $dU^{(1,2)}/da$ to the stress intensity factors given in Equation (1.5). Therefore, the complexity of evaluating the $M^{(1,2)}$ integral presented by Equation (1.6) is circumvented.

By this formulation stress intensity factors or the G_T components of the superimposed states can be related to the total strain energy values. This relation provides an additional algebraic equation to the existing one (Equation (1.2)) and enables evaluation based upon only energy considerations for mixed mode situations.

III. Applications

A brief explanation related to the superimposed equilibrium states follows.

An auxiliary equilibrium state is superimposed on an unknown mixed mode situation. By selecting the auxiliary state as a pure mode I or pure mode II situations, the final algebraic equations yield simple relations. The unknown state, pure mode I state and pure mode II state are denoted by the superscripts 2, α and β, respectively. By expressing the stress intensity factors $K_I^{(2)}$ and $K_{II}^{(2)}$ in terms of $G_I^{(2)}$ and $G_{II}^{(2)}$, Equations (1.7c) and (1-5) lead to the following relations for an orthotropic material system [14]:

$$G_I^{(2)} = \frac{\left(\frac{dU^{(\alpha,2)}}{da}\right)^2}{4G_I^{(\alpha)}} \qquad (2.1)$$

$$G_{II}^{(2)} = \frac{\left(\frac{dU^{(\beta,2)}}{da}\right)^2}{4G_{II}^{(\beta)}} \qquad (2.2)$$

where $G_I^{(\alpha)}$ and $G_{II}^{(\beta)}$ are the pure mode I and mode II contributions from the auxiliary solutions. The coupled strain energy between the auxiliary pure mode I state and the unknown mixed mode state is $U^{(\alpha,2)}$. Similarly, $U^{(\beta,2)}$ represents the coupled strain energy between the auxiliary pure mode II state and the unknown mixed mode state.

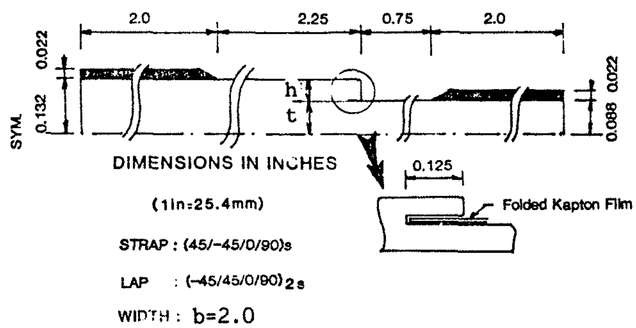
The coupled strain energy can be obtained by computing the work done by the forces from one equilibrium state on the displacements from the other. An alternative way consists in using Equation (1.1) from which the coupled strain energy can be evaluated as:

$$U^{(1,2)} = U^{(0)} - U^{(1)} - U^{(2)} \qquad (2.3)$$

Applications which utilize analytical results obtained by the SD model, as well as numerical results obtained by the FEM are presented in the next two sections.

A. Shear Deformation Model Solution

The SD model and the previous approach are used to estimate the G_T components for the double cracked-lap-shear (DCLS) specimen (Figure 1).



Properties of Graphite/Epoxy Tape AS4/3501-6

Material Tape	E_{11} GPa(MSI)	E_{22} GPa(MSI)	G_{12} GPa(MSI)	ν_{12}	Nominal Fiber Volume %	Nominal Ply Thickness mm(inches)
AS4/3501-6	139.3(20.2)	11.7(1.70)	5.8(0.85)	0.3	65%	0.1397(0.0055

Figure 1. The DCLS Specimen-Geometry And Properties

The two superimposed states for this solution are described in Figure 2. An auxiliary solution was selected as a pure mode I situation, denoted by the superscript α. The unknown mixed mode situation is denoted by the superscript 2 ($G_I^{(2)}$, $G_{II}^{(2)}$) while, $U^{(\alpha,2)}$ denotes the coupled strain energy between the two states.

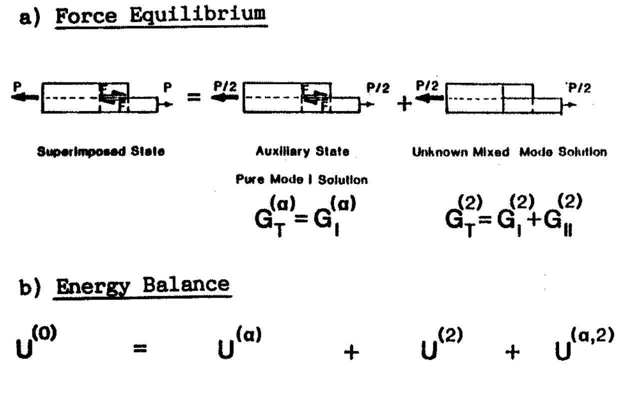


Figure 2. The Two Superimposed States

In this application, the analysis is orientated to find the variables associated with Equation (2.1).

For a linear,elastic material [15] the total energy release rate G is given as

$$G_T = \frac{P^2}{2b} \frac{dC}{da} \qquad (2.4)$$

The first step is to use Equation (2.4) for the auxiliary pure mode I situation:

$$G_T = G_I^{(\alpha)} = \frac{P^2}{2b} \frac{dC^{(\alpha)}}{da} \qquad (2.5)$$

where $C^{(\alpha)}$ is the specimen compliance under a pure mode I situation.

The analytical methodology to obtain $C^{(\alpha)}$ by using the SD model is described in [14].By substituting its expression into Equation (2.5), the following form is obtained:

$$G_I^{(\alpha)} = \frac{P}{4E_{11}t} \left\{ N_o\left(1 - \frac{h+t}{4t+h}\right) + N_{1j}^{*}\left[\operatorname{sech}^2 s_j(L-a) + \frac{1-\cosh ra}{\sinh^2 ra}\right] + \frac{N_o(2t-h)}{(4t+h)} \frac{(1-\cosh ra)}{\sinh^2 ra} \right\}$$

$$(j=1,2) \qquad (2.6)$$

Summation over the index is implied in the second term of Equation (2.6). The constants s_j and r- are the roots of the characteristic equations which represent the system of the coupled ordinary differential equations for the sublaminate analysis [14].The P is the applied load while N_{1j}^* are the interlaminar force resultants obtained from the boundary conditions.The elastic properties and the laminate thickness are E_{11}, h and t (Figure 1).

The second step is to evaluate the coupled strain energy between the auxiliary state α, and the mixed mode state 2. This is established by computing the work done by the forces from the auxiliary state on the displacement from the mixed mode state. The final expression for the change in the coupled strain energy with respect to the crack length is

$$\frac{dU^{(\alpha,2)}}{da} = \frac{P^2}{4A_H}\left[\frac{L_{12}M_{1j}\operatorname{sech}^2 s_j(L-a)}{(M_{11} - M_{12})} - \left(1 - \frac{A_H}{(A_{11})_3}\right)\right]$$

$$(j=1,2) \qquad (2.7)$$

Summation over the index is implied in the first term of Equation (2.7). The constants M_{1j},are the moment resultants obtained from the boundary conditions while A_H, $(A_{11})_3$ and L_{12} represent the elastic properties of the sublaminates [14]. Now that the values for $G_I^{(\alpha)}$ and $(dU^{(\alpha,2)}/da)$ have been found, their substitution into Equation (2.1) yields $G_I^{(2)}$. In order to obtain the sliding unknown mode $G_{II}^{(2)}$ the the following formulation is used:

$$G_{II}^{(2)} = G_T^{(2)} - G_I^{(2)} \qquad (2.8)$$

B. Finite Element Model Solution

In this section, the coupled strain energy approach utilizes FEM results to evaluate the G_T components for the DCLS specimen. The strain energy values required in the coupled strain energy formulation are supplied as an ordinary output by the EAL [16] program.

As in the previous section, Figure 2 represents the equilibrium states under consideration. A very coarse mesh was used in the FEM simulations, exploiting the insensitivity of the strain energy differences to the number of DOF as described [4],[7],[14]. To obtain the first equation(2.1), a pure mode I situation is selected as the auxiliary solution.

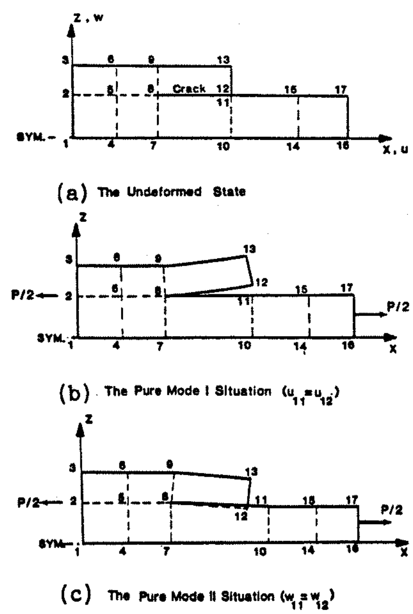


Figure 3. Finite Element Representation

Figure 3a shows the schematic representation for this configuration. The mode II suppression is achieved by placing a very stiff beam element in x direction between the coincident nodes, 11 and 12. Under the applied load, this stiff element constrains the relative displacements in x direction between these nodes. In this case, a pure mode I situation is obtained as presented in Figure 3b. For this auxiliary case, the strain energy $U^{(\alpha)}$ is obtained and two "runs" make possible its derivative with respect to the crack length.This provides the value $G_I^{(\alpha)}$. An additional finite element simulation is conducted in order to evaluate the total strain energy for the superimposed state. Finally, all the values involved in $G_I^{(2)}$ estimation as presented by Equation (2.1) have been defined.

The choice of mode II supression as an auxiliary solution causes a penetration of the upper crack surface into the lower one, as described by Figure 3c. As a result, a contact situation is created between the crack surfaces. In order to solve this contact problem, an iterative solution has to be conducted which is beyond the scope of this study. This situation is circumvented by using Equation (2.8).

IV. Results and Discussion

The results obtained using the coupled strain energy approach are presented in Tables 1 and 2. A systematic comparison with results obtained by different numerical based methods and by analytical models is given in these tables for the DCLS specimen. This is done in order to compare methods and to present the benefits associated with each method and model.

Method	DOF	G_I [J/m²]	Error%	G_{II}	Error%	G_T	Error%	Remarks
Coupled Strain Energy	482	257.	1.5	431.	1.1	686.	0.0	- Uses an auxiliary equilibrium state
	1484	256.	1.1	432.	1.0	688.	0.0	- Coarse mesh
Virtual Crack Extention	482	—		—		689.	0.0	- Limited to G_T predictions
	1484	—		—		688.	0.0	- Coarse mesh
	5490	—		—		688.	0.0	
Crack Closure [12]	482	278.	9.8	411.	5.7	689.	0.0	- Dependent on mesh and Δa size
	1484	269.	6.3	416.	4.5	685.	0.0	
	3968	264.	4.3	424.	2.7	688.	0.0	- Needs refinement at the crack tip
	5490	261.	3.0	426.	2.2	687.	0.0	
• Sensitivity • Approach [14]	2 Terms	254.	0.0	434.	0.0	688.	0.0	- Independent of mesh and Δa size
	3 Terms	253.	0.0	435.	0.0	688.	0.0	- Needs numerical G_I and G_{II} results for application
	4 Terms	253.	0.0	436.	0.0	688.	0.0	

Table 1. The Energy Release Rate Components Obtained By Numerical Based Methods

The G component results presented in Table 1 are obtained using numerical based methods. The values used by the different methods are achieved by using a FEM simulation with an assumed crack extenssion equal to 10 percent of the total crack length. Lacking an exact solution, the acceptability of the computed results is determined in this context by comparison with the sensitivity method [14]. The results predicted by this method are independents of mesh size and crack extension size.

A difference of 6 and 8 percent in G_I and G_{II} values, respectively, is obtained between the coupled strain energy method and the sensitivity method. The difference in G_T is only 3 percent. The same difference in G_T is obtained using the virtual crack extension method. However, this method does not predict the G_T components. By using the crack closure technique, these differences are within 10 percent for G_I and G_{II} at 482 DOF and decrease to about 2.5 percent at 5490 DOF.

Method	Model	G_I [J/m²]	Error%	G_{II}	Error%	G_T	Error%	Remarks
Coupled Strain Energy	SD	231.	8.4	457.	4.8	688.	0.0	Uses an auxiliary equilibrium state
[16] Crack Closure	SD	235.	7.1	454.	4.1	689.	0.0	Crack step size based upon boundary layer decay length
[14] Crack Closure	M	220.	13.0	469.	7.5	689.	0.0	Crack step size based upon boundary layer decay length
Irwin's [3] Definition	CL	—		—		689.	0.0	Limited to G_T prediction Least computational effort

Table 2. The Energy Release Rate Components Obtained By Analytical Methods

Several remarks associated with each method are summarized in this table. The advantage --- accuracy and efficiency represented by the coarse mesh-- is fully exploited by the coupled strain energy method. This is clear by comparing the number of DOF used.The results presented in Table 2 are based upon methods which utilize analytical models to evaluate the G_T components. A comparison between G_I and G_{II} values obtained by the "sensitivity" method and those obtained by the coupled strain energy method which utilizes the SD model exhibits a 8.4 and 4.8 percent difference, respectively. Almost the same difference is observed by using the crack closure method[16] with the SD model. However, the advantage achieved by the coupled strain energy by eliminating the requirement of "boundary decay length definition" is considerable.

V. Concluding Remarks

A new method based upon the total strain energy of two superimposed equilibrium states is presented. It's main contribution consists in providing an equation for G_T components in addition to the existing one supplied by the virtual crack method. This method utilizes several properties of the conservation J integral and its equality to the energy release rate for linear elastic situations.

This approach can utilize numerically provided results as well as analytical solutions to determine the $G_T$components. The total energy considerations involved in this approach make it very attractive for cases in which the strain energy values are supplied by numerical methods such as the FEM. These values are substituted in simple algebraic equations which yield reliable estimations of $G_T$components.

In the cases that analytical solutions are used, this method circumvents the need of a boundary layer decay length definition and leads to solutions for G_T components independent of this parameter.

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