STUDIES ON OPTIMIZATION OF LAMINATED COMPOSITE PLATES

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Abstract

This paper deals with the study of minimum weight design of laminated composite plates subjected to inplane and transverse loading. Angle ply laminates with orthotropic layers and antisymmetric plies are considered. Thickness of plies and the corresponding fiber orientations are treated as design variables. Numerical results have been obtained for different aspect ratios and biaxial inplane loadings. This study indicates that the fiber orientations of plies near the mid-plane have little effect on the optimum design. There exists a particular fiber orientation angle for the overall thickness of the laminate which results in optimum design for a plate of given aspect ratio under a given set of loading.

I. Introduction

During the last two decades or so, fiber reinforced laminates have found increasing applications in many engineering structures. The high strength to weight and stiffness to weight ratio render them to be an ideal structural material for Aerospace work. This material has an advantage that its properties can be tailored through fiber orientation, thickness of laminae and stacking sequence. This gives the designer an added degree of flexibility to achieve the desired strength or stiffness in any direction.

The advent of high modulus/high strength fiber reinforced composites, such as carbon/epoxy and graphite/epoxy has resulted in an increase in the use of laminated fiber reinforced plates and other structural shapes as primary structural members.

The growing use of composites have stimulated interest in the development of optimal design of structures made of composite materials. The earliest attempt in this direction seems to be due to Mroz(1) who attempted to obtain the optimal reinforcement using strength as a design criteria. Bryzgalin(2) and Love and Melchers(3) employed a stiffness criteria to obtain an optimal design using constant thickness composites.

Lai and Achenbach(4) used a direct search procedure to obtain an optimal design for minimum tensile stress at the interface in a layered structure subjected to time harmonic and transient loads. Khot et. al(5) suggested an efficient optimization method based on strain energy distribution and a numerical search for the minimum weight design of structures. The procedure takes into account multiple loading conditions and displacement constraints on the structure.

Schmit and Farshi(6,7) presented a technique for minimum weight design of symmetric composite laminates subjected to multiple inplane loading conditions. The problem is cast as a nonlinear programming problem with preassigned fiber orientations and treating the thickness of the ply as a design variable. Hayashi(8) optimized the fiber volume fractions for columns and orientation angle for plates and cylinders for the corresponding buckling strength.

Chao et.al(9) determined the optimum orientation for a symmetric composite laminate with inplane loading. The buckling results presented are restricted to specially orthotropic laminates.

Hirano(10) optimised the buckling load of laminated plates under uniaxial and biaxial compression. The plate is made up of plies of equal thickness. The fiber orientation of each ply is treated as a design variable while the number of plies is preassigned.

Recently Sharma et.al(11) have shown that if the laminate is made up of variable thickness plies, the degrading effects of coupling can be eliminated. This results in an increase in the strength of the laminate.

Thus, it is observed that not all the design variables have been exploited to arrive at an optimal design. Either fiber orientation or thickness of a ply is treated as a design variable.

In this paper, an attempt is made to incorporate the thickness of each ply and fiber orientation as design variables, while the number of plies is used as a design parameter. Two types of problems are studied:

1. Maximising the buckling load of a layered composite plate of a given dimension.
2. Minimum weight design of a laminate subjected to inplane and transverse loading.

The problems are cast as a nonlinear programming problem. The constraint problem is transformed into an unconstrained problem. The unconstrained minimization is performed using McCormick's modification of the Fletcher-Powell method. Results have been obtained for varying aspect ratios and uniform biaxial in-plane loading.

II. Problem Formulation & Analysis

II.1 Optimal Design of Laminated Plates Under Uniform Axial Compression

A composite laminated plate, with each lamina having orthotropic property is considered for the analysis. This orthotropy is achieved by placing in a lamina equal number of fibers at $+\alpha$ and $-\alpha$ angles with respect to the structural axis. The study is carried out for the following two cases:

(i) maximisation of the buckling load with constraint on thickness of each lamina to be within a preassigned thickness.

(ii) maximisation of the buckling load with a constraint on overall thickness of the laminate.

The objective function is

$$\text{Max } N_x(\alpha_i, t_i), \quad i = 1, \ldots N \tag{1}$$

such that

$$0 \leq \alpha_i \leq \pi/2 \tag{2}$$

and

$$\text{(i)} \quad t_i \leq t_i^*, \quad i = 1, N \tag{3}$$

$$\text{(ii)} \quad \sum_{i=1}^{N} t_i \leq t^* \tag{4}$$

The problem as stated above is a constrained optimization problem. This constrained problem is transformed into a series of unconstrained problems by introducing a penalty function associated with the constraints. The problem then reduces to

$$P(r, \alpha_i, t_i) = N_x(\alpha_i, t_i) - r \sum_{j=1}^{m} f_j(\alpha_i, t_i) \tag{5}$$

where

$$f_j(\alpha_i, t_i) = \ln(g_j(\alpha_i, t_i)) \quad \text{if } g_j(\alpha_i, t_i) \geq 0 \tag{6}$$

in which g_j's are the constraints given by equations (3) and (4). The transformed problem is solved using Fletcher and Powell method.[12]

The optimization problem requires the knowledge of buckling load and the derivatives of the buckling load with respect to α_i and t_i. For this purpose, the plate is assumed to be simply supported on all sides. The three governing equations along u, v, and w are[13]

$$A_{11}u_{,xx} + A_{66}u_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} - (B_{12} + 2B_{66})w_{,xyy} = 0 \tag{7a}$$

$$A_{22}v_{,yy} + A_{66}v_{,xx} + (A_{12} + A_{66})v_{,xy} - B_{22}w_{,yyy} - (B_{12} + 2B_{66})w_{,xxy} = 0 \tag{7b}$$

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - B_{11}u_{,xxx} - (B_{12} + 2B_{66})u_{,xyy} - (B_{12} + 2B_{66})v_{,xxy} - B_{22}v_{,yyy} + N_x w_{,xx} + N_y w_{,yy} = 0 \tag{7c}$$

where A_{ij}, B_{ij} and D_{ij} are axial, coupling and bending stiffnesses respectively.

Defining $N_y = KN_x$, and assuming a solution for u, v and w, which satisfies the boundary conditions, the buckling load N_x is obtained as[14]

$$N_x = \frac{1}{\pi^2 a^2 (m^2 + Kn^2p^2)} \times \left[T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right] \tag{8}$$

where

$$\begin{aligned} T_{11} &= A_{11}m^2\pi^2 + A_{66}n^2\pi^2p^2 \\ T_{22} &= A_{66}m^2\pi^2 + A_{22}n^2\pi^2p \\ T_{12} &= (A_{12} + A_{66})mn\pi^2p \\ T_{13} &= B_{11}m^3\pi^3 + (B_{12} + 2B_{66})mn^2\pi^3p^2 \\ T_{23} &= (B_{12} + 2B_{66})m^2n\pi^3p + B_{22}n^3\pi^3p^3 \\ T_{33} &= D_{11}m^4\pi^4 + 2(D_{12} + 2D_{66})m^2n^2\pi^4p^2 + D_{22}n^4\pi^4p^4 \end{aligned} \tag{9}$$

m and n are the number of half sine waves along x and y direction and p = (a/b) is the aspect ratio of the plate.

Derivatives of the buckling load with respect to design variables are obtained in closed form. Ref. 14 discuss these in detail.

II.2 Optimal Design of Plates Under Inplane and Transverse Loading

As stated in introduction, minimum weight is an important criterion for structural design for aerospace applications. Composites exhibit better fatigue and corrosion properties. Besides minimum weight requirements, structures are often designed for stiffness, strength, small deflection and high buckling loads. The study is carried out for the following two cases:

(i) minimum weight design of composite plate subjected to deflection constraint.
(ii) minimum weight design of a composite plate under requirements of high buckling load and minimum deflection.

The objective function is

$$\text{Min } W'(= W/A) = \text{Min} \sum_{i=1}^{N} t_i \qquad (10)$$

such that

$$0 \leq \alpha_i \leq \pi/2, \quad t_i \geq 0, \quad i = 1,N$$

and

$$w \leq w^*$$
$$N_x \geq N_x^* \qquad (11)$$

The three governing equations, which describe the behaviour of the plate under transverse loads are

$$A_{11}u_{,xx} + A_{66}v_{,yy} + (A_{12} + A_{66})v_{,xy} - B_{11}w_{,xxx} - (B_{12} + 2B_{66})w_{,xyy} = 0 \qquad (12a)$$

$$A_{22}v_{,yy} + A_{66}v_{,xx} + (A_{12} + A_{66})u_{,xy} - B_{22}w_{,yyy} - (B_{12} + 2B_{66})w_{,xxy} = 0 \qquad (12b)$$

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} - B_{11}u_{,xxx} - B_{22}v_{,yyy} - (B_{12} + 2B_{66})(u_{,xyy} - v_{,xxy}) = q(x,y) \qquad (12c)$$

The plate is assumed to be simply supported on all sides and subjected to a uniformly distributed load. Following the method discussed in Ref. (14), the deflections w_{mn}, u_{mn} and v_{mn} are given by

$$u_{mn} = \frac{q_{mn}}{a^5\bar{D}}(T_{22}T_{13} - T_{12}T_{23}) \qquad (13a)$$

$$v_{mn} = \frac{q_{mn}}{a^5\bar{D}}(T_{11}T_{23} - T_{13}T_{12}) \qquad (13b)$$

and

$$w_{mn} = \frac{q_{mn}}{a^5\bar{D}}(T_{22}T_{11} - T_{12}^2) \qquad (13c)$$

where

$$\bar{D} = \frac{(T_{11}T_{22} - T_{12}^2)}{8} \times \left[T_{33} + \frac{2T_{12}T_{13}T_{23} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right]$$

and

$$q_{mn} = \frac{16q_o}{2} \cdot \frac{1}{mn}$$

m and n are number of half sine waves along x and y directions. T_{11}, T_{12}, etc. are given by Eq. (9).

In addition to constraints described in Section II.1, following additional constraints are introduced

$$g_1 = 1 - \frac{W}{W^*} \qquad (14a)$$

$$g_2 = (N_x/N_x^*) - 1 \qquad (14b)$$

Here again, we obtain analytical expressions for the derivatives of the constraints with respect to the design variables.

III. Numerical Computations and Results

III.1 Laminated Plates Under Uniform Axial Load

Numerical studies have been carried for rectangular composite plates with number of plies (N), aspect ratio (p) and biaxial loading ratio (K) as parameters for Boron/Epoxy composites, whose material properties are as follows:

$E_1 = 2.11\text{x}10^4$ kg/m^2, $E_2 = 2.11\text{x}10^3$ kg/mm^2

$\nu_{12} = 0.3$, $G_{12} = 7.03\text{x}10^2$ kg/mm^2

Table 1, shows the effect of number of plies on the buckling for a square plate. It is observed that at the optimum point fiber orientation of all the plies is same, which is found to be 45°. This may be attributed to the fact that the term T33 in the buckling load relation (see Eq. 8) contributes maximum to the buckling load. T33 depends on D11, D12, D22 and D66. The bending stiffness coefficient D_{11} decreases with increase in fiber orientation from 0° to 90°. Further D_{22} increases with orientation from 0° to 90°. At 45° D_{11} and D_{22} are equal. In view of this variation, T_{33} attains a maximum value when fibers are oriented at 45° (m = 1 and n = 1 is the primary mode of buckling).

The variation of maximum buckling load with aspect ratio is shown in Table 2. The optimum fiber orientation increases from 0° to 90° with aspect ratio for uniaxial loading. However, for biaxial loading, the variation in fiber orientation is approximately from 19° to 62° for a variation in

No. of plies	K	Optimum fiber orientation (degrees)						Reduced critical load	Mode of buckling
		α_1	α_2	α_3	α_4	α_5	α_6		
2	0	45.0	45.0	-	-	-	-	21.977	1:1
3	0	44.9	44.6	44.9	-	-	-	21.977	1:1
		45.0	45.0	45.0	-	-	-	21.977	1:1*
4	0	45.3	43.9	43.9	45.1	-	-	21.975	1:1
		45.0	45.0	45.0	45.0	-	-	21.977	1:1*
6	0	44.5	45.3	45.5	45.5	45.4	44.5	21.975	1:1
		45.0	45.0	45.0	45.0	45.0	45.0	21.977	1:1*

* analysis at the optimum point.

TABLE 1. EFFECT OF NUMBER OF PLIES ON MAXIMUM BUCKLING LOAD (SQUARE PLATE)

aspect ratio for 0.5 to 2.0. This behaviour could be explained in the following manner. In the expression for T_{33}, D_{22} is multiplied by p^4 while the second term is multiplied by p^2. For $p < 1$, contribution of D_{22} decreases with decreasing aspect ratio. As a result for $p < 1$, the optimum fiber orientation is between 0° and 45°. For $p > 1$, the contribution of D_{22} increases, so that the optimum fiber orientation is between 45° to 90°.

The variation of buckling load with fiber orientation for a laminated plate with p = 1.25 and subjected to a uniaxial load is shown in Fig. 1. It is observed that for fiber orientation up to 50°, the mode shape is 1:1, that is, one half sine wave along x direction and one half sine wave along y direction. Beyond 50°, the mode shape associated with the maximum buckling load changes. Detailed analysis shows that the following relationship holds at the optimum fiber orientation

$$(U_2 \sin 2\alpha^* - 2U_3 \sin 4\alpha^*)p^4 + 12U_3 \sin 4\alpha^* p^2 - (U_2 \sin 2\alpha^* + 2U_3 \sin 4\alpha^*) = 0 \qquad (15)$$

where $U_2 = \dfrac{Q_{11} - Q_{22}}{2}$, $U_3 = (Q_{11} + Q_{22} - Q_{12} - 4Q_{66})/8$ and α^* is the fiber orientation. Thus, for a square plate (p = 1), Eq. (15) reduces to

$$\sin 4\alpha^* = 0 \qquad (16)$$

Therefore, $\alpha^* = \pi/4$ at the optimum point. For p = 1.25, Eq. (15) yields $\alpha^* = 50.5$ (Ref. Table 2). At this fiber orientation, the number of half sine waves changes from 1 to 2 at an aspect ratio of 1.25.

Fig. 2 shows the variation of buckling load with fiber orientation for a rectangular plate subjected to biaxial loading. It is observed that at small values of fiber orientation, the mode shape associated with the buckling load is one-half sine wave along x direction and two-half sine wave along y direction. The mode shape changes at the fiber orientation of approximately 10°.

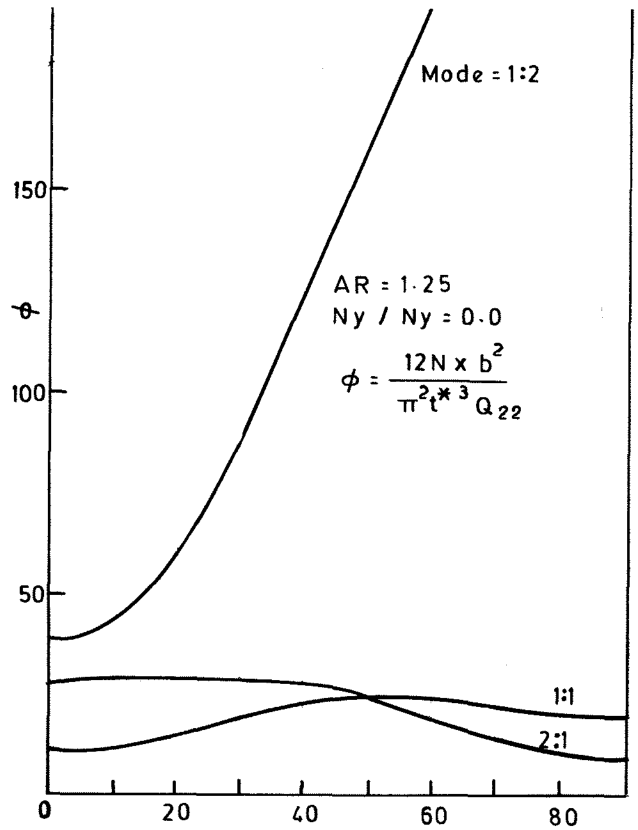

FIG.1 VARIATION OF BUCKLING LOAD WITH FIBER ORIENTATION

III.2 Plates Under Inplane and Transversed Loading

Numerical computations have been done for Boron/Epoxy composite. The upper bounds on the constraints are taken as

$$W^* = 2 \text{ mm}$$

and $N_x^* = 5.00$ kg/mm

Table 3, gives the results for optimum ply thickness and corresponding fiber orientation for the minimum weight design. It is observed for a square plate, at the optimum point, fibers in all the plies are

Aspect ratio	K	Optimum fiber orientation (degrees)						Reduced critical load	Mode of buckling
		α_1	α_2	α_3	α_4	α_5	α_6		
0.5	0.0	3.0	5.2	15.5	15.5	4.2	1.8	42.020	1:1
		0	0	0	0	0	0	42.126	1:1*
0.8	0.0	38.0	38.6	26.2	26.2	38.6	38.0	23.086	1:1
		38.0	38.0	38.0	38.0	38.0	38.0	23.130	1:1*
1.25	0.0	52.9	49.2	25.2	24.8	48.7	51.0	22.840	1:1
		50.0	50.0	50.0	50.0	50.0	50.0	23.090	1:1*
2.0	0.0	44.9	45.0	44.9	44.9	45.0	44.9	21.977	2:1
		45.0	45.0	45.0	45.0	45.0	45.0	22.000	2:1*
1.0	0.5	45.2	45.1	48.2	51.8	44.0	45.2	14.640	1:1
		45.0	45.0	45.0	45.0	45.0	45.0	14.650	1:1
2.0	0.5	61.3	66.5	69.5	69.4	66.2	61.3	12.560	1:1
		68.0	51.0	25.0	25.0	51.0	68.0	12.240	1:1

* analysis at the optimum point.

TABLE 2. VARIATION OF MAXIMUM BUCKLING LOAD WITH ASPECT RATIO (N = 6)

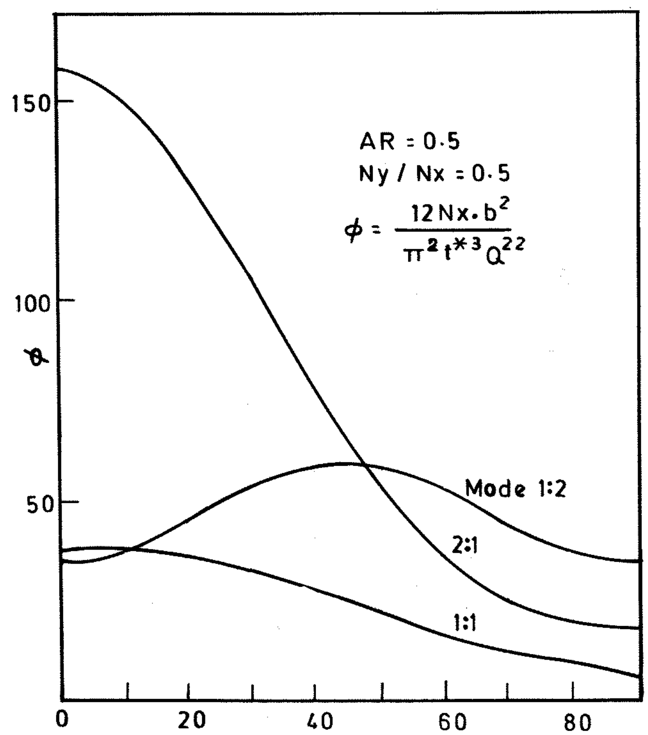

FIG.2 VARIATION OF BUCKLING LOAD WITH FIBER ORIENTATION

oriented at an angle of 45°. For aspect ratios less than one, the fiber orientations of plies closer to mid plane are closer to 45°.

The optimum ply thickness and the corresponding fiber orientations for plates subjected to biaxial loading are presented in Table 4. The optimum fiber orientation angle increases with an increase in the biaxial loading ratio for aspect ratios less than one. For a square plate, the fiber orientation angle remains constant at 45° with increase in load ratio. The contribution of the plate thickness near the mid plane is not considerable. In fact, the fiber orientation of the plies near the mid-plane do not attain the optimum value.

The variation of the maximum deflection with fiber orientation is plotted in Fig. 3. The maximum deflection attains a minimum value at higher fiber orientations as aspect ratio increases. Fig. 4 shows the variation of optimum weight with aspect ratio for different biaxial loading ratios. The optimum weight is higher for higher biaxial loading ratio. Fig. 5 shows the active constraint for a given aspect ratio and biaxial loading. The figure also gives the values of the other constraints. From Fig. 6 it is observed that the optimum weight does not depend upon the biaxial loading ratio if the constraint on deflection is active. It increases with increase in biaxial loading ratio, since the buckling load decreases with increase in load ratio, the thickness of the laminate has to increase.

IV. General Conclusions

1. The buckling load is maximum when the total thickness of the plate assumes the optimum fiber orientation for a given aspect ratio and biaxial loading ratio.

Aspect ratio	No. of plies	t_1/α_1	t_2/α_2	t_3/α_3	t_4/α_4	t_5/α_5	t_6/α_6	Optimum weight
		Optimum values						
0.5	2	0.97	0.97	-	-	-	-	1.94
		0.21	0.21	-	-	-	-	
	4	0.92	0.04	0.04	0.93	-	-	1.94
		0.09	39.65	39.65	0.05	-	-	
	6	0.83	0.11	0.05	0.03	0.10	0.82	1.95
		5.10	33.12	43.88	45.69	31.89	5.28	
1.0	2	2.0	2.0	-	-	-	-	4.01
		45.00	45.00	-	-	-	-	
	4	1.0	1.0	1.0	1.0	-	-	4.01
		45.00	44.99	44.99	45.00	-	-	
	6	0.67	0.67	0.67	0.67	0.67	0.67	4.01
		45.00	45.00	45.00	45.00	45.00	45.00	
1.25	2	2.29	2.28	-	-	-	-	4.56
		52.37	52.38	-	-	-	-	
	4	1.56	0.72	0.72	1.56	-	-	4.56
		53.03	46.54	46.53	53.02	-	-	
	6	0.84	0.88	0.57	0.54	0.89	0.83	4.57
		52.69	52.29	46.58	46.56	52.10	52.70	

TABLE 3. OPTIMUM VALUES OF PLY THICKNESS AND FIBER ORIENTATION FOR A PLATE (ONLY UNIFORMLY DISTRIBUTED LOAD)

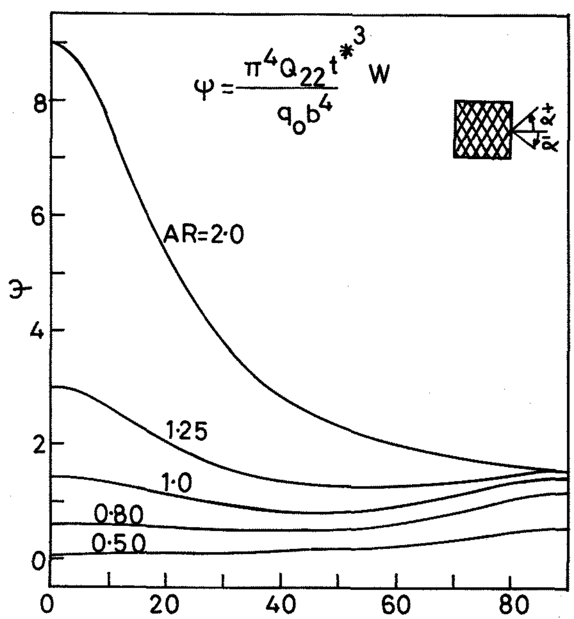

FIG. 3 MAXIMUM DEFLECTION OF SIMPLY SUPPORTED LAMINATED PLATE VS FIBER ORIENTATION

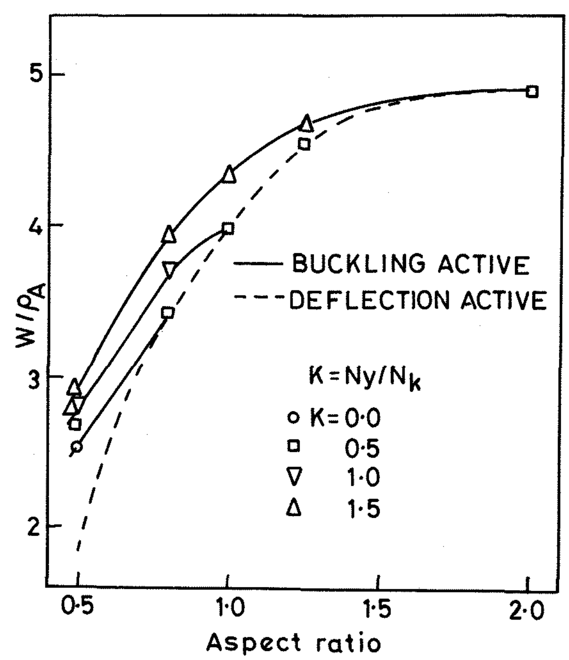

FIG. 4 VARIATION OF OPTIMUM WEIGHT WITH ASPECT RATIO

Aspect ratio	K	No. of plies	Optimum values t_1/α_1	t_2/α_2	t_3/α_3	t_4/α_4	Optimum weight
0.5	0.5	2	1.34 9.54	1.34 9.54	- -	- -	2.68
		4	0.87 6.86	0.48 39.43	0.49 39.47	0.35 6.84	2.68
0.5	1.0	2	1.40 18.18	1.41 18.18	- -	- -	2.81
		4	0.84 16.50	0.57 40.85	0.57 40.85	0.84 16.50	2.81
0.5	1.5	2	1.46 23.32	1.46 23.32	- -	- -	2.93
		4	1.33 22.70	0.02 52.90	0.04 53.00	1.54 23.90	2.93
1.0	0.5	2	2.00 45.00	2.00 45.00	- -	- -	4.01
		4	1.00 45.00	1.00 45.00	1.00 45.00	1.00 45.00	4.01
1.0	1.0	2	2.01 45.00	2.01 45.00	- -	- -	4.02
		4	1.00 45.00	1.00 45.00	1.00 45.00	1.00 45.00	4.02
1.0	1.5	2	2.16 45.00	2.16 45.00	- -	- -	4.33
		4	1.08 45.00	1.08 45.00	1.08 45.00	1.08 45.00	4.33
2.0	0.5	2	2.45	2.45	-	-	-

TABLE 4. VARIATION OF OPTIMUM VALUES WITH ASPECT RATIO AND BIAXIAL LOADING RATIO

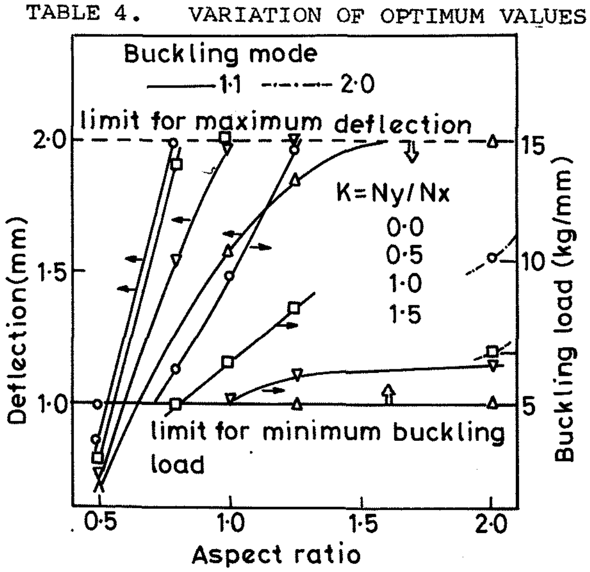

FIG. 5 VARIATION OF DELECTION AND BUCKLING LOAD WITH ASPECT RATIO FOR OPTIMUM WEIGHT DESIGN

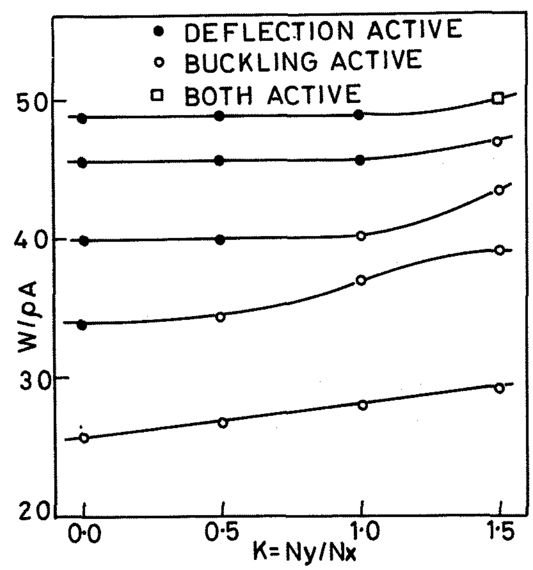

FIG. 6 VARIATION OF OPTIMUM WEIGHT WITH BIAXIAL LOADING RATIO

2. The designes of the composite plate can work with relaxed manufacturing tolerances, as far as plies near the mid plane are concerned.

3. The weight per unit area of the plate increases with increasing aspect ratio. However, it tends to approach an asymptotic value as aspect ratio increases.

References

1. Mroz, Z., Optical design of structures of composite materials, Int. J. Solids & Struct., 6, 559-870, 1970.
2. Bryzalin, G.J., On some optimal design criteria of inhomogenous anisotropic bodies, J. Appl. Math. & Mech., 36, 716-723, 1972.
3. Love, P.G., and Melchers, R.E., On the theory of optimal constant thickness fiber reinforced plates, Int. J. Mech. Sc., 19, 311-324, 1972.
4. Lai, Y.S., and Achenbach, J.D., Optimal design of layered structures under dynamic loading, Int. J. Comp. & Struct., 3, 559-572, 1973.
5. Khot, N.S. et.al., Optimization of fiber reinforced composite structures, Int. J. Solids & Struct., 9, 1225-1236, 1973.
6. Schmit, Jr., L.A., and Farshi, B., Optimum laminate design for strength and stiffness, Int. J. Numer. Meth. Engg., 7, 519-534, 1973.
7. Schmit, Jr., L.A., and Farshi, B., Optimum design of laminated fiber composite plates, Int. J. Numer. Meth. Engg., 4, 623-640, 1977.
8. Hayashi, T., Optimization for elastic buckling strength of fiber reinforced composite structures-columns, plates and cylinders, Proc. Soc. of Mat. Sc., Kyoto, Japan, 399-405, 1974.
9. Chao, C.C. et.al., Optimization of buckling and yield strengths of laminated composites, J. AIAA, 13, 1131-1132, 1975.
10. Hirano, Y., Optimum design of laminated plates under axial compression, J. AIAA, 17, 1017-1019, 1979.
11. Sharma, S. et.al., Some comments on coupling of angle-ply laminates, J. Struct. Mech., 7, 473-482, 1979.
12. Iyengar, N.G.R., and Gupta, S.K., Programming Methods in Structural Design, Edward Arnolds, U.K., 1981.
13. Jones, R.M., Mechanics of Composite Materials, Scripta Book Co., 1975.
14. Joshi, S.P., Studies on Optimization of Laminated Composite Plates, M.Tech. Thesis, Dept. of Aero. Engg., I.I.T. Kanpur, August 1981.