ICAS-88-1.3.2 OPTIMAL GUIDANCE FOR HIGH ORDER AND ACCELERATION CONSTRAINED MISSILE

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Abstract

Explicit formulas of optimal guidance laws for linear, time invariant, arbitrary order and acceleration constrained missile are derived. These formulas are given in terms of the missile's transfer function and acceleration constraint. Optimal, full state feedback guidance laws are synthesized and compared to 1st order approximation and the Proportional Navigation for minimum and non-minimum missile's dynamics. Simulations on a third order missile's model show the relative gain from using the full order guidance law versus the acceleration constraint as well as some robustness tests.

I. Introduction

The optimal control theory has been used to derive modern guidance laws which have improved performance. This improved performance is achieved by considering the detailed dynamics of the target and missile. However, it comes at the expense of increased complexity in realization, sensitivity to knowledge of various parameters, etc.

An extensive study of the literature on guidance laws (GL) in general, and on optimal guidance laws in particular, is performed in[1]. In various references [2-6] optimal guidance laws are derived for 1st and 2nd order missiles, respectively. In our previous paper[8] the structure of optimal guidance laws for linear, arbitrary high order missile was considered. Mainly, we derived the closed loop, general structure formulas of the guidance law. Further, we studied the behaviour of the gains for minimum and non-minimum phase missiles and compared the performance of some suboptimal approximations of the guidance laws (GL).

The effects of the acceleration constraint (imposed by the structural or aerodynamic limitations) on guidance laws and performance for 1st order missile are systematically treated by Anderson[7].

In this paper we derive an optimal guidance law, on collision course, for linear, time invariant, arbitrary order and acceleration constrained missile. It is shown that for minimum phase missile the optimal guidance law is the guidance law for unconstrained missile with saturation on the commanded acceleration. However, for non-minimum phase missile this is only a suboptimal guidance law and the optimal controller is more complicated.

In the paper comparison of the Proportional Navigation, 1st order approximation and full order optimal guidance laws is performed on a third order minimum and non-minimum phase acceleration constrained missile. The comparison is performed on a common basis. Moreover, the robustness of these guidance laws is subject to an analysis, namely, the sensitivity to uncertainty/variation in parameters, radome refraction slope and acceleration constraint is checked for minimum and non-minimum phase airframes.

The main conclusions are that for minimum phase missile the full order guidance law does not give improved performance with respect to the 1st order approximation, while for non-minimum phase missile there are situations, (combinations of pole and zero placement, and acceleration constraint) when the full order guidance law is worth being used.

II. General Problem and Solution

In order to derive an optimal guidance law let us consider the minimization of the following quadratic performance index. (This performance is equivalent to "any other" quadratic unconstrained index, as shown in Appendix A).

$$J = \frac{1}{2} x^T(t_f)Gx(t_f) + \int_t^{t_f} u^T(\tau)Ru(\tau)d\tau \tag{1}$$

where $x(\cdot)$ is the state vector, $u(\cdot)$ is the control vector; $G\geq 0$ and $R\geq 0$ are weighting matrices; and t_f is the time of flight. All vectors and matrices are of the appropriate dimensions. The minimization of the performance index is subject to the linear differential equation constraint

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

and a constraint on the input

$$u^Tu \leq U_o^2 \quad . \tag{3}$$

In appendix B the following solution is obtained*

$$u(t) = -U_o \,\mathrm{Sat}[\frac{1}{U_o} R^{-1}B^T\Phi^T(t_f,t)Gx(t_f)] \quad , \tag{4}$$

where the terminal state is given implicitly by the integral equation

$$x(t_f) = \Phi(t_f,t)\, x(t) - \int_t^{t_f} \Phi(t_f,\tau)BU_o \cdot \mathrm{Sat}\,[\frac{1}{U_o}R^{-1}B^T\Phi^T(t_f,\tau)Gx(t_f)]d\tau \tag{5}$$

and

$$\dot{\Phi}(t,t_o) = A\Phi(t,t_o), \quad \Phi(t_o,t_o) = I \quad . \tag{6}$$

This optimal solution is usually difficult to implement. It may be approximated by the following practical solution:

$$u(t) = -U_o \mathrm{Sat}\{\frac{1}{U_o}R^{-1}B^T\Phi^T(t_f,t)G[I+\int_t^{t_f} \Phi(t_f,\tau) \cdot BR^{-1}B\Phi^T(t_f,\tau)G\, d\tau]^{-1}\, \Phi(t_f,t)x(t)\} \quad . \tag{7}$$

This practical solution is for some cases, that will be described in the sequel, the optimal solution, when (5) has at most one solution. For other cases, for which (5) has more than one solution, it is only a suboptimal solution.

* The saturation vector function is defined

$$\mathrm{Sat}[x] = \begin{cases} x & \|x\|\leq 1 \\ \frac{x}{\|x\|} & \|x\|>1 \end{cases} \quad .$$

III. Optimal Guidance Law Derivation

The intercept geometry is shown on Figure 1. Here we use the same methodology and notation as in[8]. The linearized kinematics are given by the differential equations:

$$\dot{y} = v$$
$$\dot{v} = a_T - a_m \quad . \tag{8}$$

The dynamics of the n-th order missile are

$$\frac{d}{dt}\begin{bmatrix} a_m \\ \underline{\rho}_m \end{bmatrix} = \begin{bmatrix} a_{11} & \underline{a}_{12} \\ \underline{a}_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_m \\ \underline{\rho}_m \end{bmatrix} + \begin{bmatrix} b_1 \\ \underline{b}_2 \end{bmatrix} u(t) \quad . \tag{9}$$

This is a partition of the state variables where the missile's acceleration, a_m, is the 1st state variable and $\underline{\rho}_m$ are the rest n-1 state variables. a_{11}, b_{11} are scalars; $\underline{a}_{21}$, $\underline{a}_{12}^T$, $\underline{b}_2$ are $(n-1)\times 1$ vectors; a_{22} is a $(n-1)\times(n-1)$ matrix; and $u(t)$ is a scalar input. Consequently, the system equation (2) is:

$$\frac{d}{dt}\begin{bmatrix} y \\ v \\ a_m \\ \underline{\rho}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{11} & \underline{a}_{12} \\ 0 & 0 & \underline{a}_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y \\ v \\ a_m \\ \underline{\rho}_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ \underline{b}_2 \end{bmatrix} u(t). \tag{10}$$

The contribution of deterministic target's acceleration, a_T, is treated in [6,11], therefore it will not be considered here and is left for the example.

Further, let us assume that we are interested only in the minimization of the final miss, $y(t_f)$, i.e.:

$$G = \mathrm{diag}\,[g, 0 \ldots. 0], \quad R = 1 \quad . \tag{11}$$

A. General Solution

The substitution of (10, 11) in (4,5) results in

$$u(t) = -U_o \mathrm{Sat}\left\{\frac{g}{U_o} L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\Bigg|_{t_f-t} y(t_f)\right\} \tag{12}$$

where $y(t_f)$ is given implicitly by

$$y(t_f) = y + (t_f-t)\,v -$$

$$\left[L^{-1}\left\{\frac{1}{s^2}\frac{a_m(s)}{a_m(o)}\right\}\Bigg|_{t_f-t} \quad L^{-1}\left\{\frac{1}{s^2}\frac{a_m(s)}{\underline{\rho}_m(o)}\right\}\Bigg|_{t_f-t}\right]\begin{bmatrix} a_m \\ \underline{\rho}_m \end{bmatrix}$$

$$-\int_t^{t_f} L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\Bigg|_{t_f-\tau} \cdot \tag{13}$$

$$U_o \mathrm{Sat}\left\{\frac{g}{U_o} L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\Bigg|_{t_f-\tau} y(t_f)\right\} d\tau$$

where

$\frac{a_m(s)}{u(s)}$ - missile transfer function,

$\frac{a_m(s)}{a_m(o)}$ - missile acceleration response to initial condition in the acceleration state,

$\frac{a_m(s)}{\underline{\rho}_m(o)}$ - missile acceleration response to initial conditions in the states $\underline{\rho}_m$, $(n-1)\times 1$ vector*.

The initial condition responses $a_m(s)/a_m(o)$ and $a_m(s)/\underline{\rho}_m(o)$ are evaluated from the homogeneous equation of the missile (9).

B. Minimum Phase Missile

This section deals with a minimum phase missile. The ramp response of minimum phase missile is monotonously increasing**, i.e. $L^{-1}\left\{\frac{1}{s}\frac{a_m(s)}{u(s)}\right\}\Big|_t > 0$. Figure 2 describes the behaviour of the argument of the saturation function in (12,13). From this figure one can see that for $t_1<t<t_f$ no saturation occurs and (12,13) can be solved explicitly. At $t=t_1$ saturation occurs simultaneously in (12,13). This saturation sustains for $t<t_1$. The polarity of the saturation depends on $y(t_f)$. Consequently, for stable minimum phase missile the optimal GL is

$$u(t) = -U_o \,\mathrm{Sat}\left[\frac{1}{U_o}\Lambda(t_f-t)\,Z(t_f-t)\right] \tag{14}$$

where $\Lambda(\ )$, the guidance gain and $Z(\ )$, the zero effort miss, are given by:

$$\Lambda(t_f-t) = \frac{L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\Big|_{t_f-t}}{\frac{1}{g} + \int_t^{t_f}\left\{L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{u(s)}\right]\Big|_{t_f-\tau}\right\}^2 d\tau} \tag{15}$$

$$Z(t_f-t) = y + (t_f-t)v - \left[L^{-1}\left\{\frac{1}{s^2}\frac{a_m(s)}{a_m(o)}\right\}\Bigg|_{t_f-t} \quad L^{-1}\left\{\frac{1}{s^2}\frac{a_m(s)}{\underline{\rho}_m(o)}\right\}\Bigg|_{t_f-t}\right]\begin{bmatrix} a_m \\ \underline{\rho}_m \end{bmatrix} \tag{16}$$

this is exactly the practical solution (7).

C. Non-Minimum Phase Missile

For non-minimum phase missile the ramp response is not monotonously increasing***, as may be seen on figure 3, for missile with one zero at RHP, which describes the behaviour of the argument of the saturation function in (12,13). For $t_2<t<t_f$, (14, 15, 16) is the optimal solution, however, for $t<t_2$ it is not. Close inspection of the problem will discover that the gain given by (14, 15, 16) is smaller than the optimal and the switching times t_1, t_2, t_3 are given only implicitly. Since the exact optimal guidance law is complicated to solve and implement, in the sequel only the suboptimal/ practical guidance law will be considered.

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* The following notation is used:

$$\frac{\underline{v}}{\underline{x}} \triangleq \left[\frac{v}{x_1}, \frac{v}{x_2}, \ldots \frac{v}{x_k}\right] \quad .$$

** This section applies as well for the more general class of missiles with monotonously non-decreasing ramp response.

*** To be precise, this section applies only for the more restricted class of missiles with non-monotonous ramp response.

IV. Example

As an example we will consider here guidance of missile whose airframe and autopilot model is described by a 3rd order transfer function with one zero, i.e.

$$H(s) = \frac{a_m(s)}{u(s)} = \frac{-s\tau_z+1}{(s\tau_1+1)\,(s\tau_2+1)\,(s\tau_3+1)} \quad . \qquad (17)$$

In this example it is assumed that the target's acceleration, a_T, is described by a step function whose initiation time is uniformly distributed over the flight time.

Figure 4 shows the schematic diagram of the guidance loop. It includes the target model, where $w_T(t)$ represents the random target's maneuver [13], the kinematics, glint noise - $w_g(t)$, steady state Kalman Filter which produces the best estimates of the state variables (y, v, a_T) [11, Appendix B], the guidance law (c_1, c_2, c_3, c_4, c_5, c_6), saturation function and model of the airframe + autopilot.

We consider/compare the performance of three guidance laws:

a) The full order suboptimal GL;
b) First order approximation of the GL;
c) Proportional Navigation.

The comparison is performed by a computation of RMS miss due to the target's acceleration and glint by the SLAM analysis.[9]

A. Full Order Suboptimal/Practical GL

In order to derive the full order suboptimal/practical GL we use the following state space description of the missile (derived from (17))

$$\frac{d}{dt}\begin{bmatrix} a_m \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\tau_z \\ 0 & 0 & 1 \\ -\frac{1}{a_3} & -\frac{\tau_z+a_1}{a_3} & -\frac{a_2}{a_3} \end{bmatrix} \begin{bmatrix} a_m \\ \rho_2 \\ \rho_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a_3} \end{bmatrix} u(t) \qquad (18)$$

where

$$a_1 = \tau_1 + \tau_2 + \tau_3$$
$$a_2 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$$
$$a_3 = \tau_1\tau_2\tau_3 \ .$$

The full order GL is then given by

$$c_1 = 1$$

$$c_2 = t_f - t$$

$$c_3 = \tfrac{1}{2}(t_f-t)^2$$

$$c_4 = -L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{a_m(o)}\right]\Bigg|_{t_f-t} \qquad (19)$$

$$c_5 = -L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{\rho_2(o)}\right]\Bigg|_{t_f-t}$$

$$c_6 = -L^{-1}\left[\frac{1}{s^2}\frac{a_m(s)}{\rho_3(o)}\right]\Bigg|_{t_f-t}$$

$$\Lambda(t_f-t) = \frac{L^{-1}\left[\frac{1}{s^2}H(s)\right]\Big|_{t_f-t}}{\frac{1}{g}+\int_t^{t_f}\left\{L^{-1}\left[\frac{1}{s^2}H(s)\right]\Big|_{t_f-\tau}\right\}^2 d\tau}, \quad g = 10^{10}. \qquad (20)$$

The commanded acceleration, before the saturation, is

$$A_c = \Lambda(t_f-t)[c_1y+c_2v+c_3a_T+c_4a_m+c_5\rho_2+c_6\rho_3] \qquad (21)$$

and

$$\frac{a_m(s)}{a_m(o)} = \frac{a_3s^2+a_2s+a_1+\tau_z}{a_3s^2+a_2s^2+a_1s+1}$$

$$\frac{a_m(s)}{\rho_2(o)} = \frac{a_3s+a_2+\tau_z(\tau_z+a_1)}{a_3s^3+a_2s^2+a_1s+1} \qquad (22)$$

$$\frac{a_m(s)}{\rho_3(o)} = \frac{a_3(1-s\tau_z)}{a_3s^3+a_2s^2+a_1s+1}$$

The coefficient of the target's acceleration c_3 is taken as in[11]. Although the coefficients c_4, c_5, c_6 can be solved explicitly we precomputed them by integrating backward as part of the SLAM analysis.

B. First Order Approximate GL

Here in order to derive the guidance law we approximate the missile transfer function by a single pole, i.e.

$$H(s) \approx \frac{1}{s\tau_a+1} \qquad (23)$$

where $\tau_a = \tau_1+\tau_2+\tau_3+\tau_z$. Then c_1, c_2 and c_3 are unchanged, $c_5 = 0$, $c_6 = 0$ and

$$c_4 = \tau_a^2\ \left[e^{-T}+T-1\right], \quad T \triangleq \frac{t_f-t}{\tau_a}, \qquad (24)$$

and

$$\Lambda(t_f-t) = \frac{L^{-1}\left[\frac{1}{s^2}\frac{1}{s\tau_a+1}\right]\Big|_{t_f-t}}{\frac{1}{g}+\int_t^{t_f}\left\{L^{-1}\left[\frac{1}{s^2}\frac{1}{s\tau_a+1}\right]\Big|_{t_f-\tau}\right\}^2 d\tau} \qquad (25)$$

which gives [11, eq.(6)], for $g \to \infty$.

C. Proportional Navigation

Proportional Navigation is derived if the missile is assumed to have instantaneous response, i.e. $a_m(s)/u(s) = 1$. Then we have c_1 and c_2 unchanged, $c_3 = 0$, $c_4 = 0$, $c_5 = 0$, $c_6 = 0$, and

$$\Lambda(t_f-t) = \frac{3}{(t_f-t)^2}, \quad g \to \infty \ . \qquad (27)$$

D. Results

This section presents representative results of the performance for the missile's model and GL's previously described.

Figure 5 presents curves of the effective navigation ratio, N', ($N' \triangleq (t_f-t)^2 \Lambda(t_f-t)$), versus time-to-go, t_{go}, ($t_{go} = t_f-t$), of the 1st order approximate GL and the 3rd order GL for minimum and non- minimum phase autopilots, respectively. One can see that the effective navigation ratio goes to the positive infinity ($g \to \infty$) for the 1st order approximation and 3rd order minimum phase case. For 3rd order non-minimum phase case the effective navigation ratio behaves more "wildly" and goes to the negative infinity.

The following analysis is presented for target's acceleration of 5g uniformly distributed over 5 sec and glint with spectral density of 1 m^2/Hz.

Figure 6 shows the RMS miss distance versus the value of the missile's autopilot zero, for unconstrained missile. One can see, as may be expected, the uniform performance of the 3rd order full state feedback GL. For minimum phase case the difference between the 1st order approximation and the 3rd order GL is minor. However, for the non- minimum phase case one can see the superiority of the full order GL.

Figures 7 and 8 present the RMS miss distance versus the autopilot's zero placement for missile's constraint of 50g and 30g, respectively. One can see that the acceleration constraint degrades the performance. The 3rd order GL with the constraint is superior now for smaller range of the zero placement.

Figures 9, 10 and 11 present results for constrained missile in a normalized form, i.e. curves of the normalized miss, RMS miss/RMS miss without constraint and full order GL , versus the normalized missile's maneuverability, maximal missile's acceleration/target's acceleration, ($a_{M_{max}}/a_T$) for a_T = 5g.
From Figure 9, one can see that for minimum phase missile the full order GL is no better than the 1st order approximation. In other words, for such airframe the 1st order approximation is sufficient and the higher order GL gives negligible improvement. However, for non-minimum phase missile, one can see from figures 10 and 11, for $a_{M_{max}}/a_T > 10$ (i.e. $a_{M_{max}} > 50$ at a_T = 5g), the gain from using the 3rd order full state feedback GL. By comparison of Figures 10 and 11, one can trade off between maneuverability and response time (the missile on Figure 11 has shorter response). The same results can be deduced from Figures / and 8 as well. (The RMS miss without constraint and full order GL for missile on Figures 9 and 10 is 2.56m, and on Figure 11 is 2.36m).

Figures 12, 13 and 14 present sensitivity/ robustness studies with respect to the radome refraction slope, for missiles acceleration constraint of 30g. Figure 12 is for minimum phase airframe. One can see that the difference between the 1st and 3rd order GL's is practially negligible. For non-minimum phase airframe, Figures 13 and 14, for τ_α = 1, 4 sec, 14, for τ_α = 1, 4 sec, respectively, one can see the improved robustness in respect of the radome refraction slope of the 3rd order GL.

Appendix A

Let us consider the following two optimization problems on finite interval.

Problem I:

$$J_1 = \frac{1}{2} x_f^T P_f x_f + \frac{1}{2} \int_t^{t_f} [x^T Q x + 2u^T S x + u^T R u]\, d\tau$$

$$\min J_1, \text{ subject to } \dot{x} = Ax + Bu$$

Problem II:

$$J_2 = \frac{1}{2} x_f^T W_f x_f + \frac{1}{2} \int_t^{t_f} u^T V u\, d\tau$$

$$\min J_2, \text{ subject to } \dot{x} = Fx + Hu$$

Lemma: Optimization Problems I and II are equivalent if

$$\begin{aligned} F &= A - BR^{-1}S - BR^{-1}B^T\bar{P} \\ H &= B \\ W_f &= P_f - \bar{P} \\ V &= R \end{aligned}$$

where $\bar{P}$ is the solution of the Algebraic Matrix Riccati Equation

$$(A-BR^{-1}S)^T\bar{P}+\bar{P}(A-BR^{-1}S)-\bar{P}B^TR^{-1}B\bar{P}+Q-S^TR^{-1}S = 0$$

Proof: a) The solution of Problem I is

$$u = R^{-1}(S + B^T P)x$$

$$-\dot{P} = (A-BR^{-1}S)^T P+P(A-BR^{-1}S)-\bar{P}B^TR^{-1}B\bar{P}+Q-S^TR^{-1}S;$$

$$P(t_f) = P_f,\ t \le t_f,$$

the solution P is then

$$P(t)=\bar{P}+\Phi^T(t_f,t)(P_f-\bar{P})\cdot$$

$$\{I+\int_t^{t_f}\Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)(P_f-P)d\tau\}^{-1}\Phi(t_f,t);$$

$$t \le t_f;\ \frac{d}{dt}\Phi(t,t_o) = (A-BR^{-1}S - B^TR^{-1}B\bar{P})\,\Phi(t,t_o)$$

$$\Phi(t_o,t_o) = I\ ,$$

and

$$(A-BR^{-1}S)^T\bar{P}+\bar{P}\,(A-BR^{-1}S)-\bar{P}B^TR^{-1}B\bar{P}+Q-S^TR^{-1}S = 0\ .$$

Then the closed loop is

$$\dot{x} = \{A-BR^{-1}S-BR^{-1}B^T\bar{P}-BR^{-1}B\Phi^T(t_f,t)(P_f-\bar{P})\cdot$$

$$[I+\int_t^{t_f}\Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)(P_f-\bar{P})d\tau]^{-1}\Phi(t_f,t)\}x$$

b) The solution of Problem II is

$$u = -V^{-1}H^T W\, x$$

$$-\dot{W} = F^T W + WF - WH^T V^{-1} HW\ ,\quad W(t_f) = W_f,\ t \le t_f,$$

the solution W is

$$W(t) = \psi^T(t_f,t)W_f\{I+\int_t^{t_f}\psi(t_f,\tau)HV^{-1}H^T\psi^T(t_f,\tau)W_f d\tau\}^{-1}$$

$$\cdot\psi(t_f,t);\ \frac{d}{dt}\psi(t,t_f) = F\psi(t,t_o);\quad \psi(t_o,t_o) = I,$$

and

$$\dot{x}=\{F-HV^{-1}H\psi^T(t_f,t)W_f$$

$$[I+\int_t^{t_f}\psi(t_f,\tau)HV^{-1}H^T\psi^T(t_f,\tau)W_f d\tau]^{-1}\psi(t_f,t)\}x.$$

and the equivalence directly follows.

Appendix B: General Problem and Solution

The problem being considered here is to minimize the quadratic performance index (1) subject to the constraints (2) and (3).

The solution is obtained by the minimization of the Hamiltonian[10,12]

$$H(x,p,u) = \frac{1}{2} u^T Ru + p^T[Ax+Bu] \qquad (B.1)$$

with the constraint $u^Tu \le U_o{}^2$ where p is the costate vector.

1) First let us assume that $u^Tu < U_o{}^2$ so that H is derivable. Then the solution is derived from

$$H_u = 0 \qquad (B.2a)$$

$$H_x^T = -\dot{p} \qquad (B.2b)$$

$$p(t_f) = \frac{\partial}{\partial x(t_f)}[\tfrac{1}{2}x^T(t_f)Gx(t_f))] = Gx(t_f) \qquad (B.2c)$$

then from (B.2a) the feedback is

$$u = -R^{-1}B^Tp \qquad (B.3)$$

and we have the following two-point boundary value problem

$$\dot{x} = Ax - BR^{-1}B^Tp \ ; \quad x(t_o) = x_o \qquad (B.4a)$$

$$\dot{p} = -A^Tp \qquad ; \quad p(t_f) = G\,x(t_f) \qquad (B.4b)$$

From (B.4a) we have

$$x(t_f) = \Phi(t_f,t)x(t)-\int_t^{t_f}\Phi(t_f,\tau)BR^{-1}B^Tp(\tau)d\tau \qquad (B.5)$$

where

$$\dot{\Phi}(t,t_o) = A\Phi(t,t_o) \ ; \quad \Phi(t_o,t_o) = I \qquad (B.6)$$

and from (B.4b)

$$p(t) = \Phi^T(t_f,t)\,G\,x(t_f) \quad . \qquad (B.7)$$

Next, substitution of (B.7) into (B.5) and rearrangement give

$$\{I+\int_t^{t_f}\Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)Gd\tau\}x(t_f) = \Phi(t_f,t)x(t) \qquad (B.8)$$

and finally (B.8), (B.7) and (B.3) give

$$u(t) = -R^{-1}B^T\Phi^T(t_f,t)G\cdot \qquad (B.9)$$

$$\{I+\int_t^{t_f}\Phi(t_f,\tau)BR^{-1}B^T\Phi^T(t_f,\tau)Gd\tau\}^{-1}\Phi(t_f,t)x(t) \ .$$

2) Now, let us assume that u reached the constraint (3) so that the Hamiltonian (B.1) is underivable and one should look for a solution by direct minimization of H according to Pontriagyn's minimum principle[12], i.e.

$$\min \quad H(x,\ p,\ u) \quad , \quad u^Tu \le U_o{}^2 \qquad (B.10)$$

and it is necessary that

$$H(x^*,\ p^*,\ u^*) \ \le \ H(x^*,\ p^*,\ u) \qquad (B.11)$$

where $(\ )^*$ denotes the values at optimum.

So we find that

$$\frac{1}{2}\,u^{*T}Ru^*+p^{*T}(Ax^*+Bu^*] \le \frac{1}{2}\,u^TRu+p^{*T}[Ax^*+Bu] \qquad (B.12)$$

and we have

$$u^{*T}[B^Tp^* + \frac{1}{2}\,Ru^*] \le u^T[B^Tp^* + \frac{1}{2}\,Ru] \quad . \qquad (B.13)$$

The optimal control must minimize the scalar product

$$< u,\ \ B^Tp^* + \frac{1}{2}\,Ru > \quad , \qquad (B.14)$$

i.e. u and $B^Tp^* + \frac{1}{2}\,Ru$ should be parallel and in opposite direction. Figure B.1 shows a geometric interpretation of equation (B.14).

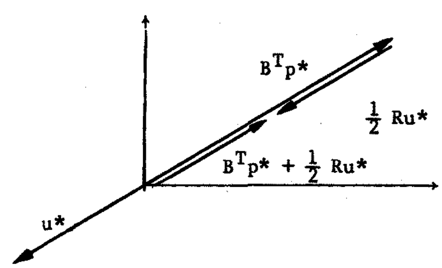

Figure B.1

Since $u^{*T}u^* = U_o{}^2$ we claim that the optimal feedback is

$$u^* = -U_o\,\frac{R^{-1}Bp^*}{\|R^{-1}B^Tp^*\|} \quad . \qquad (B.15)$$

Since the Hamiltonian is unconstrained with respect to $x(t_f)$ and p, (B.2b) and (B.2c) must be satisfied, i.e. we have the two-point boundary value problem

$$\dot{x} = Ax + Bu \quad ; \quad x(t_o) = x_o \qquad (B.16a)$$

$$\dot{p} = \quad -A^Tp \quad ; \quad p(t_f) = Gx(t_f) \qquad (B.16b)$$

where u is taken from (B.15).

3) Finally, from the previous section we deduce that the solution to the problem considered here is the two-point boundary value problem

$$\dot{x} = Ax + Bu \quad ; \quad x(t_o) = x_o \qquad (B.17a)$$

$$\dot{p} = -A^Tp \quad ; \quad p(t_f) = Gx(t_f) \qquad (B.17b)$$

$$u = -U_o\ \mathrm{Sat}\ \{\ \frac{1}{U_o}\,R^{-1}B^Tp\ \} \quad . \qquad (B.17c)$$

From (B.17b) we have

$$p(t) = \Phi^T(t_f,t)\ Gx(t_f) \qquad (B.18)$$

and from (B.17a) we have the implicit equation for $x(t_f)$

$$x(t_f) = \Phi(t_f,t)x(t) - \int_t^{t_f} \Phi(t_f,\tau)BU_o \cdot \mathrm{Sat}\{\frac{1}{U_o} R^{-1}B^T\Phi^T(t_f,\tau)Gx(t_f)\}d\tau \qquad (B.19)$$

and

$$u(t) = -U_o \ \mathrm{Sat}\{ \frac{1}{U_o}R^{-1}B^T\Phi^T(t_f,t)Gx(t_f)\} \ . \qquad (B.20)$$

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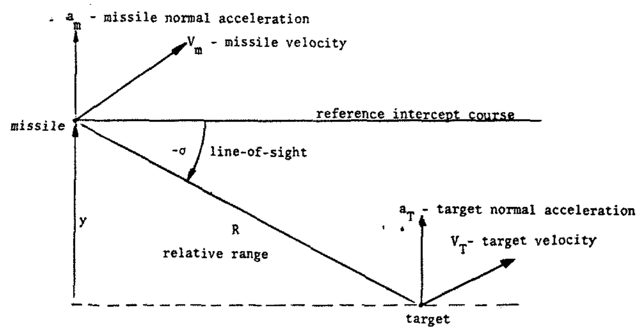


Figure 1: Intercept Geometry.

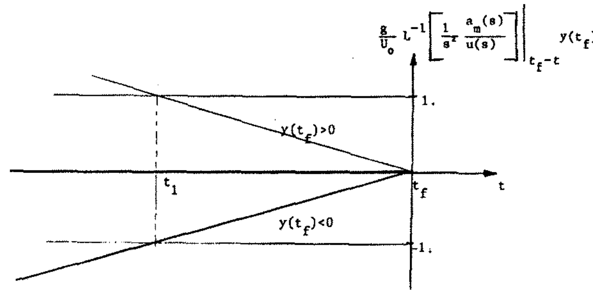


Figure 2: Argument of the Saturation Function in (12,13) versus time for minimum phase missile.

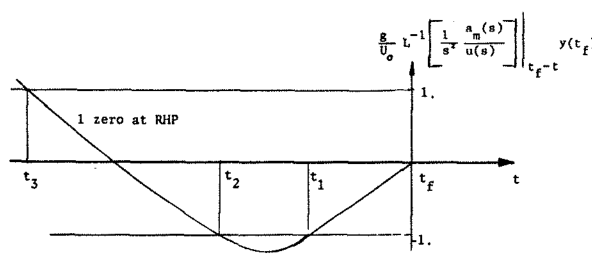
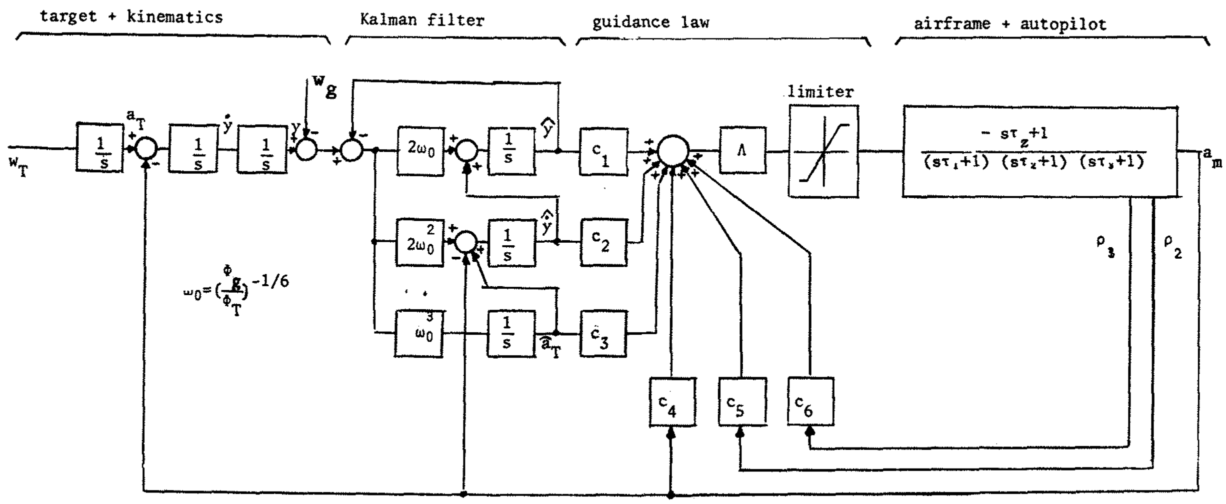


Figure 3: Argument of the Saturation Function in (12,13) versus time for non-minimum phase missile with one zero at RHP.



Φ_g - spectral density of the target's glint(w_g).

Φ_T - spectral density of the target's maneuver(w_T).

Figure 4: Schematic Diagram of the Guidance Loop.

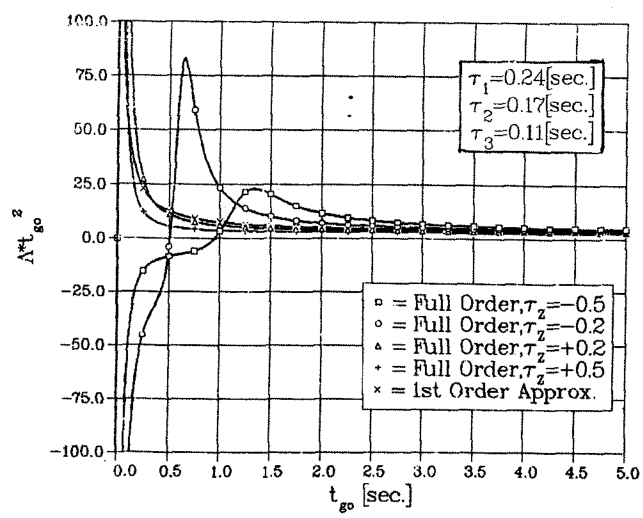


Figure 5: The effective Navigation Ratio versus time-to-go, for 1st Order Approximate Guidance Law and 3rd Order Guidance Law for Minimum and Non-minimum Missiles.

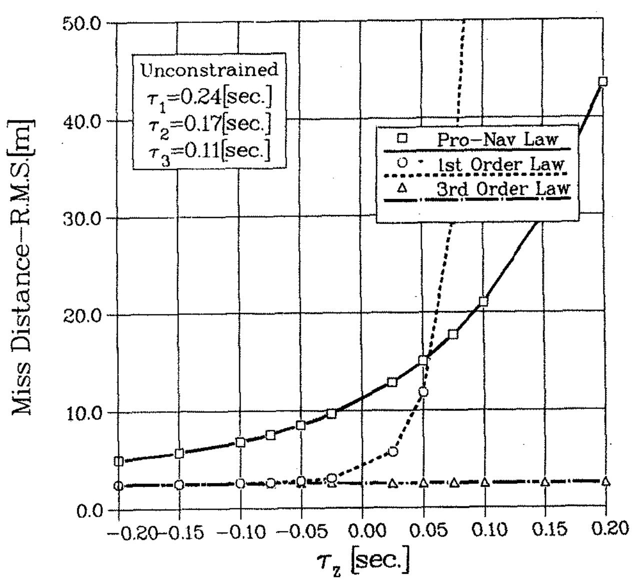
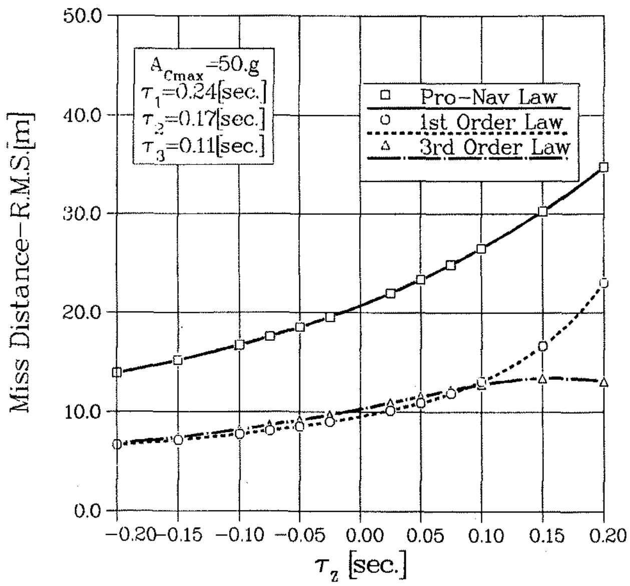


Figure 6: The RMS Miss Distance versus The Zero Placement for Unconstrained Missile.



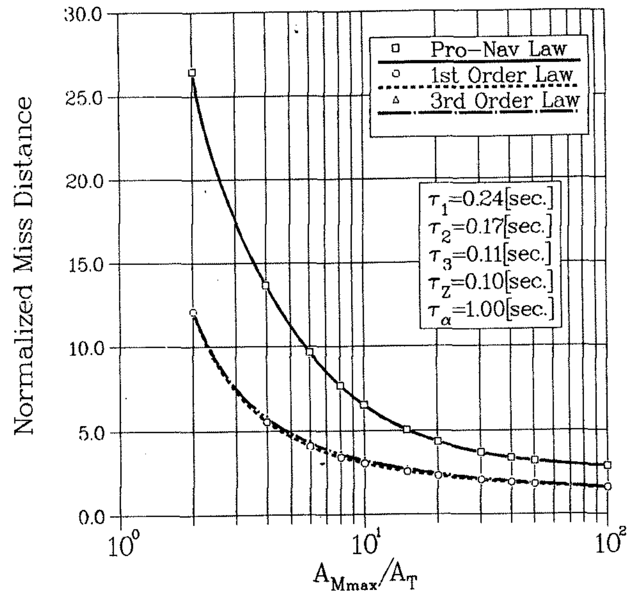
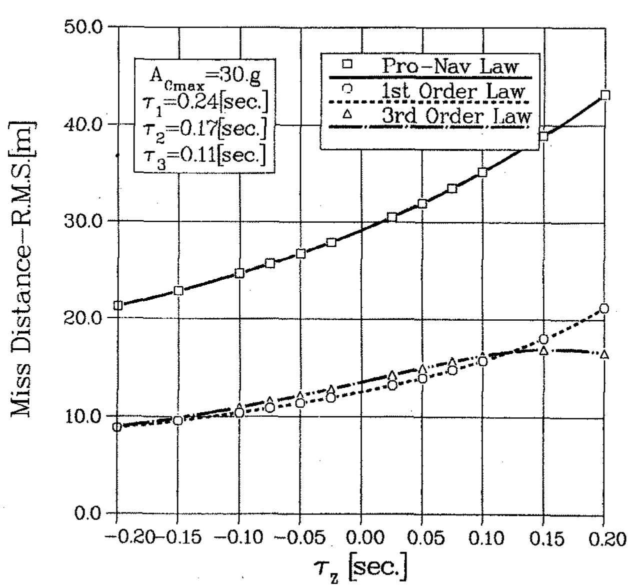


Figure 7: The RMS Miss Distance versus The Zero Placement for Missile's Acceleration Constraint of 50g.

Figure 9: Normalized Miss versus The Normalized Missile's Maneuverability for Minimum Phase Missile.



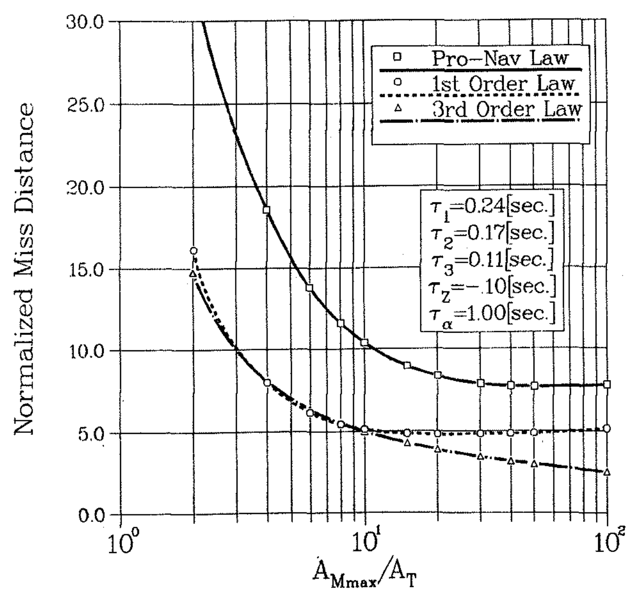


Figure 8: The RMS Miss Distance versus The Zero Placement for Missile's Acceleration Constraint of 30g.

Figure 10: Normalized Miss versus The Normalized Missile's Maneuverability for Non-Minimum Phase Missile.

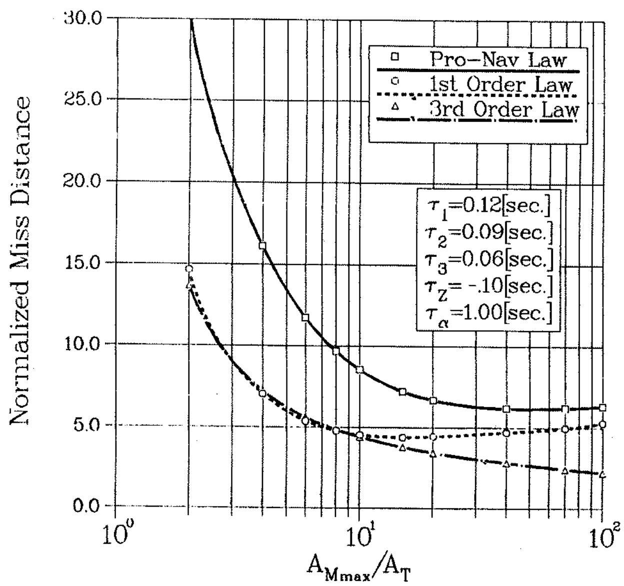


Figure 11: Normalized Miss versus The Normalized Missiles Maneuverability for Non-Minimum Phase Missile.

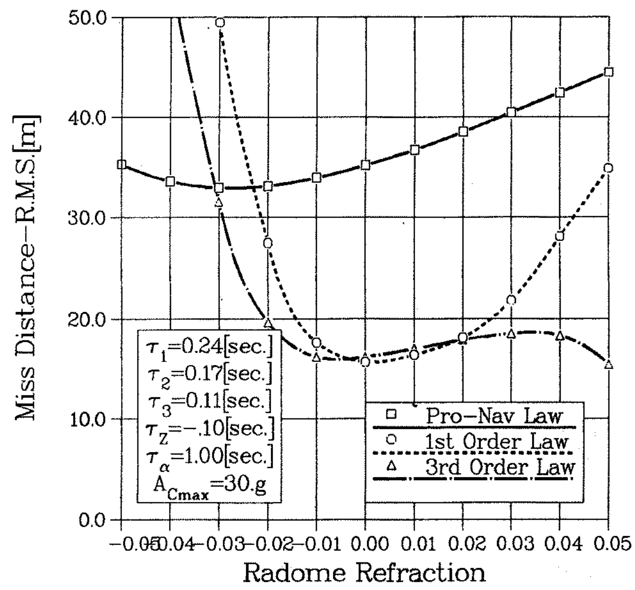


Figure 13: Sensitivity of the RMS Miss to the Radome Refraction Slope for Non-Minimum phase missile and τ_α = 1sec.

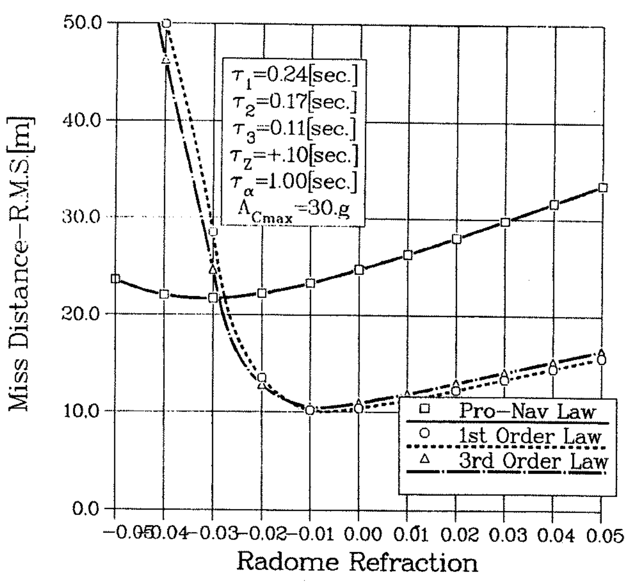


Figure 12: Sensitivity of the RMS Miss to the Radome Refraction Slope for Minimum Phase Missile and τ_α = 1sec.

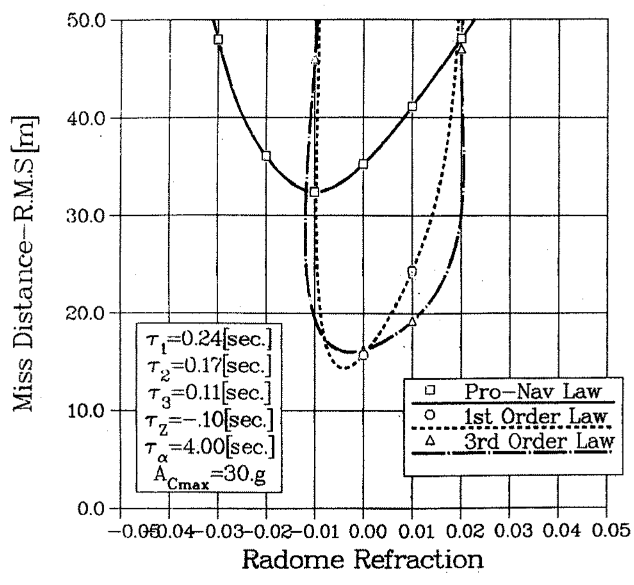


Figure 14: Sensitivity of the RMS Miss to the Radome Refraction Slope for Non-Minimum Phase Missile and τ_α = 4sec.