LIFTING SURFACE APPROACH OF OSCILLATING WINGS IN WEAK SHEAR FLOW

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Abstract

The importance of investigating unsteady wing problem in shear flow is increasing. However, nonpotential property of the flow prevents using a perturbation velocity potential. In this paper, a theory of wings which are oscillating in a weak shear flow is presented. The flow is assumed to be incompressible and inviscid, and the nonuniform velocity distribution is normal to the wing surface. The potential lifting surface theory is extended into the shear flow case by the method of successive approximations. The integral equation for the lift distribution to the first order approximation is derived by the double Fourier transform, and it is solved numerically by the mode function method. Calculations regarding oscillating rectangular wings with heaving and pitching modes in a shear flow are presented as examples. Generalized forces which can be easily related with unsteady lift forces and moments are obtained. Results show that the shear flow decreases all forces in amplitudes.

I. Introduction

The problem calculating lift and pitching moment on a wing, especially in an unsteady condition, is fundamentally important for several kinds of studies of aeronautics. In almost all of the cases discussed, the flow is assumed to be uniform (potential flow). In engineering applications, however, the upstream condition is not always uniform, but rather nonuniform, being disturbed by several obstacles ahead of the wing. It is possible to divide the types of shear flow along the spanwise direction (the upper figure of Fig.1), and the shear flow normal to the wing surface (the lower figure of Fig.1).

For the former type, the author[1] presented the results of calculations in which rectangular wings are submerged in the uniform shear flow along the spanwise direction; unsteady lifting surface theory was applied. Due to the linearity of the flow velocity distribution and the rectangular wing, the integral equation for the lift distribution reduced to a well known form which could be solved by the already developed method of mode functions[2].

For the latter type of shear flow, some theoretical works have been already published. Tsien[3] obtained the exact solution of a uniform shear flow past a symmetrical Joukowski airfoil. Chen[4] used the second order theory in calculating the pressure distribution over the airfoil in uniform shear flow. Chow et al.(5) treated an airfoil in nonuniform flow, and solved nonlinear differential equations numerically using the finite difference method. Ventres[6], Yates[7] and Chi[8] succeeded in obtaining the effect of boundary layer on wing surface by the lifting surface theory. Nishiyama and Hirano[9],[10] analyzed the unsteady

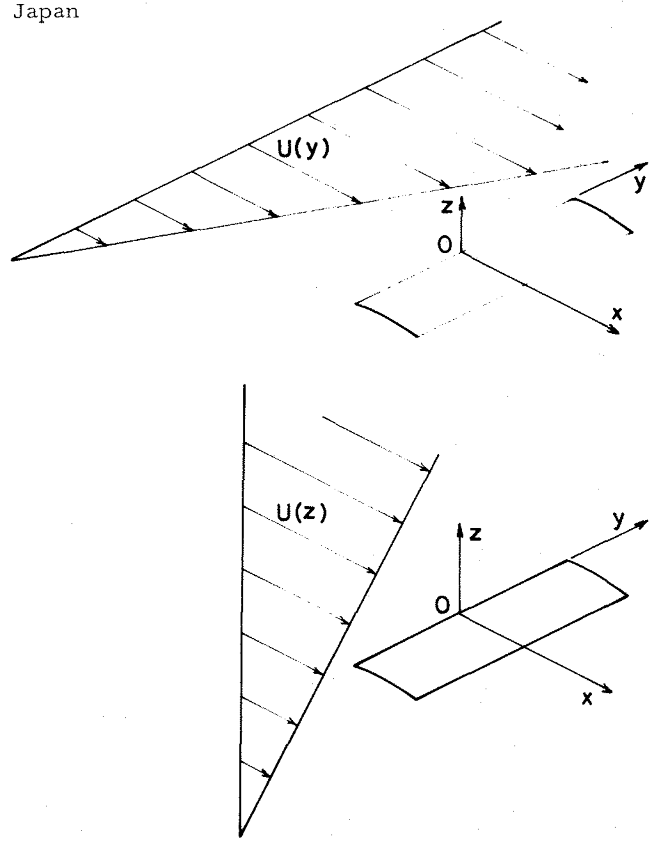

Fig.1 Two Types of Shear Flow

characteristics of an airfoil in the shear flow.

In this paper, the latter type of shear flow is analyzed, and an unsteady lifting surface theory based on an inviscid and incompressible shear flow model is presented. Since the velocity distribution of laminar wake (see Fig.2) is adopted, the method of Ref.1 cannot be applied. A weak shear model is used in order to extend the linearized potential theory by successive approximations. Accordingly, an approaching flow is assumed to be slightly disturbed from the uniform flow on the plane normal to the wing surface, and then a small positive quantity ε representing a measure of this deviation is introduced. By a double Fourier transform, the partial differential equation for the pressure is transformed into an ordinary differential equation. The integral equation for the lift distribution is derived and solved by the mode function method. Generalized forces are calculated to the first order approximation by the collocation method.

II. Basic Equations

Let a thin wing be submerged in incompressible and inviscid flow as illustrated in Fig.2. If it is assumed that the disturbed velocities are much smaller than the free stream

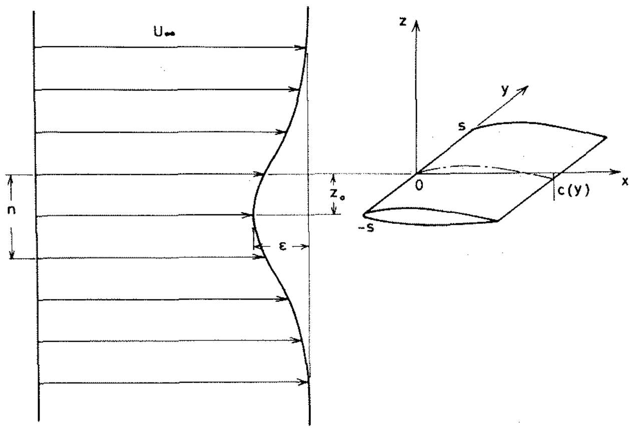

Fig.2 Wing in Weak Shear Flow

velocity $U(z)$, the equations of motion can be linearized by the small perturbation theory as

$$\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} + w\frac{dU(z)}{dz} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad \text{(a)}$$

$$\frac{\partial v}{\partial t} + U(z)\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \qquad \text{(b)} \qquad (1)$$

$$\frac{\partial w}{\partial t} + U(z)\frac{\partial w}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0, \qquad \text{(c)}$$

and the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (2)$$

After differentiations and algebraic manipulations between Eqs.(1) and (2), the following partial differential equation for pressure can be derived

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\nabla^2 p - 2\frac{dU(z)}{dz}\frac{\partial^2 p}{\partial x \partial z} = 0. \qquad (3)$$

Assuming simple harmonic motion, p is written by

$$p(x,y,z,t) = \bar{p}(x,y,z)e^{i\omega t}. \qquad (4)$$

Therefore, Eq.(3) can be rewritten as

$$\left[i\omega + U(z)\frac{\partial}{\partial x}\right]\nabla^2 \bar{p} - 2\frac{dU(z)}{dz}\frac{\partial^2 \bar{p}}{\partial x \partial z} = 0. \qquad (5)$$

Now, we can transform (x,y)-space into (α,β)-space by using a double Fourier transform expressed by

$$\bar{p}^*(\alpha,\beta;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}(x,y,z)e^{-i(\alpha x+\beta y)}dxdy. \qquad (6)$$

As is well known, the inverse transform of Eq.(6) is given by

$$\bar{p}(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}^*(\alpha,\beta;z)e^{i(\alpha x+\beta y)}d\alpha d\beta. \qquad (7)$$

Applying the double Fourier transform to both sides of Eq.(5), the following ordinary differential equation is obtained.

$$\frac{d^2\bar{p}^*}{dz^2} - 2\frac{\alpha}{\omega+U(z)\alpha}\frac{dU(z)}{dz}\frac{d\bar{p}^*}{dz} - R^2\bar{p}^* = 0, \qquad (8)$$

where $R^2=\alpha^2+\beta^2$. Eq.(8) is a linear differential equation of the second order. Here, we confine our attention to the weak shear case, in which the velocity distribution is given by

$$U(z) = U_\infty\{1 + \varepsilon f(z)\} \qquad (9)$$

where ε is a small quantity of the first order. In this case, $\bar{p}^*$ may be expanded into a power series of ε as follows:

$$\bar{p}^* = \bar{p}^*_0 + \varepsilon\bar{p}^*_1 + \varepsilon^2\bar{p}^*_2 + \cdots, \qquad (10)$$

where the $\bar{p}_i^*(i=0,1,\ldots)$ are functions of z, α and β. Substituting Eqs.(9) and (10) into Eq.(8), successive differential equations can be obtained by collecting the same order coefficients of ε as follows:

$$\frac{d^2\bar{p}^*_0}{dz^2} - R^2\bar{p}^*_0 = 0 \qquad : O(1) \qquad (11)$$

$$\frac{d^2\bar{p}^*_1}{dz^2} - R^2\bar{p}^*_1 = \frac{2U_\infty\alpha f'}{\omega+U_\infty\alpha}\bar{p}^*_0 \qquad : O(\varepsilon) \qquad (12)$$

$\cdots\cdot,$

where the following approximation is made.

$$\frac{2\alpha}{\omega + U_\infty\{1+\varepsilon f(z)\}\alpha}U_\infty\varepsilon f'(z)$$
$$= \frac{2\varepsilon U_\infty\alpha f'}{\omega + U_\infty\alpha}\{1 - \varepsilon\frac{U_\infty\alpha f}{\omega + U_\infty\alpha} + O(\varepsilon^2)\}.$$

It is clear that Eq.(11) expresses the potential flow case, and, therefore, in order to solve Eqs.(11) and (12), we can use the method of successive approximations; i.e. (i) solve Eq.(11) first, (ii) solve Eq.(12) by substituting $\bar{p}^*_0$ into the right hand side.

III. Derivation of the Integral Equation and Method of Solution

Before deriving the integral equation for the lift distribution, it is convenient to obtain the relation between $\bar{p}^*$ and $\bar{w}^*$. The Fourier transform is applied to both sides of Eq.(1c).

$$i\{\omega + U(z)\alpha\}\bar{w}^* + \frac{1}{\rho}\frac{d\bar{p}^*}{dz} = 0 \qquad (13)$$

follows. Substituting Eqs.(9), (10) and $\bar{w}^*$, expanded as

$$\bar{w}^* = \bar{w}^*_0 + \varepsilon\bar{w}^*_1 + \varepsilon^2\bar{w}^*_2 + \cdots, \qquad (14)$$

into Eq.(13), the following successive equations are obtained in a manner similar to that of Eqs.(11) and (12).

$$i(\omega + U_\infty\alpha)\bar{w}^*_0 + \frac{1}{\rho}\frac{d\bar{p}^*_0}{dz} = 0 \qquad : O(1) \qquad (15)$$

$$i(\omega + U_\infty\alpha)\bar{w}^*_1 + iU_\infty\alpha f\bar{w}^*_0 + \frac{1}{\rho}\frac{d\bar{p}^*_1}{dz} = 0 \quad : O(\varepsilon) \qquad (16)$$

At this point, the boundary conditions can be expressed in (α,β)-space as follows:
(i) $\bar{p}^* \longrightarrow 0$ at $z \longrightarrow \pm\infty$; i.e. any disturbance should dsiappear at plus and minus infinity.
(ii) $\bar{w}^*(\alpha,\beta,+0) = \bar{w}^*(\alpha,\beta,-0)$ (this is the condition of tangential flow.)

First, let us consider the $O(\varepsilon)$-case, Eqs.(12) and (16). Eq.(12) can be easily solved as

$$\bar{p}^*_1(\alpha,\beta,z) = C_{1-}e^{Rz} + C_{1+}e^{-Rz} + \frac{G^*(\alpha)}{z}\int^z f'(t)\bar{p}^*_0(\alpha,\beta,t)\sinh R(z-t)dt \qquad (17)$$

where

$$G^*(\alpha) = \frac{2U_\infty\alpha}{\omega + U_\infty\alpha} \qquad (18)$$

By the boundary conditions, the two integration constants are related as

$$C_{1+} = -\{C_{1-} + \frac{G^*(\alpha)}{R}(\int_{-\infty}^{0} + \int_{0}^{\infty}) f'(t)\bar{p}_0^*(\alpha,\beta,t)\cosh Rt dt\} \quad \text{(a)}$$

$$C_{1-} = \frac{1}{4}\rho U_\infty^2 l_1^* + \frac{G^*(\alpha)}{2R}\int_{-\infty}^{\infty} f'(t)\bar{p}_0^*(\alpha,\beta,t)\sinh Rt dt - \frac{G^* \alpha}{2R}\int_{-\infty}^{\infty} f'(t)\bar{p}_0^*(\alpha,\beta,t)\cosh Rt dt \quad \text{(b)} \quad (19)$$

where l_1^* is the lift force by the shear flow as follows:

$$l_1^* = \frac{\bar{p}_{1-}^*(\alpha,\beta,t) - \bar{p}_{1+}^*(\alpha,\beta,t)}{\frac{1}{2}\rho U_\infty^2} \quad (20)$$

Combining Eqs.(16) and (17) with Eqs.(19) and (20),

$$\frac{\bar{w}_1^*(\alpha,\beta,0)}{U_\infty} + \frac{U_\infty \alpha}{\omega + U_\infty \alpha} f(0)\frac{\bar{w}_0^*(\alpha,\beta,0)}{U_\infty} = -\frac{U_\infty R}{4i(\omega+U_\infty\alpha)}l_1^*(\alpha,\beta) - \frac{U_\infty G^*(\alpha)}{8i(\omega+U_\infty\alpha)}\{\int_0^{\infty} f'(t)e^{-2Rt} \times l_0^*(\alpha,\beta)dt - \int_{-\infty}^{0} f'(t)e^{2Rt}l_0^*(\alpha,\beta)dt\} \quad (21)$$

is derived, where $l_0{}^*$ is the lift distribution of the potential flow case; this is the equation for the lift distribution $l_1{}^*$ in (α,β)-space. The integral equation for l_1 in (x,y)-space can be obtained after inverting Eq.(21) and some algebraic manipulations (see Appendix A):

$$\frac{\tilde{w}_1(x,y)}{U_\infty} + f(0)\frac{\tilde{w}_0(x,y)}{U_\infty} + S_1(x,y) + S_2(x,y) = -\frac{1}{8\pi}\iint_S \tilde{l}_1(x',y')e^{i\bar{\nu}x_0}\bar{c}^2 K(x_0,y_0)dx'dy' \quad (22)$$

where $x_0 = x - x'$, $y_0 = y - y'$, $\bar{\nu} = \omega\bar{c}/U_\infty$ and

$$\bar{w}_1(x,y) = \tilde{w}_1(x,y)e^{-i\bar{\nu}x}$$
$$\bar{l}_1(x,y) = \tilde{l}_1(x,y)e^{-i\bar{\nu}x} \quad (23)$$

$$S_1(x,y) = -i\bar{\nu}f(0)\{\int_{-\infty}^{x}\frac{\tilde{w}_0(x',y)}{U_\infty}dx' - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\tilde{w}_0(x',y)}{U_\infty}dx'\}$$

$$S_2(x,y) = -\frac{1}{8\pi}\iint_S \tilde{l}_0(x',y')e^{i\bar{\nu}x_0}\bar{c}^2 T(x_0,y_0)dx'dy'$$

$$T(x_0,y_0) = \int_0^{\infty}\{f'(z)+f'(-z)\}\frac{e^{-i\bar{\nu}x_0}}{2}[(1-i\bar{\nu}x_0)\frac{1}{\bar{c}}\frac{\partial}{\partial z} \times\{\int_{-\infty}^{x_0}\frac{e^{i\bar{\nu}\lambda}}{\sqrt{\lambda^2+y_0^2+(2z)^2}}d\lambda - \frac{1}{2}K_0(\bar{\nu}\sqrt{y_0^2+(2z)^2})\} + 4i\bar{\nu}z\{\frac{1}{\bar{c}}\frac{e^{i\bar{\nu}x_0}}{\sqrt{x_0^2+y_0^2+(2z)^2}} - i\frac{\bar{\nu}}{\bar{c}}\int_{-\infty}^{x_0}\frac{e^{i\bar{\nu}\lambda}}{\sqrt{\lambda^2+y_0^2+(2z)^2}}d\lambda + \frac{1}{2}i\frac{\bar{\nu}}{\bar{c}}K_0(\bar{\nu}\sqrt{y_0^2+(2z)^2})\}] \quad (24)$$

It is easy to ascertain that the integral equation for l_0 (the O(1)-case, the potential flow case) can be derived by applying the same process used in the O(ε)- case to Eqs.(11) and (15). The process is omitted here. Consequently, l_0 can be obtained by solving the potential flow case, and then l_1 can be obtained by solving Eq.(22) using the value of l_0.

Several kinds of methods have been presented in order to solve the integral equation in lifting surface theory. Among these methods the 'mode function method' is applied in this paper. The detail of this method can be found in Ref.2. The points different from the usual method are explained in Appendix B.

IV. Examples of Numerical Calculations and Discussion

Numerical calculations are performed by the collocation method. The wing surface is divided into many small panels as follows:

$$x_{p\nu} = x_l(\eta_\nu) + \frac{c(\eta_\nu)}{2}(1 - \cos\phi_p),\ p = 1,\cdots,N$$
$$\eta_\nu = \frac{y_\nu}{s} = -\cos\theta_\nu,\ \nu = 1,\cdots,m, \quad (25)$$

where

$$\phi_p = \frac{2\pi p}{2N+1},\ \theta_\nu = \frac{\nu\pi}{m+1}. \quad (26)$$

Then, the integral equations are transformed into a set of linear equations as follows:

$$-\frac{\tilde{w}_0(p,\nu)}{U_\infty} = \sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}^0\Omega_q(p,\nu,r) \ :\ O(1) \quad (27)$$

$$-\frac{\tilde{w}_1(p,\nu)}{U_\infty} - f(0)\frac{\tilde{w}_0(p,\nu)}{U_\infty} - S_1(p,\nu) - S_2(p,\nu) = \sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}^1\Omega_q(p,\nu,r) \ :\ O(\varepsilon) \quad (28)$$

where

$$\Omega_q(p,\nu,r) = \sum_{\lambda=1}^{\Lambda+1}\{\bar{R}_q(p,\nu,\lambda)\kappa_{r\lambda}\} + P_q(p,\nu)\rho_{\nu r} + P_q'(p,\nu)\sigma_{\nu r} + (\frac{s}{c_\nu})^2 E_q(p,\nu)\tau_{\nu r}$$

In the above equations $\bar{R}_q$, P_q, P_q', E_q, $\kappa_{r\lambda}$, $\rho_{\nu r}$, $\sigma_{\nu r}$ and $\tau_{\nu r}$ take the same forms as those in Ref.2. Therefore, if the upwash velocities w_0 and w_1 are given on collocation points, the simultaneous equations Eqs.(27) and (28) can be solved, and the lift distributions can be calculated. In order to obtain the unsteady response in the shear flow, it is effective to calculate the generalized force which is defined by

$$Q_{ij} = -\frac{1}{2dD}\iint_S z_i l_j dxdy, \quad (29)$$

where i denotes force mode, and j oscillation mode.

In this paper, rectangular wings with zero thickness are adopted as examples of numerical calculations, where the aspect ratios are 3 and 6. The number of collocation points is $N = 3$ in the chordwise direction, and $m = 11$ in the spanwise direction. Some oscillation modes are given subsequently. Since a simple harmonic motion is assumed, the heaving oscillation $j = 1$, and the pitching oscillation $j = 2$ are used, i.e.

$$z_{j=1} = -h,\ z_{j=2} = -\alpha_0 x \quad (30)$$

Both heaving amplitude h and pitching amplitude α_0 are taken as unity in calculations. Therefore, upwash velocities are given by

$$[\frac{\tilde{w}_0(x,y)}{U_\infty}]_j = e^{i\bar{\nu}x}\{\frac{\partial z_j(x,y)}{\partial x} + i\frac{\bar{\nu}}{\bar{d}}z_j(x,y)\}$$

$$= e^{i\bar{\nu}x}(-\frac{i\bar{\nu}}{d})_{j=1},\ e^{i\bar{\nu}x}(-1-\frac{i\bar{\nu}x}{d})_{j=2}$$

$$[\frac{\tilde{w}_1(x,y)}{U_\infty}]_j = e^{i\bar{\nu}x}f(0)\frac{\partial z_j(x,y)}{\partial x} = 0_{j=1},\ -e^{i\bar{\nu}x}f(0)_{j=2}, \tag{31}$$

and force modes are given by

$$z_{i=1} = -d\ ,\ z_{i=2} = -x\ , \tag{32}$$

where i = 1 denotes a half of the lift coefficient, and i = 2 a half of the nose-down pitching moment coefficient about the leading edge. Finally, the following velocity profile $f(z)$ is assumed as

$$f(z) = -\exp\{-(\frac{z+z_0}{n})^2\}\ , \tag{33}$$

where z_0 is the wing position, and n is the degree of spread of the shear flow. This profile simulates the wake produced by an object ahead of the wing. Generalized force given by Eq.(29) are obtained for the first order approximation at

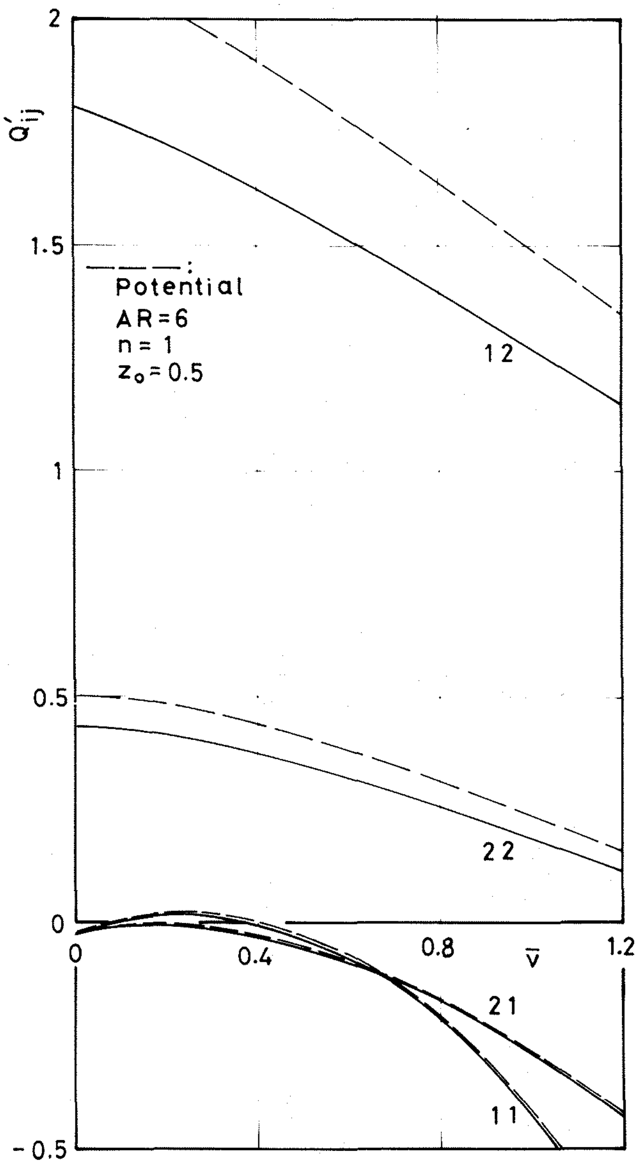

Fig.3 Stiffness and Damping Derivatives against Frequency Parameter in Heaving Mode

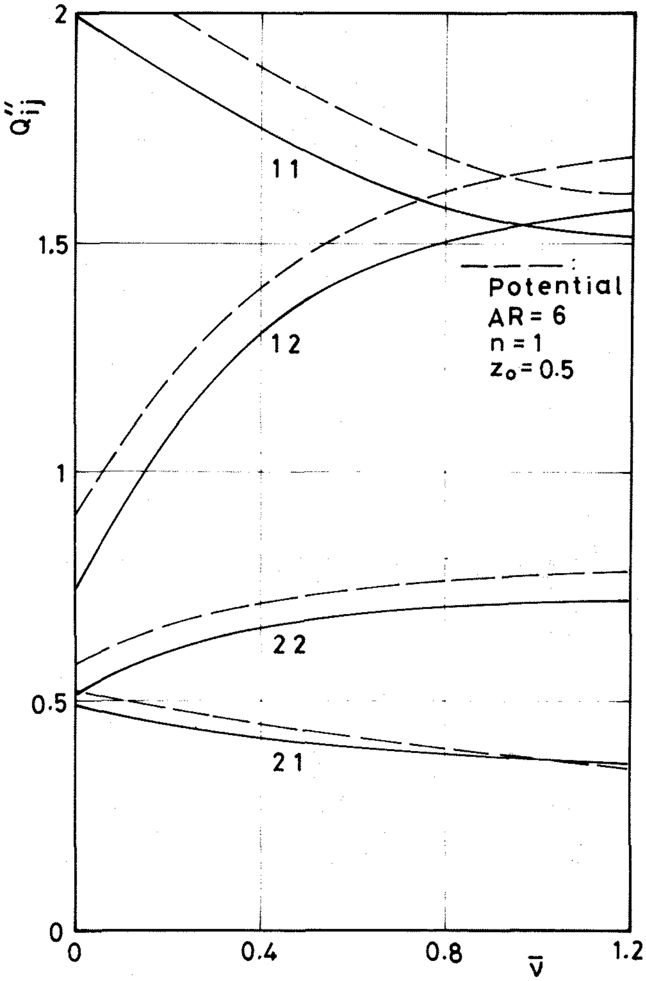

Fig.4 Stiffness and Damping Derivatives against Frequency Parameter in Pitching Mode

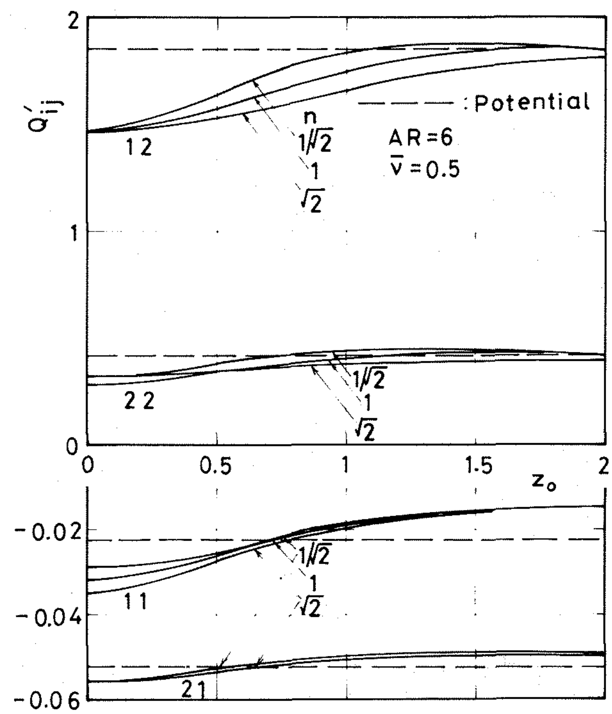

Fig.5 Stiffness and Damping Derivatives against Wing Position in Heaving Mode

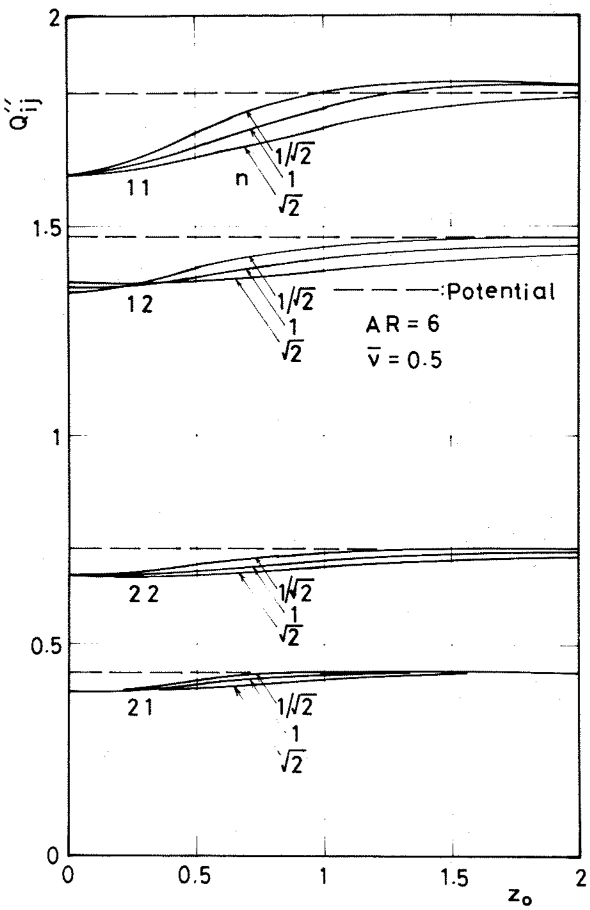

Fig.6 Stiffness and Damping Derivatives against Wing Position in Pitching Mode

ε = 0.1, i.e. the maximum loss of the approaching flow velocity is 10% of the reference velocity U_∞. Complex generalized forces in the unsteady state can be rewritten as

$$Q_{ij} = Q'_{ij} + i\bar{\nu}Q''_{ij} \tag{34}$$

where Q'_{ij} and Q''_{ij} are the stiffness and damping derivatives respectively (2). The former is related to the vertical and angular position of the cases i = 1 and 2 respectively.

In Figs.3 and 4, these derivatives in two oscillatory modes are shown for the wing with aspect ratio 6. The broken line shows the potential flow results (O(1)-case). The parameters of the shear flow are taken as n = 1.0 and z_0= 0.5. It is clear from these figures that the loss of flux in the shear flow decreases all force coefficients except the values for Q_{ij} in the heaving mode. The difference between results of the potential flow and those of the shear flow is not so affected by in $\bar{\nu}$ the range of these calculations. As described above, subscripts (1,1) and (2,1) are related to a half of the lift coefficient and nose-down moment coefficient about the leading edge in the heaving respectively, and (1,2) and (2,2) are those in the pitching.

In Figs.5 and 6, generalized forces are plotted against z_0 for three values of n. Since they are symmetric with respect to z_0 = 0, and have a peak

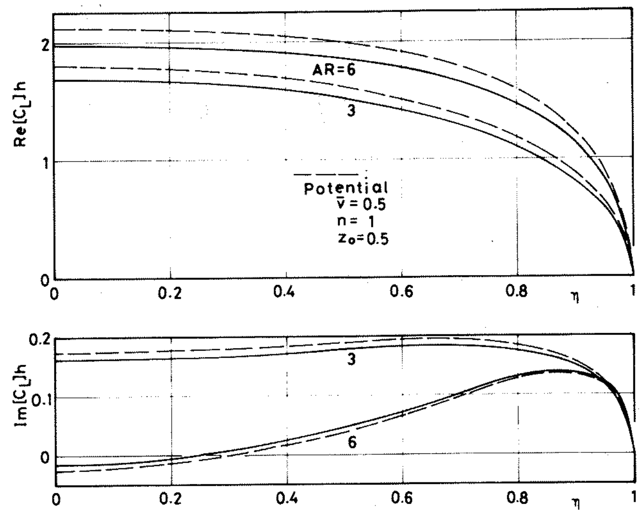

Fig.7 Spanwise Lift Distribution in Heaving Mode

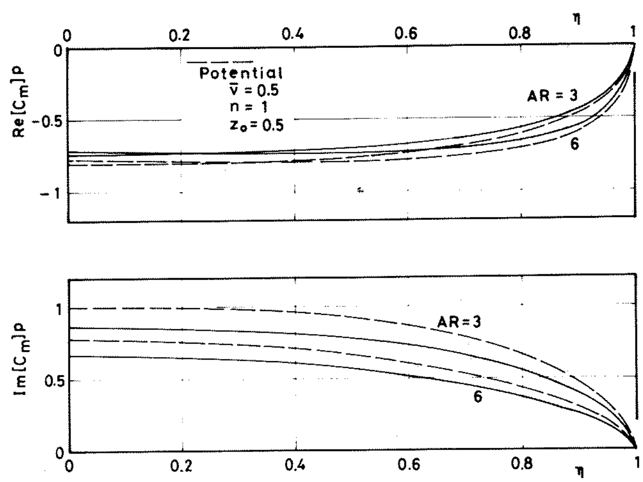

Fig.8 Spanwise Pitching Moment Distribution in Pitching Mode

at z_0 = 0, the results of the positive region are illustrated. These tendencies can be predicted by the form of $S_2(x,y)$ given in Eq.(24), because it includes the second order derivative of $f(z)$, which is symmetric with respect to z. Although the shear flow decreases all force coefficients near the center of the wake, these coefficients take the same values at the center of the wake except for Q''_{11} and Q''_{12}, and they gradually approach the potential flow values as z_0 becomes large. The larger n becomes, which means the wake extends in the xz-plane with the constant maximum loss ε, the wider the influence of the shear flow becomes. It seems that not the total loss of flux but the velocity gradient with respect to z becomes dominant at small n. $S_2(x,y)$ is more affected by n than by any other parameter. In Figs.7 and 8, the spanwise aerodynamic force distributions are shown for aspect ratio 3 and 6 in the case of $\bar{\nu}$ = 0.5, n = 1.0 and z_0= 0. By these figures, results already established in the potential flow case are obtained.

V. Conclusions

The response of oscillating wings in the weak shear flow are obtained by unsteady lifting surface theory. The flow is assumed to be inviscid and incompressible, but has no potential. The partial differential equation for the pressure is transformed into an ordinary differential equation by the double Fourier transform. In order to simplify the problem, a method of successive approximation is used, i.e. all variables are expanded into power series of ε, which is the degree of strength of the shear flow. Moreover, it is formulated in arbitrary modes of oscillation of wings.

Numerical calculations are performed for the velocity profile which simulates the wake caused by an obstacle ahead of the wing. Generalized forces are obtained in heaving and pitching oscillatory modes of rectangular wings with aspect ratios 3 and 6. Lift coefficients and pitching moment coefficients for each mode can be easily calculated by the generalized forces. The loss of flux in the approaching flow decreases the amplitude of all force coefficients. Unfortunately, no experimental results have been found. However, these results seem to agree well with those described in the two-dimensional steady flow case.

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Appendix A

The double Fourier transform and its inverse transform are expressed as

$$F[\cdot] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot \, e^{-i(\alpha x+\beta y)} dx dy \quad (A.1)$$

$$F^{-1}[\cdot] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot \, e^{i(\alpha x+\beta y)} d\alpha d\beta \quad (A.2)$$

respectively. Therefore, inverse transform of $G^*(\alpha)$, e^{-2Rt} and $R/i\cdot 2(\omega+U_\infty\alpha)$ can be obtained as follows:

$$F^{-1}[G^*(\alpha)]=2\delta(y)\{\delta(x)-\frac{i}{2}(\frac{\omega}{U_\infty})e^{-i\frac{\omega x}{U_\infty}}\mathrm{sgn}(x)\} \quad (A.3)$$

$$F^{-1}[e^{-2Rt}]= \frac{1}{\pi}\frac{t}{\{x^2+y^2+(2t)^2\}^{3/2}} \quad (A.4)$$

$$F^{-1}[\frac{R}{i\cdot 2(\omega+U_\infty\alpha)}]= \frac{1}{4\pi U_\infty}\lim_{t\to 0}\frac{\partial^2}{\partial t^2}e^{-i\frac{\omega x}{U_\infty}}\int_{-\infty}^{x}\frac{e^{i\frac{\omega\lambda}{U_\infty}}}{\sqrt{\lambda^2+y^2+t^2}}d\lambda \quad (A.5)$$

The right hand side of Eq.(A.5) is the kernel function of the integral equation for the lift distribution in potential flow.

Next, the convolution formula of the inverse transform is given by

$$F[F_1^*\cdot F_2^*] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-x',y-y')F_2(x',y')dx'dy' \quad (A.6)$$

Using Eqs.(A.3), (A.4), (A.5) and (A.6), following results can be obtained.

$$F^{-1}[G^*(\alpha)\bar{w}_0(\alpha,\beta)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-x')\delta(y-y')\bar{w}_0(x',y')dx'dy'$$

$$=2[\bar{w}_0(x,y)-i\frac{\omega}{U_\infty}e^{-i\frac{\omega}{U_\infty}x}\{\int_{-\infty}^{x} e^{i\frac{\omega}{U_\infty}x'}\bar{w}_0(x',y)dx'$$

$$-\frac{1}{2}\int_{-\infty}^{\infty} e^{i\frac{\omega}{U_\infty}x'}\bar{w}_0(x',y)dx'\}] \quad (A.7)$$

$$F^{-1}[\frac{G^*(\alpha)}{i\cdot 2(\omega+U_\infty\alpha)}e^{-2Rt}l_0^*(\alpha,\beta)]$$

$$=-\frac{1}{\pi U_\infty}\int\int_{-\infty}^{\infty} e^{-i\frac{\omega x_0}{U_\infty}}[(1-\frac{\omega x_0}{U_\infty})\frac{\partial}{\partial t}\{\int_{-\infty}^{x_0}\frac{e^{i\frac{\omega\lambda}{U_\infty}}}{\sqrt{\lambda^2+y_0^2+(2t)^2}}d\lambda$$

$$-\frac{1}{2}K_0(\frac{\omega}{U_\infty}\sqrt{y_0^2+(2t)^2})\}+\frac{4\omega t}{U_\infty}\{\frac{e^{i\frac{\omega x_0}{U_\infty}}}{\sqrt{x_0^2+y_0^2+(2t)^2}}$$

$$-i\frac{\omega}{U_\infty}\int_{-\infty}^{x_0}\frac{e^{i\frac{\omega\lambda}{U_\infty}}}{\sqrt{\lambda^2+y_0^2+(2t)^2}}d\lambda+\frac{i}{2}\frac{\omega}{U_\infty}K_0(\frac{\omega}{U_\infty}\sqrt{y_0^2+(2t)^2})\}]$$

$$\times l_0(x',y')dx'dy' \quad (A.8)$$

Appendix B

Let l_1 be expanded into following series.

$$l_1(x',y') = \frac{8s}{\pi c(y')}\sum_{q=1}^{N}\Gamma_q^1(\eta')\Psi_q(\phi') , \quad (B.1)$$

where

$$\Psi_q(\phi') = \frac{\cos(q-1)\phi' + \cos q\phi'}{\sin\phi'} \quad (B.2)$$

Substituting Eq.(24) into Eq.(22), the integral equation is rewritten as

$$\frac{\tilde{w}_1(x,\eta)}{U_\infty}+f(0)\frac{\tilde{w}_0(x,\eta)}{U_\infty}+S_1(x,\eta)+S_2(x,\eta)$$
$$=\frac{1}{2\pi}\int_{\eta'=-1}^{1}\frac{1}{(\eta-\eta')^2}\sum_{q=1}^{N}\Gamma^1(\eta')F_q(x,\eta;\eta')d\eta' \qquad (B.3)$$

where

$$S_1(x,\eta)=-i\bar{\nu}f(0)\{\int_\infty \frac{x\tilde{w}_0(x',\eta)}{U_\infty}dx'-\frac{1}{2}\int_\infty^\infty\frac{\tilde{w}_0(x',\eta)}{U_\infty}dx'\} \qquad (a)$$

$$S_2(x,\eta)=-\frac{1}{8\pi}\iint_S\frac{8s}{\pi c(\eta')}\sum_{q=1}^{N}\Gamma_q^0(\eta')\Psi_q(\phi)e^{i\bar{\nu}x_0}\bar{c}^2$$
$$\times T(x_0,y_0)\frac{c(\eta')}{2}\sin\phi d\phi s d\eta' \qquad (b)(B.4)$$
$$=\frac{1}{2\pi}\int_{\eta'=-1}^{1}s^2\sum_{q=1}^{N}\Gamma_q^0(\eta')S_q(x,\eta;\eta')d\eta'$$

$$S_q(x,\eta;\eta')=-\frac{1}{\pi}\int_0^\pi\bar{c}^2e^{i\bar{\nu}x_0}T(x_0,y_0)\Psi_q(\phi)\sin\phi d\phi \quad (B.5)$$

In order to calculate the right hand side of Eq.(B.5), it is convenient to divide into four parts as follows:

$$T(x_0,y_0)=\sum_{k=1}^{4}T_k(x,\eta;\eta') \qquad (B.6)$$

where

$$T_1(x,\eta;\eta')=-\frac{e^{-i\bar{\nu}x_0}}{\bar{c}}(1-i\bar{\nu}x_0)f'(0)F(0;x_0,\eta,\eta')$$

$$T_2(x,\eta;\eta')=-\frac{e^{-i\bar{\nu}x_0}}{2\bar{c}}(1-i\bar{\nu}x_0)\int_0^\infty\{f''(z)-f''(-z)\}F(z;x_0,\eta.\eta')dz$$

$$T_3(x,\eta;\eta')=\frac{e^{-i\bar{\nu}x_0}}{2\bar{c}}4\bar{\nu}^2\int_0^\infty\{f'(z)+f'(-z)\}F(z;x_0,\eta,\eta')dz$$

$$T_4(x,\eta;\eta')=\frac{e^{-i\bar{\nu}x_0}}{2\bar{c}}4i\bar{\nu}^2\int_0^\infty\{f'(z)+f'(-z)\}\frac{e^{i\bar{\nu}x_0}}{\sqrt{(\bar{\nu}x_0)^2+(\bar{\nu}y_0)^2+(2iz)^2}}dz \quad (B.7)$$

and

$$F(z;x_0,y_0)=\int_{-\infty}^{x_0}\frac{e^{i\bar{\nu}\lambda}}{\sqrt{\lambda^2+y_0^2+(2z)^2}}d\lambda-\frac{1}{2}K_0(\bar{\nu}\sqrt{y_0^2+(2z)^2})$$
$$=\int_0^{x_0}\frac{e^{i\bar{\nu}\lambda}}{\sqrt{\lambda^2+y_0^2+(2z)^2}}d\lambda+\frac{1}{2}[K_0(\bar{\nu}\sqrt{y_0^2+(2z)^2})$$
$$-i\pi\{I_0(\nu\sqrt{y_0^2+(2z)^2})-\mathit{L}_0(\nu\sqrt{y_0^2+(2z)^2})\}] \qquad (B.8)$$

Accordingly, S_q is rewritten as

$$S_q(x,\eta;\eta')=\sum_{k=1}^{4}S_q^k(x,\eta;\eta') \qquad (B.9)$$

where

$$S_q^k(x,\eta;\eta')=-\frac{1}{\pi}\int_0^\pi\bar{c}^2e^{i\bar{\nu}x_0}T_k(x,\eta;\eta')\Psi_q(\phi)\sin\phi d\phi,\quad k=1,2,3 \quad (B.10)$$

Logarithmic singularity included in S_q' is removed by

$$Z_q^1(x,\eta;\eta')=S_q^1(x,\eta;\eta')-C_q^1(x,\eta)\log|\eta-\eta'| \qquad (B.11)$$

$$C_q^1(x,\eta)=\frac{\pi}{\bar{c}}\{1+i\mu(\frac{1}{4}-X)\}f'(0)\{\frac{1}{2}\int_0^1f_q^1(X_0)dX_0$$
$$-2\int_0^Xf_q^1(X_0)dX_0\}-\frac{\pi}{\bar{c}}i\frac{\mu}{4}f'(0)\{\frac{1}{2}\int_0^1f_q^2(X_0)dX_0-2\int_0^Xf_q^2(X_0)dX_0\} \qquad (B.12)$$

$$f_q^1(X_0)=\frac{1}{\sqrt{X_0(1-X_0)}}[\cos\{(q-1)\cos^{-1}(1-2X_0)$$
$$+\cos\{q\cos^{-1}(1-2X_0)\}]$$

$$f_q^2(X_0)=\frac{1}{\sqrt{X_0(1-X_0)}}[\cos\{(q-2)\cos^{-1}(1-2X_0)$$
$$+\cos\{(q+1)\cos^{-1}(1-2X_0)\}] \qquad (B.13)$$

where

$$X_0=\frac{1}{2}(1-\cos\phi_p)$$

Finally S_2 is written as

$$S_2(p,\nu)=-s^2\sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}^0\sum_{\lambda=1}^{\Lambda+1}Z_q(p,\nu,\lambda)\kappa_{r\lambda}$$
$$-s^2\sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}^0C_q^1(p,\nu)\tau_{\nu r}$$
$$-s^2\sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}^0\sum_{\lambda=1}^{\Lambda+1}S_q'(p,\nu,\lambda)\kappa_{r\lambda} \qquad (B.14)$$

where

$$S_q'(x_{p\nu},\eta_\nu;\eta_\lambda)=\sum_{k=2}^{4}S_q^k(x_{p\nu},\eta_\nu;\eta_\lambda) \qquad (B.15)$$

and

$$Z_q^k(x,\eta;\eta')=\frac{2}{\Lambda+1}\sum_{\lambda=1}^{\Lambda+1}Z_q^k(x,\eta;\eta_\lambda)[\frac{1}{2}+\sum_{\omega=1}^{\Lambda}\cos\omega\theta_\lambda\cos\omega\theta']. \qquad (B.16)$$