REDUCING FUEL CONSUMPTION BY CYCLIC CONTROL

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Abstract

Fuel consumption in range and endurance flight is considered as an optimal cyclic control problem. In regard to range cruise, the incompressible and compressible flight regimes are treated separately because each of them shows specific effects for optimal cyclic flight. The improvements achievable in the incompressible flight regime depend on the altitude range admissible. For the compressible flight regime, it is shown that drag rise effects represent a key factor limiting the improvements possible by optimal cyclic cruise. Furthermore, results are presented for endurance flight which is more improved by optimal cyclic control than range cruise.

I. Nomenclature

C_D = drag coefficient
C_L = lift coefficient
D = drag
E = energy
G = function denoting state variable constraint
g = acceleration due to gravity
H = Hamiltonian
h = altitude
J = performance criterion
L = lift
M = Mach number
m = mass
S = reference area
T = thrust
t = time
V = speed
x = horizontal coordinate
γ = flight path angle
δ = throttle setting
λ = Lagrange multiplier
μ = Lagrange multiplier
ν = Lagrange multiplier
ρ = atmospheric density
σ = specific fuel consumption

II. Introduction

There are many efforts for reducing the fuel consumption in order to increase the flight efficiency and performance of aircraft. Some of these are concerned with trajectory optimization. Recent results show that the well-known steady-state cruise is not generally optimal but improvements may be achieved by a non-steady type of cruise (Refs. 1 - 16). This type of cruise consists of a flight path where the control and state variables behave in a periodic manner. As a result, the trajectory of the aircraft is no longer rectilinear but shows a periodic behavior consisting of repeated cycles.

It is the purpose of this paper to further develop the understanding of optimal cyclic cruise and its possible superiority to steady-state cruise. In particular, it will be shown for the range cruise problem that optimal cyclic control in the incompressible flight regime may be regarded as a problem different from the compressible flight regime. For endurance maximization, the incompressible flight regime appears to be of predominant importance.

III. Problem Formulation

The range cruise problem is to find periodic flight paths where the fuel consumed per range travelled is smaller than for the best steady-state cruise. This is equivalent to a periodic control problem consisting of minimizing the following performance criterion

$$J = - \frac{x_{cyc}}{m_f(x_{cyc})} \tag{1}$$

The expressions shown represents the ratio of a horizontal cycle length x_{cyc} and the fuel consumed in a cycle $m_f(x_{cyc})$.

For endurance flight, the performance criterion may be written as

$$\bar{J} = - \frac{t_{cyc}}{m_f(t_{cyc})} \tag{2}$$

Both criteria are subject to the equations of motion. Since the expressions developed in the following are concerned with optimizing range cruise, the equations are written in a way suitable for this problem by using the horizontal coordinate x as the independent variable. For optimizing endurance flight, the time may be retained as the independent variable and similar expressions not given here may be developed.

The equations of motion and fuel consumption may be written as

$$\frac{dV}{dx} = \frac{T-D-mg\sin\gamma}{mV\cos\gamma}$$

$$\frac{d\gamma}{dx} = \frac{L-mg\cos\gamma}{mV^2\cos\gamma}$$

$$\frac{dh}{dx} = \tan\gamma \tag{3}$$

$$\frac{dm_f}{dx} = \frac{\dot{m}_{f0}+\sigma T}{V\cos\gamma}$$

The mass of the airplane can be considered constant for one cycle, since the fuel consumed is small as compared with the total of the mass, i.e.

$$m_f(x_{cyc}) - m_f(0) << m \tag{4}$$

Periodicity of the flight path implies the following boundary conditions

$$V(x_{cyc}) = V(0),\ \gamma(x_{cyc}) = \gamma(0),\ h(x_{cyc}) = h(0) \tag{5}$$

The initial condition for the fuel mass may be written as

$$m_f(0) = 0 \tag{6}$$

The models for thrust, drag and lift are

$$T = T_{min}(h) + \delta[T_{max}(h) - T_{min}(h)]$$

$$L = C_L(\rho/2)V^2S \tag{7}$$

$$D = C_D(\rho/2)V^2S$$

where

$$C_D = C_D(C_L, M) \tag{8}$$

The atmospheric model which is used for air density, speed of sound and thrust dependence on altitude corresponds to the ICAO Standard Atmosphere (Ref. 17).

The control variables are the lift coefficient C_L and the throttle setting δ which are subject to the following inequality constraints

$$C_{L_{min}} \leq C_L \leq C_{L_{max}}$$
$$0 \leq \delta \leq 1 \tag{9}$$

The periodic control problem can now be stated as to find the control histories C_L and δ, the initial states ($V(0)$, $\gamma(0)$, $h(0)$) and the periodic cycle length x_{cyc} which minimize the performance criterion $J = -x_{cyc}/m_f(x_{cyc})$ subject to the dynamic system described by Eq.(3), the boundary conditions given by Eq.(5) and the inequality constraints of Eq.(9) for the control variables.

IV. Optimality Conditions

Necessary conditions for optimality can be determined by applying the minimum principle. For this purpose, the Hamiltonian is defined as

$$H = \lambda_V\frac{T-D-mg\sin\gamma}{mV\cos\gamma} + \lambda_\gamma\frac{L-mg\cos\gamma}{mV^2\cos\gamma} + \lambda_h\tan\gamma + \lambda_f\frac{\dot{m}_{f0}+\sigma T}{V\cos\gamma} \tag{10}$$

where the Lagrange multipliers $\lambda^T = (\lambda_V, \lambda_\gamma, \lambda_h, \lambda_f)$ have been adjoined to the dynamic system of Eq.(3). The Lagrange multipliers are determined by*)

$$\frac{d\lambda_V}{dx} = \lambda_V\frac{T-D-mg\sin\gamma-V(T_V-D_V)}{mV^2\cos\gamma} + \lambda_\gamma\frac{2(L-mg\cos\gamma)-VL_V}{mV^3\cos\gamma} + \lambda_f\frac{\sigma(T-VT_V)}{V^2\cos\gamma}$$

$$\frac{d\lambda_\gamma}{dx} = \lambda_V\frac{(D-T)\sin\gamma+mg}{mV\cos^2\gamma} - \lambda_\gamma\frac{L\sin\gamma}{mV^2\cos^2\gamma} - \frac{\lambda_h}{\cos^2\gamma} - \lambda_f\frac{(\dot{m}_{f0}+\sigma T)\sin\gamma}{V\cos^2\gamma} \tag{11}$$

$$\frac{d\lambda_h}{dx} = \lambda_V\frac{D_h-T_h}{mV\cos\gamma} - \lambda_\gamma\frac{L_h}{mV^2\cos\gamma} - \lambda_f\frac{\sigma T_h}{V\cos\gamma}$$

$$\frac{d\lambda_f}{dx} = 0$$

with the following boundary conditions

$$\lambda_V(x_{cyc}) = \lambda_V(0),\ \lambda_\gamma(x_{cyc}) = \lambda_\gamma(0),$$
$$\lambda_h(x_{cyc}) = \lambda_h(0) \tag{12}$$
$$\lambda_f(x_{cyc}) = x_{cyc}/m_f^2(x_{cyc})$$

The optimal controls C_L and δ are such that H is minimized. For this reason, C_L is determined either by

$$\frac{\partial H}{\partial C_L} = 0 \tag{13}$$

or by the constraining bounds of Eq.(9). In regard to throttle setting δ, H is considered

*) Partial derivatives of D, H, L, T are denoted by subscripts, e.g. $D_V = \partial D/\partial V$.

linear in δ. Thus, δ shows a bang-bang type behavior:

$$\delta = 0 \text{ for } H_\delta > 0$$
$$\delta = 1 \text{ for } H_\delta < 0 \tag{14}$$

There may be a singular arc where δ takes on intermediate values if $H\delta = 0$ for a finite interval of time. However, this case was not observed in the numerical investigation.

The system described by Eq.(3) is autonomous so that the Hamiltonian H is constant. Since furthermore the cycle length x_{cyc} is considered free, H is given by

$$H = 1/m_f(x_{cyc}) \tag{15}$$

V. Optimal Flight Paths with Altitude Constraints

There are cases of cyclic range cruise optimization, where it is necessary to introduce an upper bound for the altitude range admissible. Then cyclic cruise must be considered as a periodic optimization problem with a state variable constraint $h \leq h_{max}$. As a consequence, there are additional conditions which are presented in the following according to Ref. 18.

Basically, two possibilities exist in regard to the constraint under consideration. One is concerned with the flight path touching the constrained altitude boundary at only one point. The other possibility is characterized by the fact that the optimal flight path stays on the altitude boundary for a finite interval. Both possibilities have been observed in the numerical investigation so that the additional conditions for each of them are presented in the following.

The altitude constraint can be formulated as

$$G(x) = h - h_{max} \leq 0 \tag{16}$$

Since

$$G^{(1)}(\gamma) = \tan\gamma \tag{17a}$$

$$G^{(2)}(V, \gamma, h; C_L) = \frac{L-mg\cos\gamma}{mV^2\cos^3\gamma} \tag{17b}$$

the state variable constraint is of second order.

In regard to the first possibility described above, Eqs.(16) and (17a) yield the following conditions

$$h(x_1) = h_{max}$$
$$\gamma(x_1) = 0 \tag{18}$$

with x_1 denoting the point where the flight path touches the constraint altitude. At this point, λ_h shows a discontinuous change. Denoting by x_1^- a point just before the point under consideration and by x_1^+ immediately after, the following relation holds

$$\lambda_h(x_1^+) = \lambda_h(x_1^-) + \nu_0 \tag{19}$$

where $\nu_0 \leq 0$.

The other possibility described shows a constrained arc where the optimal flight path stays on the altitude boundary. On the constrained arc, the Hamiltonian is changed to

$$H \rightarrow H + \mu(x)G^{(2)}(V,\gamma,h;C_L) \tag{20}$$

Consequently, the following relations for the Lagrange multipliers exist

$$\frac{d\lambda_V}{dx} = -\frac{\partial H}{\partial V} - \mu(x)\frac{\partial}{\partial V}G^{(2)}$$
$$\frac{d\lambda_\gamma}{dx} = -\frac{\partial H}{\partial \gamma} - \mu(x)\frac{\partial}{\partial \gamma}G^{(2)} \tag{21}$$
$$\frac{d\lambda_h}{dx} = -\frac{\partial H}{\partial h} - \mu(x)\frac{\partial}{\partial h}G^{(2)}$$

The equation for λ_f remains unchanged since $G^{(2)}$ is independent of m_f.

The relation for $\mu(x)$ on the constrained arc is (from $H_{C_L} = 0$ and $\gamma = 0$)

$$\mu(x) = \lambda_V V\frac{\partial C_D}{\partial C_L} - \lambda_\gamma \tag{22}$$

On the unconstrained arc, $\mu(x) = 0$.

In regard to the optimal controls on the constrained arc, Eq. (17b) yields the following relation for C_L (with $\gamma = 0$):

$$C_L = \frac{2\ mg}{\rho V^2 S} \tag{23}$$

This represents the lift equation L = mg for accelerated/decelerated horizontal flight.

In regard to the entry point x_1 of the constrained arc, Eqs.(16) and (17a,b) yield the following conditions

$$h(x_1) = h_{max}$$
$$\gamma(x_1) = 0$$
$$C_L - \frac{2mg}{\rho V^2 S} = 0 \tag{24}$$

with C_L corresponding to Eq.(13).

Some of the Lagrange multipliers show discontinuous changes at the entry point. Denoting by x_1^- a point just before the entry point and by x_1^+ immediately after, the following conditions hold

$$\lambda_h(x_1^+) = \lambda_h(x_1^-) + \nu_0$$
$$\lambda_\gamma(x_1^+) = \lambda_\gamma(x_1^-) + \nu_1 \tag{25}$$

where ν_0 and ν_1 represent two additional unknowns. The Lagrange multipliers λ_V and λ_f are continuous at the entry point.

In the numerical investigation, an optimization program based on the method of multiple shooting was applied (Refs. 19, 20).

VI. Basic Characteristics of Optimal Cyclic Flight

In Fig. 1, an optimal cycle for range maximization per fuel consumed is shown in order to illustrate the basic characteristics of periodic cruise flight. As indicated in this figure, an optimal cycle may be decomposed into two phases which can be characterized by thrust behavior. In the first phase, thrust is at its maximum which, due to $T_{max} > D$, results in an increase of the energy state of the aircraft. The second phase shows thrust at its minimum where the energy state increase of the phase before is used to gain as much range as possible. Corresponding to the thrust behavior, speed level in phase 1 is high as compared with phase 2 and altitude indicating potential energy level shows an increase in phase 1 and a decrease in phase 2.

The behavior just described can be used to give a physical insight into the reasons why cyclic cruise can provide a reduction in fuel consumption. This is illustrated in Fig. 2 which shows the energy added per fuel consumed for optimal cyclic cruise and for the best steady-state flight in the altitude range considered. Due to the high speed level in phase 1, it is possible to reach values of $(dE/dm_f)_{cyc}$ which are considerably better than the best values of steady-state cruise $(dE/dm_f)_{st}$.

In the following, improvements due to cyclic cruise are shown and decisive aircraft factors are identified and evaluated. For this purpose, it is suitable to treat the incompressible and compressible flight regimes separately since the physical effects underlying optimal cyclic flight in both regimes are quite different.

VII. Optimal Cyclic Cruise in the Incompressible Flight Regime

Maximum thrust level available represents a key factor as regards cyclic cruise and its possible improvements. This is already indicated by the thrust behavior outlined in the description of the basic characteristics of optimal cyclic cruise as illustrated in Figs. 1 and 2. From the thrust behavior described it follows that the improved fuel utilization for increasing the energy state is due to the fact that excess thrust is available. An example for the effect of maximum thrust available on the trajectory is shown in Fig. 3. Characteristically, the higher thrust level results in larger amplitudes of the changes of speed and altitude, thus showing a more pronounced oscillatory behavior.

An evaluation of the effect of maximum thrust on the improvements achievable is shown in Fig. 4.

From the results presented it follows that the gains can be significantly increased when more thrust is available. Fig. 4 also includes the effect of a non-zero minimum thrust phase. The minimum thrust is related to minimum drag of steady-state horizontal flight $D_{min} = mg(C_D/C_L)_{min}$. Fig.4 shows that the minimum thrust level also has a significant effect on the gains achievable which are reduced when minimum thrust is increased.

It may be of interest to note how much the length of the optimal cyclic trajectory is increased when compared with the horizontal distance travelled which also represents the length of the steady-state cruise trajectory. This is illustrated in Fig. 5, where s_{cyc} is the actual flight path length of cyclic cruise and x_{cyc} the corresponding horizontal distance travelled. Fig. 5 shows that there is some length increase which becomes larger when more thrust is available. This is due to the fact that a higher thrust level yields an oscillatory flight profile behavior more pronounced.

In the results presented so far, an altitude constraint $h \leq h_{max}$ is imposed. This indicates a tendency for the optimal altitude range to be as high as possible. In regard to steady-state cruise, an increase of the altitude range admissible usually also yields an improvement (excluding the compressible flight regime). It is therefore of interest to know how the optimal cyclic cruise compares to the best steady-state cruise when increasing the altitude range admissible. This is illustrated in Figs. 6 and 7. Fig. 6 shows flight profiles for various values of h_{max}. Fig. 7 presents an evaluation of the effect of the altitude constraint. From this it follows that the improvements achievable with cyclic cruise are reduced when the altitude range is increased.

The tendency of the curve of Fig. 7 suggests that the improvement due to cyclic cruise may disappear if the admissible altitude is high enough. From a practical standpoint, however, this effect may be of minor importance when comparing it with the consequences resulting from the fact that the altitude range increase leads to higher speeds which eventually may approach the compressible flight regime (as indicated in Fig.7). In such a case, an altitude constraint changes its meaning for the optimal cycle and may even not become active despite sufficient excess thrust since the drag rise due to compressibility may become of predominant influence.

VIII. Optimal Cyclic Cruise in the Compressible Flight Regime

For steady-state cruise, it is well known that the drag rise due to compressibility in the high subsonic Mach number range limits the maximum Mach number economically usable. It is therefore an effect of primary importance for all flight vehicles at subsonic speeds. It will be shown in the following, that the drag rise due to compressibility also represents a key factor for cyclic cruise where it acts as a kind of barrier, too.

An example is presented in Fig. 8 which shows an optimal cylce with the altitude constraint removed and no other constraint imposed. There is again a maximum and a minimum thrust phase which can be considered as basic elements of cyclic flight as described earlier. However, the oscillatory behavior in terms of maximum changes of Mach number and lift coefficient is reduced. In particular, the maximum Mach number attained appears to be limited, despite the fact that no Mach number constraint is imposed. The reason for this type of effective Mach number limitation is due to aerodynamic characteristics because compressibility effects yield a substantial drag rise in the Mach number range of interest. As a consequence, the results for cyclic cruise in the compressible Mach number range show only small improvements when compared with results for the incompressible flight regime as described earlier.

IX. Optimal Cyclic Endurance Flight

For the range cruise problem, the improvements possible by cyclic control appear to be reduced when no altitude constraint is imposed or when the compressible flight regime is approached. By contrast, endurance flight may be significantly improved by cyclic control. An example is presented in Fig. 9 which shows the histories of state and control variables. There are similarities as regards the maximum and minimum thrust phase and the corresponding climbing and sinking flight conditions. However, there are also significant differences. The speed attained remains within the incompressible flight regime. This means for the aircraft models considered that a separate treatment of the compressible flight regime is not necessary. Another difference concerns the altitude boundary. In the numerical investigation, it was observed that the greatest endurance per fuel consumed was reached for altitudes as low as possible. Therefore, it was necessary to introduce a lower bound for the admissible altitude range.

The maximum thrust level is again of great influence. This is illustrated in Fig. 10 which shows endurance of optimal cyclic flight (t_{cyc}) as compared to the best steady-state flight (t_{st}) for the same amount of fuel. From the results presented it follows that the improvements of optimal cyclic control are significant even for comparatively low thrust levels.

X. Conclusions

Minimization of fuel consumption for a given range is considered as an optimal cyclic control problem, partially with a state variable constraint given by an upper altitude bound. The incompressible and compressible flight regimes are treated separately because each of them shows specific effects.

In regard to the basic characteristics of cyclic cruise, an optimal cycle may be decomposed into two phases of which one is a maximum-thrust increasing-energy condition and the other a minimum-thrust decreasing-energy condition. The improvements of optimal cyclic cruise are due to a better energy management.

In the incompressible flight regime, the improvements depend on the altitude range admissible. They are reduced when the thrust level is reduced or the maximum altitude admissible is increased.

In regard to the compressible flight regime, it is shown that cyclic cruise can provide an improvement which, however, is comparatively small. The drag rise due to compressibility is identified as a key factor limiting the possibilities of cyclic cruise at high subsonic Mach numbers.

Endurance flight may be more improved by cyclic control than range cruise. It is shown that the maximum thrust is again of great influence.

XI. Acknowledgment

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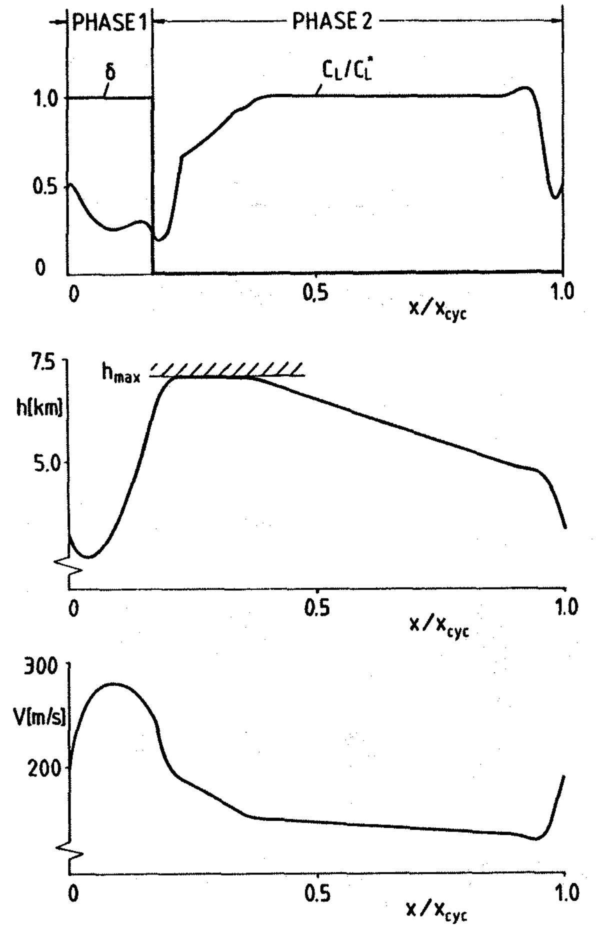

Fig. 1 Basic characteristics of optimal cyclic cruise (turbo jet type power-plant, $(T_{max})_{h=0}$=0,5 mg, x_{cyc}=81,2 km, C_L^*: lift coefficient corresponding to the maximum lift-to-drag ratio)

16% reduction of fuel consumed

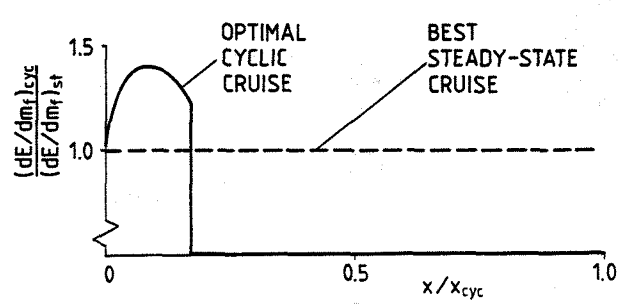

Fig. 2 Energy added per fuel consumed

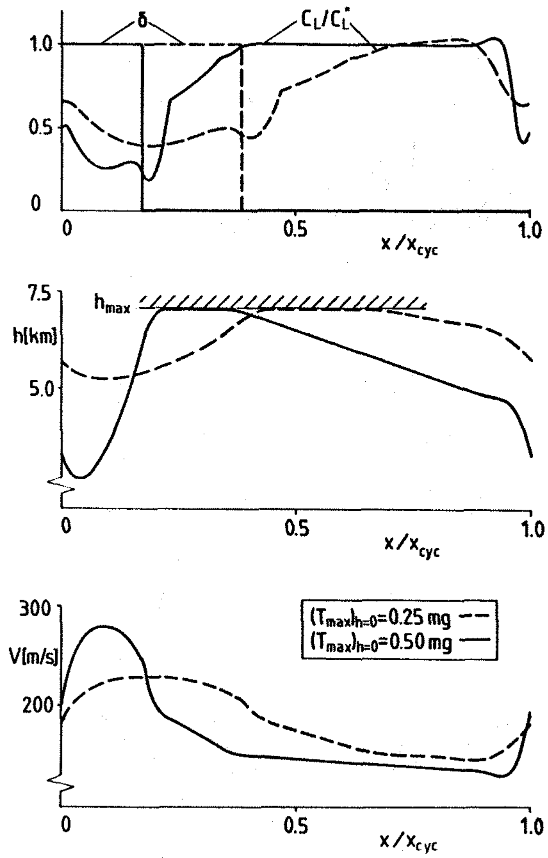

Fig. 3 Effect of maximum thrust available on optimal cyclic trajectories (C_L^*: lift coefficient corresponding to the maximum lift-to-drag ratio)

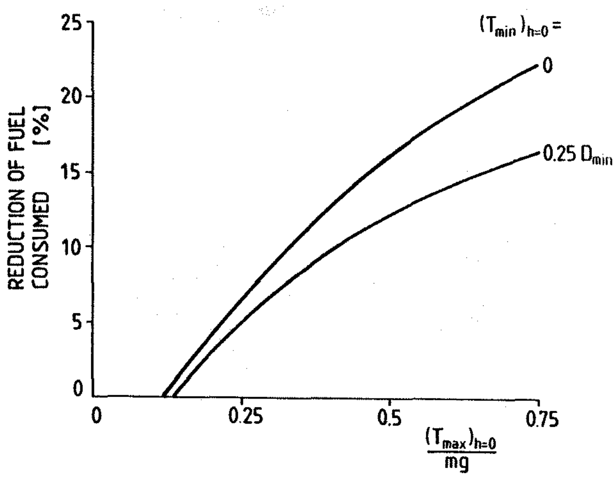

Fig. 4 Effect of maximum thrust weight ratio on the reduction of fuel consumed (turbo jet type powerplant)

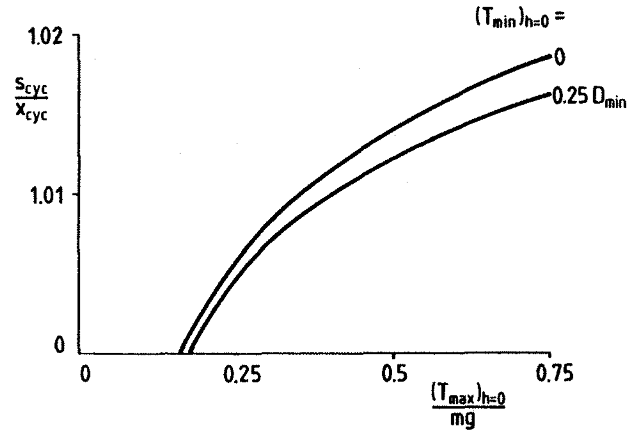

Fig. 5 Increase of flight path length (s_{cyc}) in comparison to the horizontal distance travelled (x_{cyc})

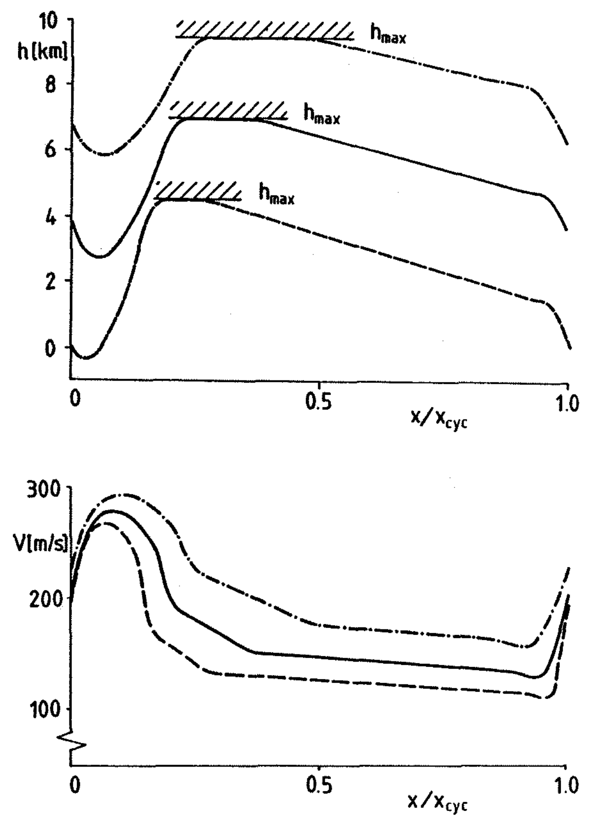

Fig. 6 Effect of an altitude constraint on optimal cyclic trajectories

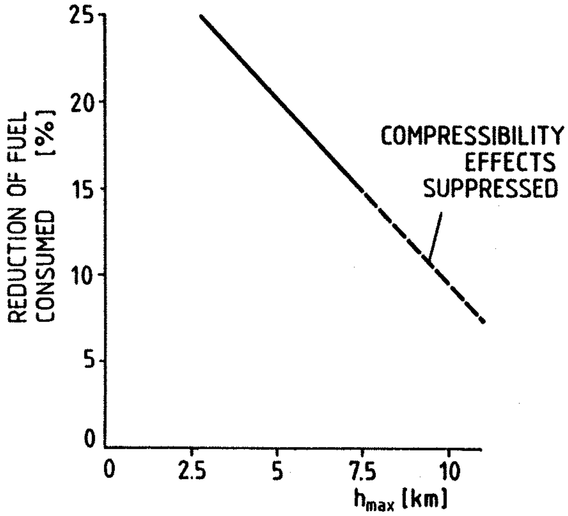

Fig. 7 Effect of an altitude constraint on fuel consumption reduction due to optimal cyclic flight

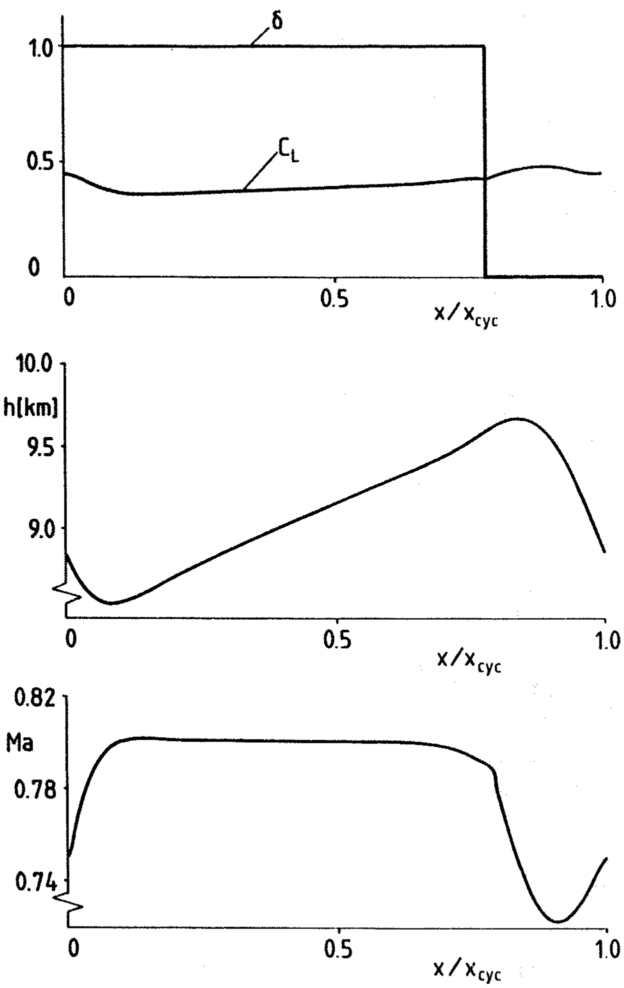

Fig. 8 Optimal cyclic cruise in the compressible flight regime (x_{cyc}=75,6 km)
0,3% reduction of fuel consumed compared to the best steady-state cruise

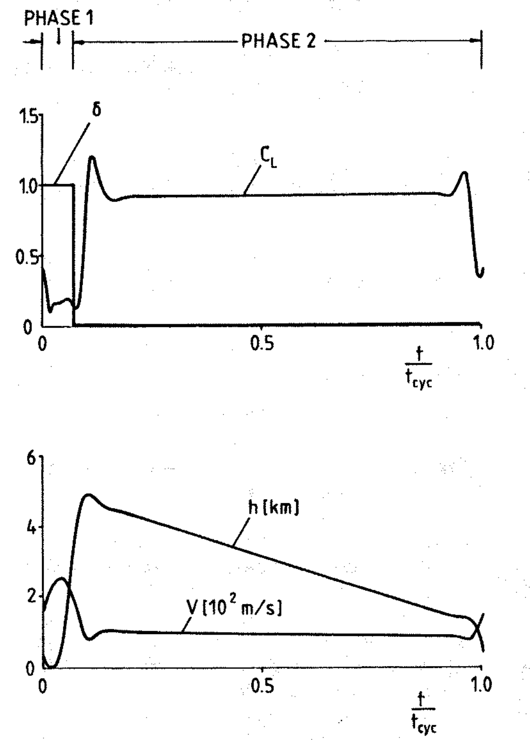

Fig. 9 Optimal cyclic endurance flight (($(T_{max})_{h=0}$=0,5 mg, t_{cyc}=11,87 min)
63,5% endurance increase compared to the best steady-state flight

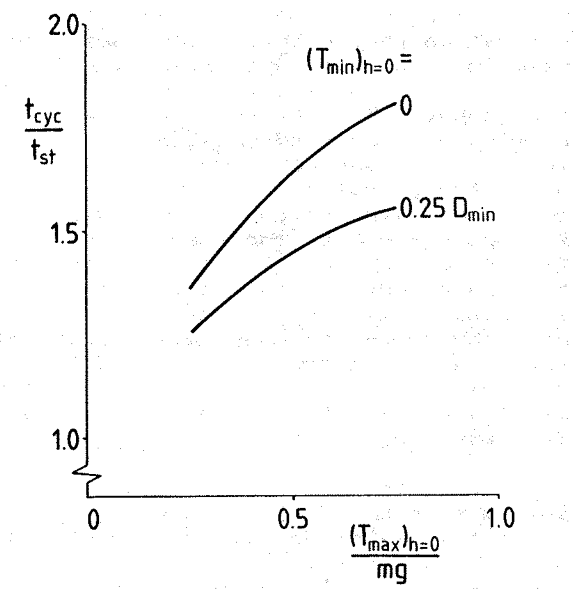

Fig. 10 Effect of maximum thrust weight ratio on endurance increase due to optimal cyclic control