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HIGH SPEED COMPUTATION OF OPTIMUM CONTROLS

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Abstract

In case of calculating the optimizing control signals derived from the maximum principle of L.S.Pontryagin, based on an iterative algorithm due to L.W.Neustadt, required for solving a two-point boundary value problem, the iterative procedure is extremely sensitive to computational error, so that the convenient digital hybrid system composed of a digital differential analyzer and a general-purpose digital computer was experimentally organized for the on-line study of a rapid convergence with high precision.

Results of the experimental digital hybrid implementation with somewhat refined algorithms show that the rapid convergence is practically possible for the 2nd- and 3rd-order system, and the 2nd-order system subject to energy constraints. Convergence processes are displayed graphically.

Introduction

The study reported on here is part of a fundamental and introductory investigation carried out experimentally on the real-time generation of the optimum control. The principal and final objective of such researches is to establish the feasibility of an optimum feedback control system, where the state of the system is continually estimated from measurements taken on the observable variables and at the same time an optimum controller (a control computer with as simple archtecture as possible) is continually generatig the optimum control to be applied by using this estimated state. The optimum control required there must accordingly be computed at a faster rate than real-time within the control computer.

The optimum control investigated in this study is derived from the maximum principle of L.S. Pontryagin,(1) which gives the optimum control process in terms of a solution of the adjoint equation of the control system. The solution is not a function of the given state of the control system, but of the time and the unknown initial condition **q** of the adjoint equation. As a result it follows that we must repeatedly and rapidly solve the two-point boundary value problem in order to find those **q** for the continuously estimated state of the control system.

The solution of the two-point boundary value problem is based on an iterative algorithm due to L.W.Neustadt,(2) where to find the proper **q** , a real valued function F will be defined such that F attains its maximum, when the independent variables are equal to the required **q** for generating the optimum control. The steepest ascent iterative procedure may also be given for computing this maximizing value.

The optimizing control signal is thus genenated corresponding to the given state of the control system. Neustadt indicates that the method of computation is particularly suitable for hybrid analogue digital computers. However, the iterative procedure accompanied is extremely sensitive to computational error, and it's hybrid analogue implementation is not always practical.(3)

The convenient digital hybrid system composed of a digital differential analyzer (i.e. an incremental computer) and a general-purpose digital computer was, therefore, experimentally organized by using resoureces on hand for a rapid convergence with high precision. The former is chiefly for the high-speed computation of integrals involved, and the latter for the total management of operation.

This paper describes the required refinement of Neustadt's iterative algorithm for convergence acceleration, the constitution of the experimental digital hybrid system developed conveniently for the purpose, and the results of digital hybrid implementation with the refined algorithm for the 2nd- and 3rd-order system, and the 2nd-order system subject to energy constraints. The rapid convergence was practically possible for given problems. Convergence processes are displayed graphically.

Neustadt's Iterative Algorithm

We consider a physical system whose state at any time is described by an n-dimensional column vector $\mathbf{x}(t)$, which may be considered to be the coordinates in phase space. Neustadt's method is applicable to systems which can be described by a linear system of ordinary differential equations of the form:

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)*\mathbf{x}(t)+\mathbf{a}(t)+\mathbf{B}(t)*\mathbf{u}(t) \qquad (1)$$

Here $\mathbf{A}(t)$ is an n*n matrix, and $\mathbf{B}(t)$ is an n*r matrix, and $\mathbf{a}(t)$ is an n-dimensional vector. These are all continuous in the time t. $\mathbf{u}(t)$ is an r-dimensional column vector. The components of $\mathbf{u}(t)$ correspond to r controllers, whose values can be adjusted to contol the state vector $\mathbf{x}(t)$. The control functions $\mathbf{u}(t)$ are constrained to take on their values in a control region U which is a closed set in an r-dimensional space.

Suppose that a scalar valued function G(u), whose domain contains U, is given. Reference will be made to the functional

$$\int_{\emptyset}^{T} G(\mathbf{u}(t))dt=J(\mathbf{u}(t)) \tag{2}$$

(T represents the duration of the control process) as the control effort associated with $\mathbf{u}(t)$.

Given an initial state and a target state in phase space, the optimization problems considered is to find the optimum control which transfers the specified initial state to the desired target state under the following conditions:

(1) Minimizing the time T to transfer.

(2) Minimizing the time T to transfer, subject to the constraint $J(\mathbf{u}(t))\leq M$, where M is a given constant.

(3) Minimizing the control effort $J(\mathbf{u}(t))$, given a time $T>\emptyset$.

An n-dimensional vector function $\mathbf{Z}(t,\mathbf{q})$ and $\mathbf{E}(t,\mathbf{q})$, and a real valued function $f(t,\mathbf{q})$ may be defined as follows:

$$\mathbf{Z}(t,\mathbf{q})=\mathbf{X}(t)^{-1}*\mathbf{r}(t)-\int_{\emptyset}^{t}\mathbf{X}(s)^{-1}*\mathbf{a}(s)ds$$
$$-\int_{\emptyset}^{t}X(s)^{-1}*\mathbf{B}(s)*\mathbf{u}(t,\mathbf{q})ds \tag{3}$$

$$E(t,\mathbf{q})=\mathbf{x}(\emptyset)-\mathbf{Z}(t,\mathbf{q}) \tag{4}$$

$$f(t,\mathbf{q})=\mathbf{q}*[\mathbf{x}(\emptyset)-\mathbf{Z}(t,\mathbf{q})] \tag{5}$$

Here $\mathbf{X}(t)^{-1}$ is an n*n matrix, which is computed as the solution of the equation

$$\dot{X}(t)^{-1}=-\mathbf{X}(t)^{-1}*\mathbf{A}(t);\ \mathbf{X}(\emptyset)^{-1}=\mathbf{I} \tag{6}$$

$\mathbf{r}(t)$ is an n-dimensional vector, which is the target state of the system. $\mathbf{u}(t,\mathbf{q})$ is the optimum control given as the solution of the adjoint equation for the initial value vector $\mathbf{q}$.

If the value of $f(t,\mathbf{q})$ is zero for t=F, then F becomes a function of $\mathbf{q}$. Neustadt has proved that

$$\text{grad } F(\mathbf{q})=E(F(\mathbf{q}),\mathbf{q}) \tag{7}$$

$\mathbf{x}(\emptyset)$ is an initial value vector of the control system for t=Ø, whose values are known. On the other hand the values of $\mathbf{q}$ are unknown.

Neustadt's iterative algorithm for finding the values of $\mathbf{q}$ is summarized below, for the time optimum case.

Let the initial estimate $\mathbf{q}^0$ be any vector such that $\mathbf{q}^0*\mathbf{x}(\emptyset)>\emptyset$. Then $\mathbf{u}(t,\mathbf{q})$, $\mathbf{Z}(t,\mathbf{q})$, $\mathbf{E}(t,\mathbf{q})$ and $f(t,\mathbf{q})$ are computed as functions of the time t for the estemated $\mathbf{q}$, and the computation is halted, when the value of $f(t,\mathbf{q})$ is zero as the time proceeds. The corresponding time is $F(\mathbf{q})$.

Then, if $\mathbf{q}^i$ is the estimate from the i-th iteration, the next estimate $\mathbf{q}^{i+1}$ is chosen to be

$$\mathbf{q}^{i+1}=\mathbf{q}^i+K*\text{grad } F(\mathbf{q}^i) \tag{8}$$

where K is a sufficiently small positive constant called 'gain'. Since grad $F(\mathbf{q})=\mathbf{E}(F(\mathbf{q}),\mathbf{q})$, it follows that

$$\mathbf{q}^{i+1}=\mathbf{q}^i+K*\mathbf{E}(F(\mathbf{q}^i),\mathbf{q}^i) \tag{9}$$

The iterative computations are continued, for instance, until $E(F(\mathbf{q}),\mathbf{q})$ which is the error of the boundary value of $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ is sufficient small in absolute value. The vector $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ will then be mapped into the initial point $\mathbf{x}(\emptyset)$.

Gain Control

Refinements of Neustadt's algorithm are required for convergence acceleration. It is assumed in Neustadt's method that K is a sufficiently small constant for the theoretical proof of convergence. On the other hand, the value of K is, in this experimental study, controlled at each iteration for a rapid convergence, which is called 'gain control'. If K is too small a constant, much more time will be spended for a convergence. If K is too large a constant, a wide amplitude of vibration and then an overflow in computation will occur. They both may bring the difficulties of convergence.

Gain controls experimentally developed under the geometrical interpretation of Neustadt's algorithm in this study are the following 1Ø schemes, which are named G.C.n (n=Ø~9) for further references. As a general rule in the following gain controls, the temporary initial value vector $\mathbf{q}^T$ instead of $\mathbf{q}^{i+1}$ of Eq.(9) is at first computed from Eq.(1Ø), and then whether $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$ at once or not is based on each gain control. The argument $F(\mathbf{q})$ in the function $E(F(\mathbf{q}),\mathbf{q})$ will henceforth be dropped for ease of notation.

G.C.0: K is not varied. This is a computational scheme of Neustadt.

G.C.1: If $F(\mathbf{q}^T)<F(\mathbf{q}^i)$ then K is multiplied by 0.5 and $\mathbf{q}^T$ is computed again from $\mathbf{q}^i$ else K is not varied and $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$.

G.C.2: In the condition of $F(\mathbf{q})$ for G.C.1, $\leq$ is used instead of $<$.

G.C.3: If $F(\mathbf{q}^T)<F(\mathbf{q}^i)$ or $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multiplied by 0.5 and $\mathbf{q}^T$ is computed again from $\mathbf{q}^i$ else K is not varied and $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$.

G.C.4: If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multipled by 0.5. $\mathbf{q}^T$ is always set to $\mathbf{q}^{i+1}$.

G.C.5: The absolute value of $\mathbf{q}$ always equals to one. K is not varied.

C.C.6: If $F(q^T)\leq F(\mathbf{q}^i)$ or $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multiplied by 0.5 else K is multiplied by 2. $\mathbf{q}^T$ is always set to $\mathbf{q}^{i+1}$.

G.C.7: If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multiplied by 0.5. If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)>0.5*\mathbf{E}(\mathbf{q}^i)*\mathbf{E}(\mathbf{q}^i)$ then K is multiplied by 2. $\mathbf{q}^T$ is always set to $\mathbf{q}^{i+1}$.

C.C.8: If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multiplied by 0.5 and $\mathbf{q}^T$ is computed from $\mathbf{q}^i$ again else $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$.

C.C.9: If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<0$ then K is multiplied by 0.5 and $\mathbf{q}^T$ is computed from $\mathbf{q}^i$ again else K is multiplied by 2 and $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$.

Here $\mathbf{q}^T$ is computed from $\mathbf{q}^i$ as follows:

$$\mathbf{q}^T=\mathbf{q}^i+K*\mathbf{E}(\mathbf{q}^i) \qquad (10)$$

Experimental Digital Hybrid System

There are two types of hybrid computation. One is an analogue hybrid and the other a digital hybrid treated here.

An analogue hybrid system which constitutes of an analogue computer for a rapid operation and a digital computer for managing the whole operation has in fact been recommended for implementing Neustadt' iteratire algorithm, where the functions $\mathbf{u}(t,\mathbf{q})$, $\mathbf{Z}(t,\mathbf{q})$, $\mathbf{E}(t,\mathbf{q})$, and $F(t,\mathbf{q})$ are computed with the analogue computer and the renewal of $\mathbf{q}$ and the whole operation control are processed with the digital computer.

However the implementation of Neustadt's algorithm depends extensively upon computational errors especially for finding the converged values of $\mathbf{q}$, which may not be overcome with the precision of the analogue operation.

A digital hybrid system as used here constitutes of a digital differential analyzer (D.D.A.) instead of an analogue computer, and a general-purpose digital computer (G.P.C.). D.D.A. is a special-purpose digital computer (S.P.C.) which is, in a sense, a digitized analogue computer, i.e. an incremental computer, and its precision can be set arbitrarily on demand.

In digital hybrid systems, an interface such as A-D converters and D-A converters between an analogue computer and a digital computer in analogue hybrid systems is not necessary. Some instructions of G.P.C. make data transfered directly between D.D.A. and G.P.C., and start and stop the operation of D.D.A.. Under appropriate conditions, D.D.A. sends interrupt signals to G.P.C., by which G.P.C. can readily control the operation of D.D.A..

Main function of D.D.A. is the same integration as that of an analogue computer. But it is not always necessary to use the time as an independent variable of the integration. D.D.A rather computes $dz=ydx$ instead of $z=\int ydx$, which makes various operations realizable. For instance multiplication is readily computed by $d(yx)=ydx+xdy$. In addition D.D.A. usually has some unigue functions called 'decision', 'servo' and such for processing non-linear special operation.

The difference between D.D.A. as a S.P.C. and G.P.C. is that the former has only two types of operation for continuously changed variables, which makes the operation of D.D.A. much more rapid than that of G.P.C. in the same precision, if similar electronic circuits are used in both machines. From this point of view, a digital hybrid system may preferentially be recommended instead of a non-hybrid G.P.C..

The experimental digital hybrid system developed for this on-line study consists of two small-scale old-fashioned computers with serial-type fixed-point arithmetic operations for D.D.A. and G.P.C.. D.D.A. has 28 integrators, whose precission is 20 bits at maximum for setting values and 30~40 bits for operations. D.D.A. generates one basic increment at each 560 micro seconds on operation.

The two fundamental operations of integrators are the integration : $dz=ydx$

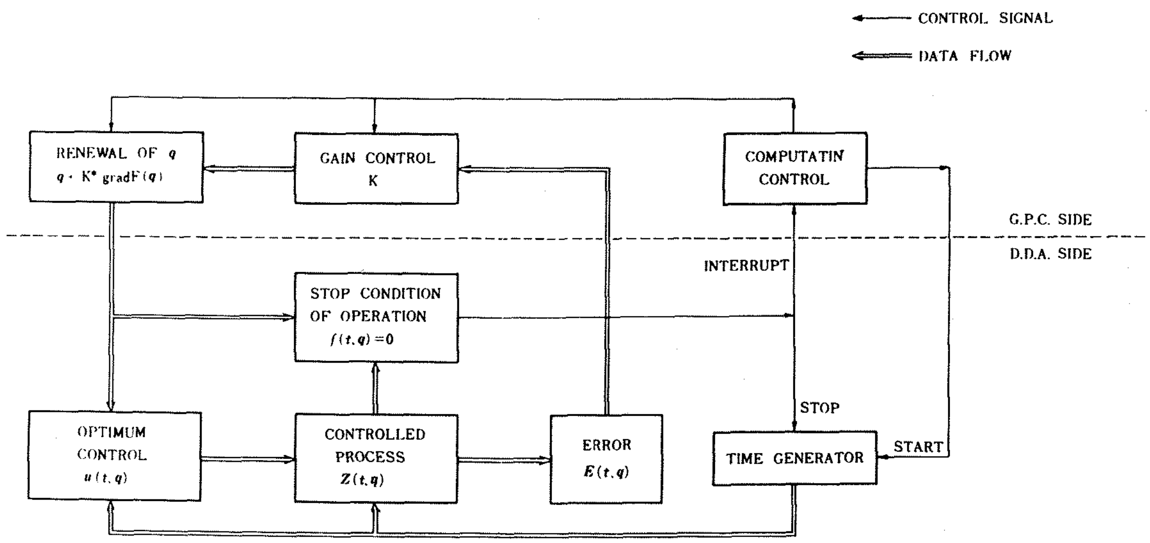

FIGURE 1. Block Diagram for Digital Hybrid Computation

and the decision : dz=(sign y)dx shown in Ex.1~2 of Figure 2. The integrator has two main registers. One is called 'R-register', and the other 'Y-counter', which contains y and accumulates dy's (over dx).

In the integration, the content of the Y-register is added to the R-register for each dx. If an overflow of the R-register occurs in this addition, then the overflow is an dz. This is a rectangular integration, and the used size of the R-register corresponds to the unit of a rectangular area for the dz.

In the decision, the R-register may not operate. According to the sign (bit) of the Y-register, dz is equal to dx or -dx for each dx. The operation of the decision is simple, but so powerful that, for instance, a discontinuons function required of the computation of the optimum control can readily be generated.

The dy's accumulated in Y-register may be the dz's sent from the several integrators, and the dx is the dz sent from the other one integrator. The integrators connect each other by a patch-board wiring similar to that of an analogue computer. The patch-board wiring, called 'mapping' in D.D.A., is part of a programming of D.D.A..

G.P.C. is standard-type and has a memory with only 256 word. One word consists of 10 bits. The momory cycle time is 20 micro seconds. Furthermore, G.P.C. has two accumulater register for arithmetic operations and data exchange with D.D.A.. The accumulater consists of 20 bits. Data of the both computer are exchanged by executing some instructions of G.P.C., of which the operation is controlled manually by a console-typewriter.

Digital Hybrid Implementation

The iterative algorithms generating the optimum control is rearranged for the digital hybrid computation, and the corresponding procedure regarding G.P.C. and D.D.A constituting the experimental digital hybrid system is described.

Consider the system governed by Eq.(1). The procedure consists chiefly of the following six steps:

Step 1: $\mathbf{u}(t,\mathbf{q})$ from the adjoint equation with the initial vector $\mathbf{q}$. (11)

Step 2: $$\mathbf{Z}(t,\mathbf{q})=\mathbf{X}(t)^{-1}*\mathbf{r}(t)-\int_{0}^{t}\mathbf{X}(s)^{-1}*\mathbf{a}(s)ds-\int_{0}^{t}\mathbf{X}(s)^{-1}*\mathbf{B}(s)*\mathbf{u}(s)ds \quad (12)$$

Step 3: $\mathbf{E}(t,\mathbf{q})=\mathbf{x}(0)-\mathbf{Z}(t,\mathbf{q})$ (13)

Step 4: $f(t,\mathbf{q})=\mathbf{q}*\mathbf{E}(t,\mathbf{q})$ (14)

Step 5: Gain control (15)

Step 6: $\mathbf{q}^{T}=\mathbf{q}^{\iota}+K*\mathbf{E}(\mathbf{q}^{\iota})$ (16)

The functions in step 1 4 are of the time t for given each $\mathbf{q}$, so that these are computed by D.D.A., speedily with high precision. Step 5 6 are processed at the end of each iteration for starting the next iteration by G.P.C. which controls the total management of the whole operation.

The whole operation of the digital hybrid system is explained by using the block diagram in Figure 1. To begin with, after starting G.P.C., K^0 and $\mathbf{q}^0$ are given anyhow, and the time generator of D.D.A. is started by G.P.C.. Then D.D.A. produces $\mathbf{u}(t,\mathbf{q})$, $\mathbf{Z}(t,\mathbf{q})$, $E(t,\mathbf{q})$ and $f(t,\mathbf{q})$, as the logical time made by the time generator proceeds. When $f(t,\mathbf{q})=0$, D.D.A. stops the time generator and at the same time interrupts the operation of G.P.C.. Interrupted G.P.C. processes the gain control and the renewal of $\mathbf{q}$, and waits for the next interrupt. This process proceeds, for instance, until $|E(\mathbf{q})|$ is sufficiently small.

Programs of D.D.A. depend so much upon problems to be solved that their conceptual extraction is demonstrated in Figure 2. Figure 3 shows an example of the definite program for the 3rd-order system described in a later section.

All operations of D.D.A. are of increment form. Including the necessary initial values, Eq.(12)~(14) are rearranged for incremental operations as follows:

$$dZ=\mathbf{X}(t)^{-1}*[\dot{\mathbf{r}}(t)-\mathbf{A}(t)*\mathbf{r}(t)-\mathbf{a}(t)$$

$$-B(t)*\mathbf{u}(t,\mathbf{q})]dt\ ;\ \mathbf{Z}(0)=\mathbf{r}(0) \qquad (17)$$

$$dE=-d\mathbf{Z}\ ;\ E(0,\mathbf{q})=\mathbf{x}(0)-\mathbf{r}(0) \qquad (18)$$

$$df=-\mathbf{q}*d\mathbf{Z}\ ;\ f(0,\mathbf{q})=\mathbf{q}*[\mathbf{x}(0)-\mathbf{r}(0)] \qquad (19)$$

In Figure 2, Eq.(17) is processed by I3, Eq.(18) by I4, and Eq.(19) by I5 and I6. The number of the integrator required in the n-dimensional case is 4*n only for those operations. In addition the integrators are still necessary for the computation of the functions contained in the right-hand term of Eq.(17).

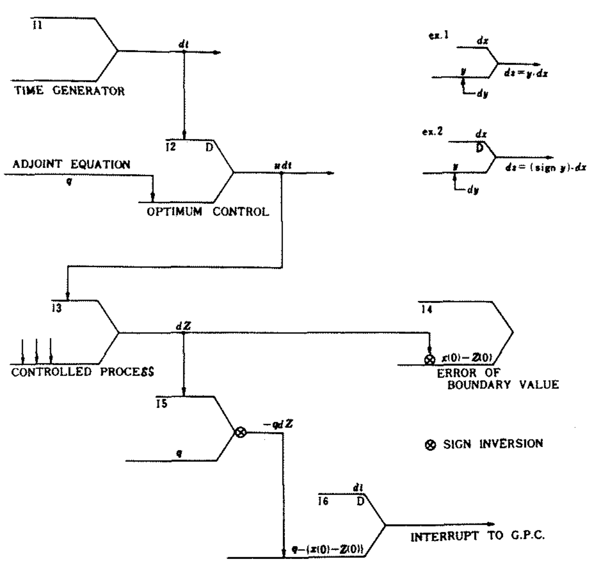

FIGURE 2. Coceptual Mapping of D.D.A.

The generation of the optimum control in many cases depends upon non-linear elements, such as on-off switches, saturations and resolvers. D.D.A. can realize these non-linear operations with the high precision, with which an analogue computer can not.

The pre-estimate K^0 and $\mathbf{q}^0$, the trial values of K and $\mathbf{q}$ for the first iteration, are 1/2 and $\mathbf{x}(0)$ respectirely, if not shown especially, in this study. Similarly, the initial values $F(\mathbf{q}^0)$ and $E(\mathbf{q}^0)$, required for the computation, are both zero.

2nd-order System

2nd-order, Undamped Oscillatory System. To begin with, the convergence behavior in the digital hybrid implementation was experimentally examined regarding the 2nd-order, undamped oscillatory system ($\ddot{x}+x=u$). This is a problem solved theoretically by Bushaw.

The atatement of the problem treated here is as follow:

"Examine the convergence behavior of $\mathbf{Z}(t,\mathbf{q})$, $\mathbf{E}(\mathbf{q})$ and $F(\mathbf{q})$ of the system where the initial point $\mathbf{x}(0)=(1,0)$ or

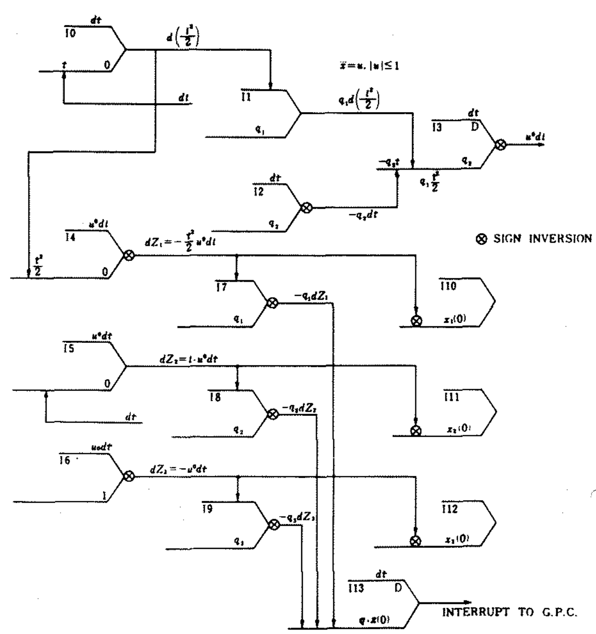

FIGURE 3. D.D.A. Mapping for 3rd-order system

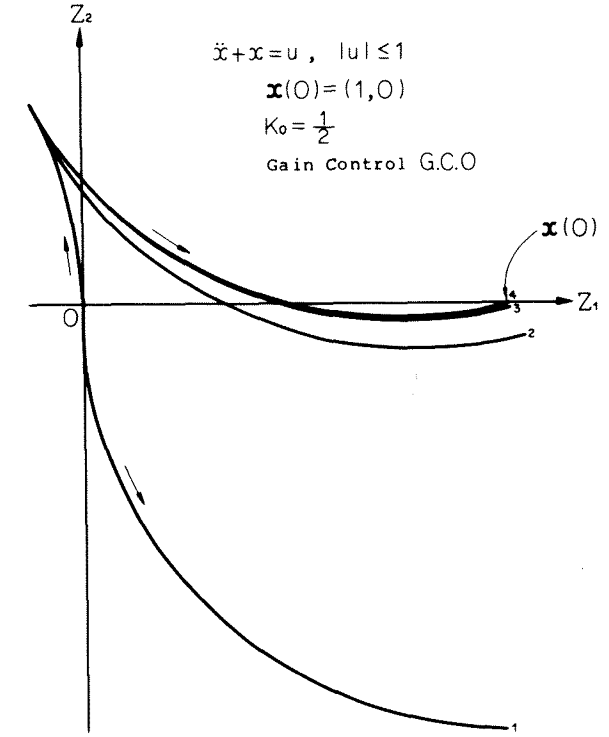

FIGURE 4. Convergence of Z-trajectories for 2nd-order, Undamped Oscillatory System

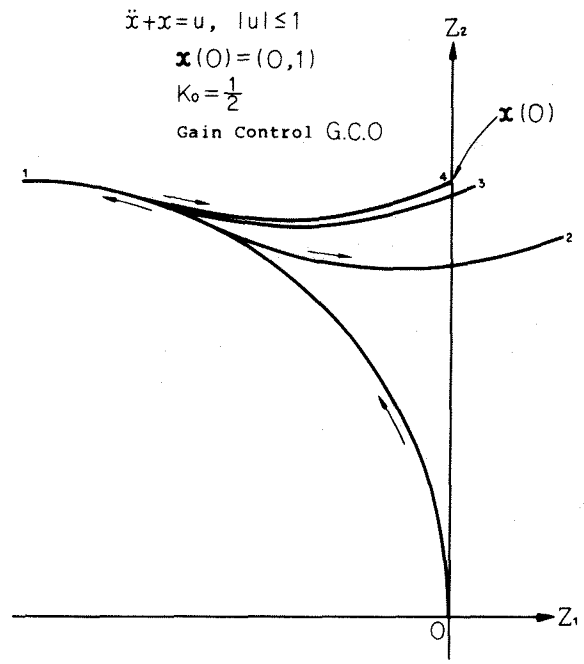

FIGURE 5. Convergence of Z-trajectories for 2nd-order, Undamped Oscillatory System

> (Ø,1) is transfered to the target point $\mathbf{r}(t)=\emptyset$ subject to $|u(t)|\leq 1$ in minimum time."

It was not necessary to control the gain K in these problems. Part of the results are displayed in Figure 4~6.

The convergence of the terminal point $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ of Z-trajectory to the initial point $\mathbf{x}(\emptyset)$ of the system is shown in Figure 4~5, where the initial point $\mathbf{Z}(\emptyset,\mathbf{q})$ is the origin because of $\mathbf{r}(F(\mathbf{q}))=\emptyset$. The convergence is monotonical and the number of iteration increases for the small value of K. A vibration occurs for the large value of K. Figure 6 shows the behavior of maximizing the optimum time $F(\mathbf{q})$ and the variation on the absolute value of the error $|\mathbf{E}(\mathbf{q})|$ of the boundary value for each iteration.

<u>2nd-order, undamped unoscillatory system.</u> The convergence behavior regarding the 2nd-order undamped unoscillatory system ($\ddot{x}=u$) is shown as simple examples where it is difficult to converge if the gain K is constant (G.C.Ø). In these cases the smaller gain K also brings the more iteration and monotonical convergence. The larger gain K for a rapid convergence makes the large amplitute of vibration and results in the convergent difficulties of the boundary error $|\mathbf{E}(\mathbf{q})|$.

Here the gain controls are introduced for the stable acceleration of convergence. The statement of the problem treated here is the same as the previous one except lack of the term of oscillation. Part of the results are displayed in Figure 7~15.

The convergence of the terminal point $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ of Z-trajectory to the initial point $\mathbf{x}(\emptyset)$ of the system is shown in Figure 7~14 where the initial point $\mathbf{Z}(\emptyset,\mathbf{q})$ is the origin because of $\mathbf{r}(F(\mathbf{q}))=\emptyset$.

Figure 7 shows the difficulties of convergence for $\mathbf{x}(\emptyset)=(1,\emptyset)$ and K=1/2 as a constant (G.C.Ø). Figure 8 however, shows

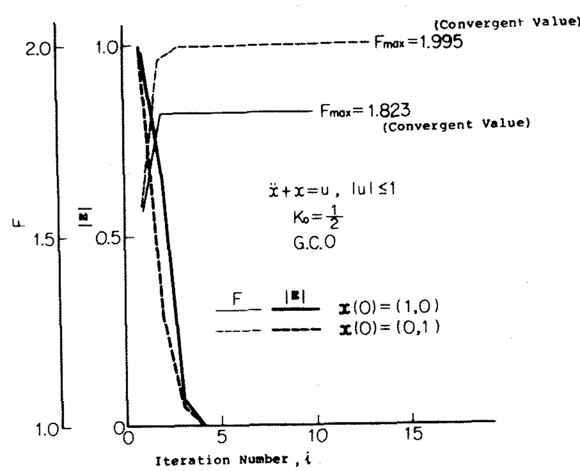

FIGURE 6. Convergence Process of 2nd-order, Undamped Oscillatory System

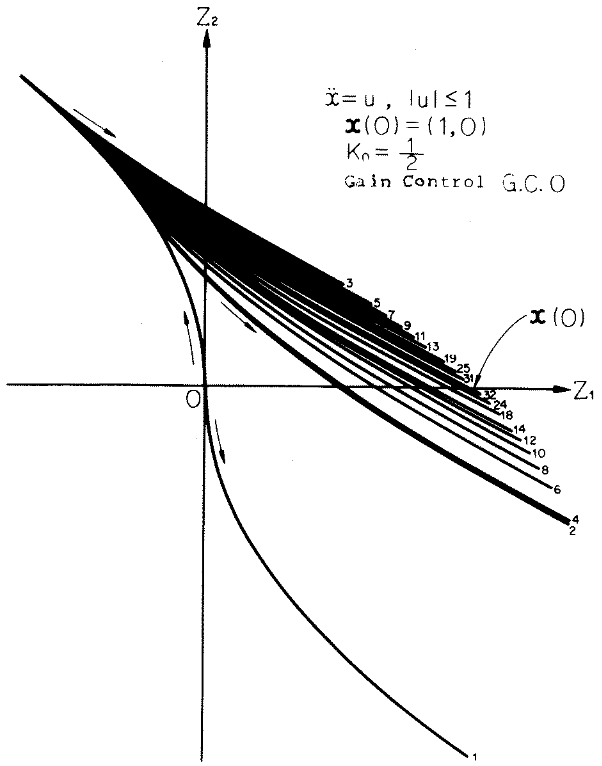

FIGURE 7. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

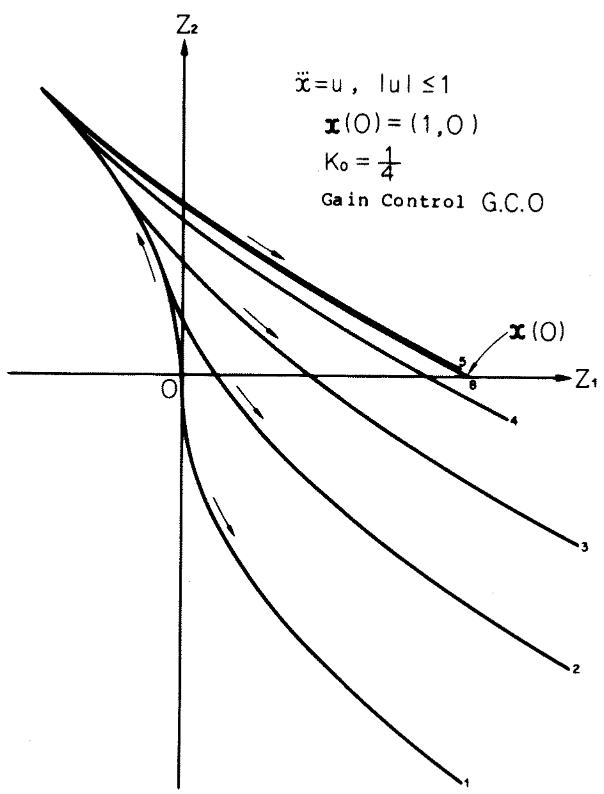

FIGURE 8. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

feasible convergence for K=1/4 (G.C.0). Figure 9 shows the difficulties of convergence for $\mathbf{x}(0)=(0,1)$ and K=3/16 (G.C.0). It is also difficult to converge for various value of K (G.C.0) in $\mathbf{x}(0)=(0,1)$.

The gain control G.C.1 for $K^0=1/2$ brings the feasible result for $\mathbf{x}(0)=(1,0)$ in Figure 10, but not for $\mathbf{x}(0)=(0,1)$ in Figure 11. The gain control G.C.1 for $K^0=3/4$ brings the feasible result for $\mathbf{x}(0)=(0,1)$.

The gain control G.C.3 was developed

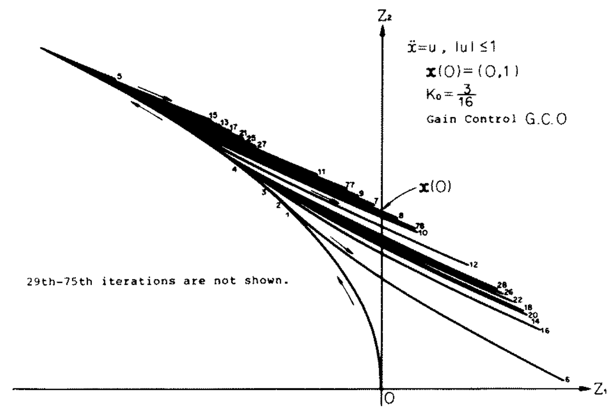

FIGURE 9. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

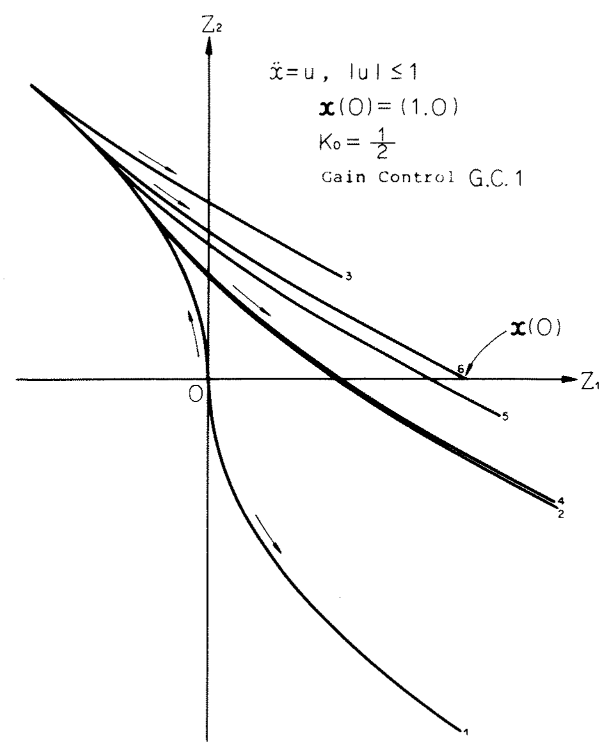

FIGURE 10. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

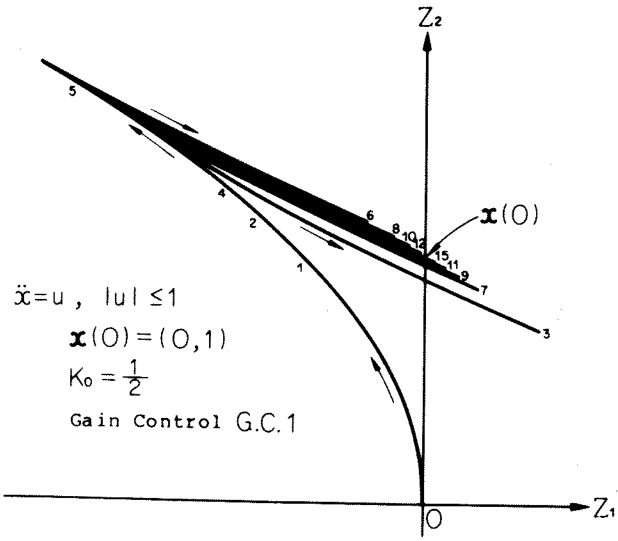

FIGURE 11. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

for the sake of avoiding the dependency of the convergence on $\mathbf{x}(\emptyset)$ and K°. G.C.3 for $K^{\circ}=1/2$ brings the feasible results for $\mathbf{x}(\emptyset)=(1,\emptyset)$ and $(\emptyset,1)$, which are shown in Figure 12~13. G.C.4 brings the same results as G.C.3.

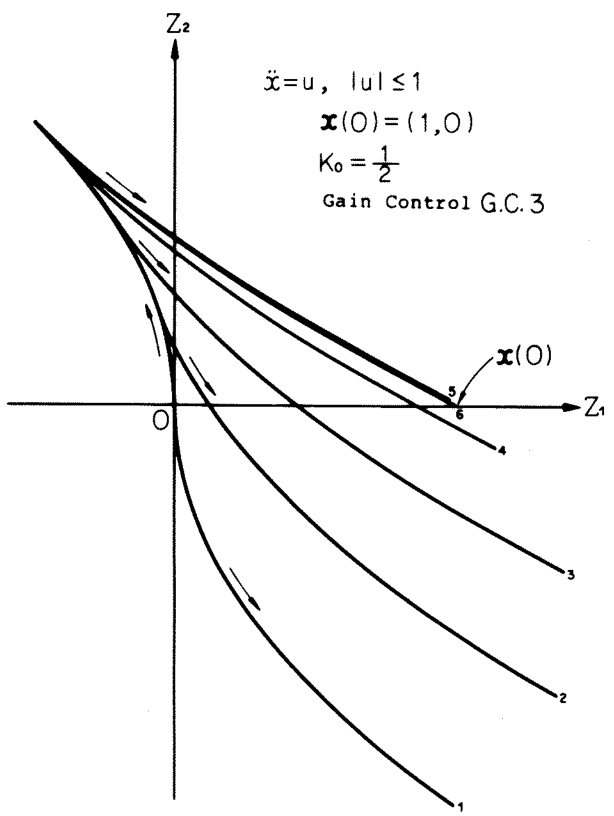

FIGURE 12. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

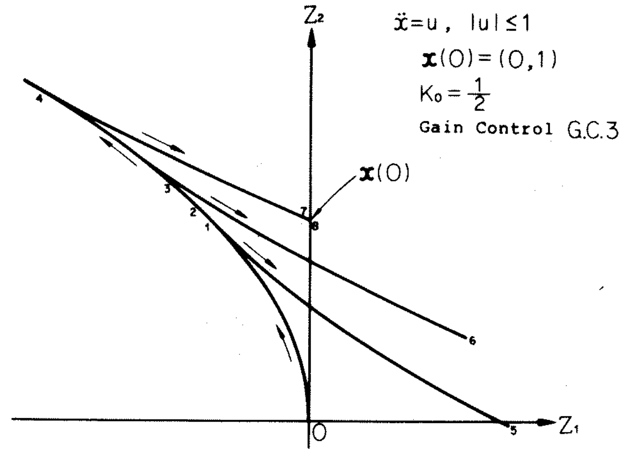

FIGURE 13. Convergence of Z-trajectories for 2nd-order Undamped Unoscillatory System

G.C.5 is a gain control based directly on the geometrical interpretation of Neustadt's algorithm and brings feasible results for the 2nd-order system. However the application of G.C.6 is restricted to the 2nd-order system because of the small capacity of the digital hybrid system.

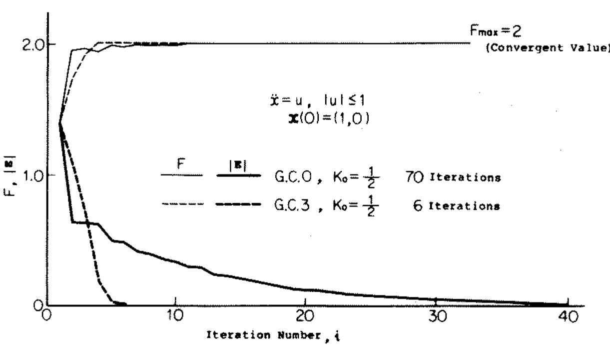

FIGURE 14. Convergence Process of 2nd-order Undamped Unoscillatory System

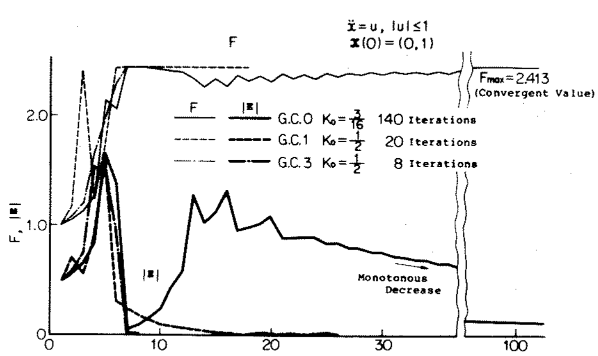

FIGURE 15. Convergence Process of 2nd-order Undamped Unoscillatory System

Figure 14~15 show the behavior of maximizing the optimum time F(**q**) and the variation on the absolute value of the error |**E**(**q**)| of the boundary value for each iteration. Convergence of F(**q**) is faster than that of |**E**(**q**)|.

3rd-order System

A characteristic of the 3rd-order system ($\dddot{x}=u$) treated here is to have only an analytical solution in the higher-order than 2nd-order system and to show the general difficulties of convergence for the multi-order system. These difficulties disappear in the 2nd-order system.

The syatement of the problem is as follow:

> "Examine the convergence behavior of Z(t,**q**), **E**(**q**) and F(**q**) of the system where the initial point **x**(Ø)=(1,Ø,Ø) is transfered to the target point **r**(t)=(Ø,Ø,Ø) subject to |u(t)|≤1 in minimum time."

The algorithm for the digital hybrid computation is formulated as follows:

$$dZ_1=-(t^2/2)*u*dt \ ; \ Z_1(\emptyset)=\emptyset \quad (2\emptyset)$$

$$dZ_2=t*u*dt \ ; \ Z_2(\emptyset)=\emptyset \quad (21)$$

$$dZ_3=-u*dt \ ; \ Z_3(\emptyset)=\emptyset \quad (22)$$

$$u(t,\mathbf{q})=\mathrm{Sgn}(-q_3+q_2*t-(q_1/2)*t^2) \quad (23)$$

where $\mathbf{q}(q_1,q_2,q_3)$ is the initial value of the adjoint vector to be determined. The D.D.A. mapping for these operations is given in Figure 3. The value of u varies discontinuonsly because of |u|=1, so that udt instead of u may be treated as a datum for internal processing for the sake of avoiding the generation of discontinuous quantity. Part of the results are displayed in Figure 16~19.

The convergence of the terminal point **Z**(F(**q**),**q**) of Z-trajectory to the initial point **x**(Ø) of the system is shown in Figure 16~18, where the initial point **Z**(Ø,**q**) is the origin because of **r**(F(**q**))=Ø. As compared with the 2nd-order system, the converge is more difficult. Figure 16~17 shows the difficulties of convergence for the gain control G.C.3 and K =1. It may be observed that each **Z**(F(**q**),**q**) for the iteration number 5~9 and 17~ moves monotonically on the same direction along a certain line little by little.

It may be thought that this is caused by too small value of the gain K. The gain control G.C.3 contains no prodecure for increasing the value of K. Therefore, the value of K may be increased manually at the starting point for such monotone moving of **Z**(F(**q**),**q**). Feasible results are not alwys shown for the gain control G.C.6 which is developed for including the procedure to increase the value of K. Figure 18 shows the result for the gain control G.C.7 which is developed for the same objective as G.C.6.

Figure 19 shows the behavior of

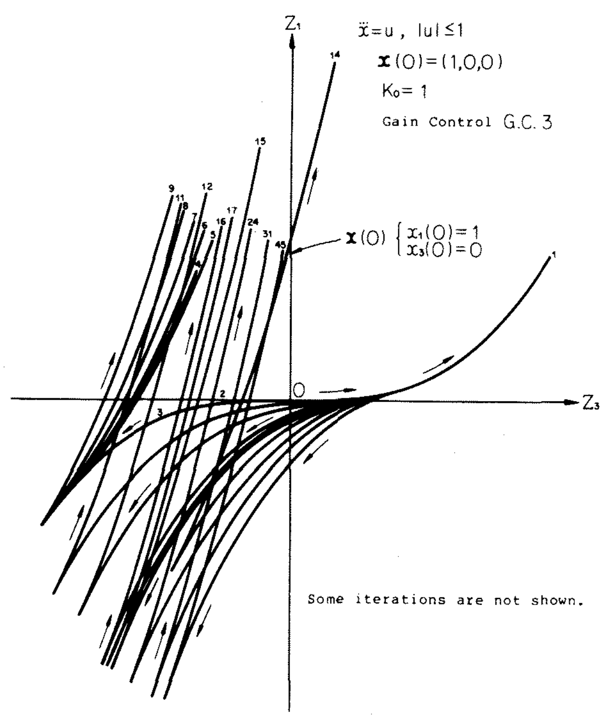

FIGURE 16. Convergence of Z-trajectories for 3rd-order System

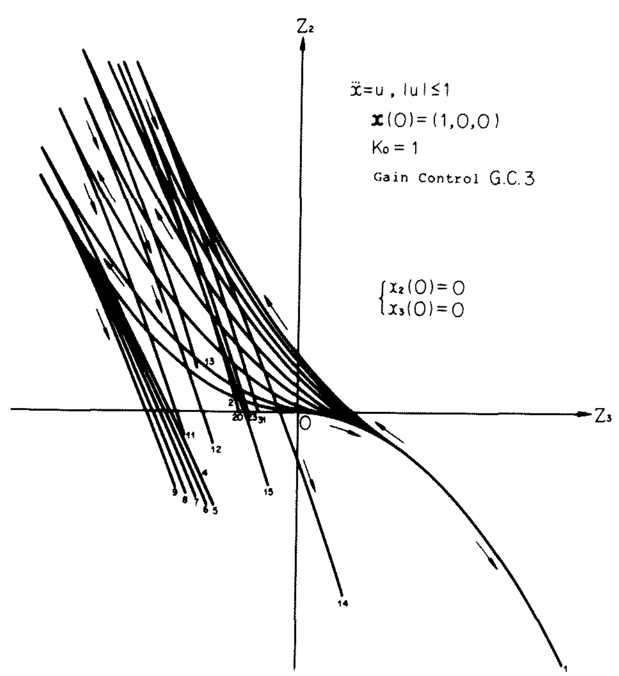

FIGURE 17. Convergence of Z-trajectories for 3rd-order System

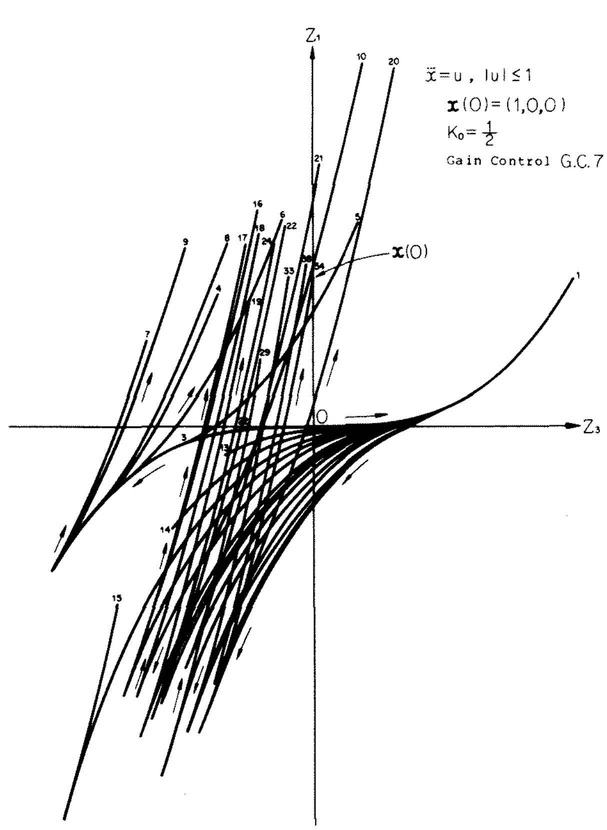

FIGURE 18. Convergence of Z-trajectories for 3rd-order System

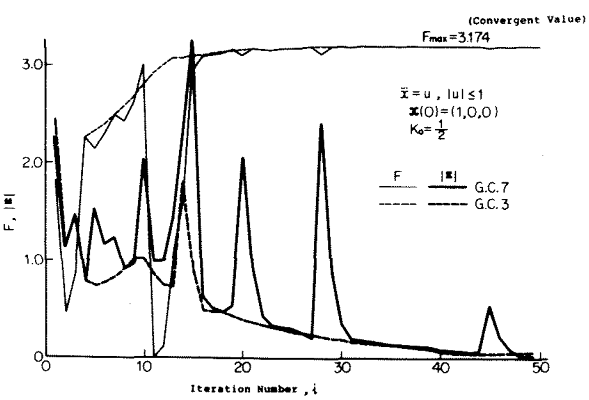

FIGURE 19. Convergence Process of 3rd-order System

maximizing the optimum time $F(\mathbf{q})$ and the variation on the absolute value of the error $|\mathbf{E}(\mathbf{q})|$ of the boundary value for each iteration.

2nd-order System subject to Energy Constraints

The convergence behavior regarding the 2nd-order, undamped unoscillatory system subject to energy constraints is examined as an example of analytically unsolvable

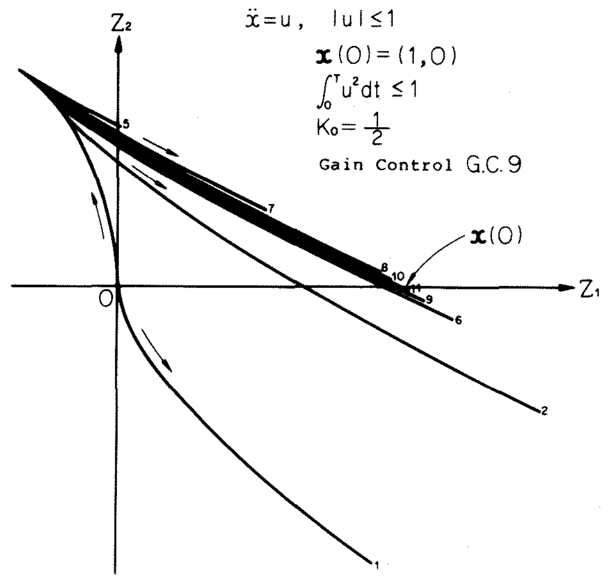

FIGURE 2Ø. Convergence of Z-trajectories for 2nd-order System subject to Energy Constraints

problems.

The statement of the problem is as follows:

" Examine the convergence behavior of $\mathbf{Z}(t,\mathbf{q})$, $\mathbf{E}(t,\mathbf{q})$, $F(\mathbf{q})$ of the system $(x=u)$ where the initial point $\mathbf{x}(\emptyset)=(1,\emptyset)$ is transfered to the target point $\mathbf{x}(F)=\mathbf{\emptyset}$ (the origin in the state-space) subject to $|u(t)|\leq 1$ and

$\int_{\emptyset}^{F} u^2\,dt \leq 1$ in minimum time."

$F=2$ and $\int_{\emptyset}^{2} u^2\,dt=2$ for the correspondingy problem without the energy constraints. This problem is formulated as the 3rd-order system, where the 3rd-state variable $x_3(t)$ is difined as follows:

$$x_3(t)=\int_{\emptyset}^{t} u^2\,dt \tag{24}$$

Here $x_3(\emptyset)=\emptyset$, and $x_3(F)=1$ from the physical meaning of the problem. Therefore the terminal point of the system is not the origin in the state-space.

The algorithm for the digital hybrid computation is formulated as follows:

$$dZ_1=t*u*dt\ ;\ Z_1(\emptyset)=\emptyset \tag{25}$$

$$dZ_2=-u*dt\ ;\ Z_2(\emptyset)=\emptyset \tag{26}$$

$$dZ_3=-u*dt\ ;\ Z_3(\emptyset)=\emptyset \tag{27}$$

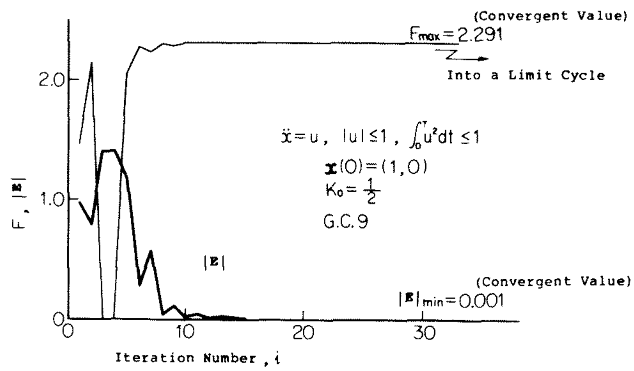

FIGURE 21. Convergence Process of 2nd-order System subject to Energy Constraints

$$u(t,\mathbf{q})=(1/2)*((q_1/q_3)*t-(q_2/q_3)) \; ; \; |u|\leq 1 \qquad (28)$$

where $\mathbf{q}(q_1,q_2,q_3)$ is the initial value of the adjoint vector to be determined.

Figure 2Ø shows the convergence of the terminal point $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ of Z-trajectory to the initial point $\mathbf{x}(\emptyset)$ of the system for the gain control G.C.9 and $K^0=1/2$. Figure 21 shows the behavior of maximizing the optimum time $F(\mathbf{q})$ and variation on the absolute value of the error $|\mathbf{E}(\mathbf{q})|$ of the bounday value for each iteration.

The values derived from convergence are the following:

$F_{max}=2.291$

$(q_1,q_2,q_3)=($ .95254, 1,Ø7983, .45581Ø$)$

Concluding Remarks

Generally speaking with regard to the results observed in this experimental study, the convergence of $F(\mathbf{q})$ is very rapid and that of $\mathbf{Z}(F(\mathbf{q}),\mathbf{q})$ late. The latest is that of the initial value $\mathbf{q}$ of the adjoint vector, because the information of the whole control process is concentrated in the $\mathbf{q}$. The considerable precision of operation is required of the convergence for $\mathbf{q}$.

As for the gain control, the control by the value of $F(\mathbf{q})$ is less effective than that of $|E(\mathbf{q})|$, because the convergence of $F(\mathbf{q})$ is very fast. It seems that the following is feasible for the generalized gain control:

> " If $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)<\emptyset$ then K is multiplied by Ø.5 and $\mathbf{q}^T$ is recomputed from $\mathbf{q}^i$ else if $\mathbf{E}(\mathbf{q}^T)*\mathbf{E}(\mathbf{q}^i)>L*\mathbf{E}(\mathbf{q}^i)*\mathbf{E}(\mathbf{q}^i)$ then K is multiplied by 2 and $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$ else $\mathbf{q}^T$ is set to $\mathbf{q}^{i+1}$, where L is a positive constant of each problem and $\mathbf{q}^T=\mathbf{q}^i+K*\text{grad } F(\mathbf{q}^i)$. "

The simple algorithms of the gain control are practically feasible for mechanization, so that the gain control only by $F(\mathbf{q})$ will be effective for centain problems.

A series of computational studies has been carried out to determine the general character of the digital hybrid implementation, of which the other computational aspects are discussed elsewhere.

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