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Abstract

A method for visual verification of aerodynamic geometries using computer-generated, color-shaded images is described. The mathematical models representing aircraft geometries are created for use in theoretical aerodynamic analyses and in computer-aided manufacturing. The aerodynamic shapes are defined using parametric bi-cubic splined patches. This mathematical representation is then used as input to an algorithm that generates a color-shaded image of the geometry. A discussion of the techniques used in the mathematical representation of the geometry and in the rendering of the color-shaded display is presented. The results include examples of color-shaded displays, which are contrasted with wire-frame-type displays. The examples also show the use of mapped surface pressures in terms of color-shaded images of V/STOL fighter/attack aircraft and advanced turboprop aircraft.

I. Introduction

Research activities in the Aircraft Aerodynamics Branch at NASA Ames Research Center include the analytical and experimental study of the aerodynamics of various aircraft concepts ranging from V/STOL fighters to advanced turboprop configurations. A critical activity in support of these research efforts is the mathematical modeling of the aircraft geometry for input to various aerodynamic-prediction computer codes. An example

of the mathematical model and the wind-tunnel model of the same aircraft is shown in Fig. 1. This same mathematical definition also is intended to be used for numerically controlled machining of wind-tunnel model components. An important part of both functions is the verification of the geometric definition to assure commonality between the mathematical models of the aircraft shape. The geometric definition used for the mathematical model of aircraft surfaces is developed, using parametric bi-cubic (PC) patches.¹⁻³ The patches are created using third-order polynomial mathematical equations. The mathematics of parametric bi-cubics allow for complete definition, including curvature, of the entire surface. This type of surface representation is preferred over that of other methods, because the technique has been well developed and documented.

The mathematical model of the aircraft geometry requires verification. Previous approaches to this problem were to either review listings of numerical input data or run the theoretical codes to find obvious errors. The disadvantages of using these methods are the personnel and computer times that are required. A more efficient method is to visually review the mathematical model, using computer graphics. This was initially accomplished by generating a wire-mesh display of the geometry; however, the detection of subtle flaws in the geometric definition by using wire-mesh displays is limited. An improved approach is to generate a color-shaded surface image of the mathematical model.

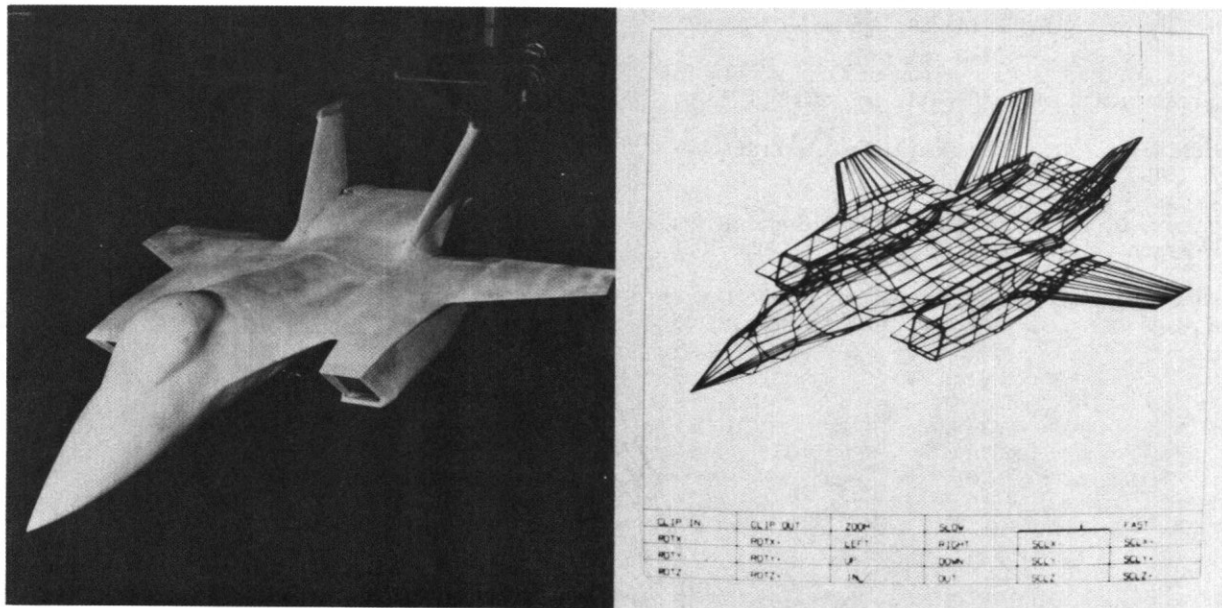


Fig. 1 A wind-tunnel model and mathematical model of the same aircraft.

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Research in computer-generated, shaded displays of three-dimensional geometries has been increasing over the past 10 years. Computer-generated shaded displays have been applied to problems in different areas; for example, in molecular analysis and displays of stress-analysis results. The method, in general, has been implemented also in the area of flight simulation displays by the aerospace industry. Aerodynamic analysis in the aerospace industry has yet to fully develop the use of color-shaded displays.

The application of color-shaded displays in the Aircraft Aerodynamics Branch at Ames Research Center progressed in two stages. The initial work covered the use of shaded displays for the verification of the mathematical model definition. The second involved the display of aerodynamic data on these geometries.

This paper discusses the techniques and facilities used for aircraft geometry definition and image generation. Geometry verification will be described, with particular emphasis on enhanced computer-generated displays of color-shaded images. Example applications of this technique will be shown, as well as the advantages gained over prior verification methods. A discussion then follows of the presentation of aerodynamic pressure data superimposed on a three-dimensional geometry as an extension of the shaded-surface technique.

II. Geometry Definition - Mathematical Modeling

The Aircraft Aerodynamics Branch has been involved in computer graphics and the mathematical modeling of three-dimensional aerodynamic geometries for about 10 years. The initial focus of the development was the generation of mathematical models of three-dimensional aerodynamic geometries. Early surface definition centered on the use of a polygonal representation. Later, techniques were chosen to accurately define surface curvature and irregularities such as discontinuity points.³ At the same time research in mathematical-surface definition was being investigated in depth. Mathematical techniques being used at that time were the Bezier, Coons, and parametric bi-cubic patches. The Bezier patch⁵ was just starting to be used by the Renault-Peugeot Company, Coons⁶ had been doing research on his patch, and Peters¹ was working with the parametric bi-cubic patch for aircraft design at the McDonnell Douglas Automation Company.

By 1972 a substantial amount of research and development work had been done by the McDonnell-Douglas Corp. on the application of parametric bi-cubic patches to three-dimensional aircraft geometries. Research by Peters¹ showed that aerodynamic geometries could be defined with the parametric bi-cubic patches. His work and the work of Timmer² readily illustrated the advantage of completely defining three-dimensional aerodynamic geometries with a mathematical model.

A brief discussion of the mathematics of parametric bi-cubics as they are implemented by the Aircraft Aerodynamics Branch at Ames Research Center is presented. A much more detailed discussion of the technique and the mathematics of the technique is given in Refs. 1-3 and 7, from which the following was summarized.

Parametric Bi-Cubic Theory

Parametric cubic geometry entities in general are described as a mathematical mapping of a point P in the x,y,z coordinate system into a u coordinate system. This definition describes the space curve (a curve in x,y,z space) shown in Fig. 2. Any point on the curve is represented by the parametric cubic equation:

$$P(u) = Au^3 + Bu^2 + Cu + D \quad (1)$$

where

$$P(u) = \begin{cases} P_x(u) = x(u) \\ P_y(u) = y(u) \\ P_z(u) = z(u) \end{cases}$$

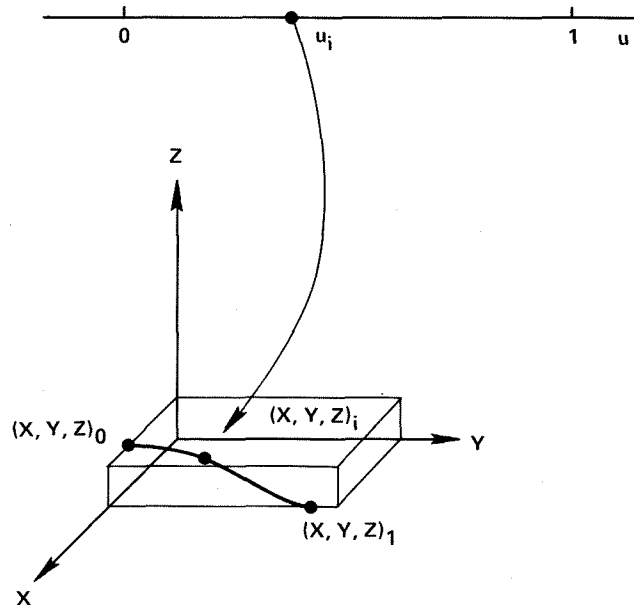


Fig. 2 The mapping of a curve from three-space to one-space.

Four equations are needed to solve for the four coefficients A, B, C, and D. The parametric derivative of Eq. (1) is expressed as the following:

$$dP/du = P'(u) = 3Au^2 + 2Bu + C \quad (2)$$

Setting $u = 0$ and $u = 1$ in Eqs. (1) and (2), the following is obtained:

$$\begin{aligned} P(0) &= D \\ P(1) &= A + B + C + D \\ P'(0) &= C \\ P'(1) &= 3A + 2B + C \end{aligned} \quad (3a)$$

in matrix form:

$$\begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \quad (3b)$$

Equation (1) is the algebraic form; however, the equation is frequently given in geometric form, because the physical significance of the coefficients to the actual geometry is easier to relate to. The geometric form is attained by solving

Eq. (3b) for the A, B, C, and D coefficients and inserting in Eq. (1):

$$P(u) = P(0)F_1(u) + P(1)F_2(u) + P'(0)F_3(u) + P'(1)F_4(u) \quad (4)$$

where

$$\begin{aligned} F_1(u) &= 2u^3 + 3u^2 + 1 \\ F_2(u) &= -2u^3 + 3u^2 \\ F_3(u) &= u^3 - 2u^2 + u \\ F_4(u) &= u^3 - u^2 \end{aligned}$$

To obtain the generalized expression for a space curve, Eq. (4) can be expanded for the x, y, z coordinates in u-space. Equation (5a) gives the general expression and Eq. (5b) is the expanded expression:

$$P(u) = [u^3 \ u^2 \ u \ 1][M][B] \quad (5a)$$

or

$$P(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix}^T = [u^3 \ u^2 \ u \ 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ 3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) & y(0) & z(0) \\ x(1) & y(1) & z(1) \\ x'(0) & y'(0) & z'(0) \\ x'(1) & y'(1) & z'(1) \end{bmatrix} \quad (5b)$$

The mathematics of the space curve is the basis for the parametric bi-cubic surface patch. As illustrated in Fig. 3, the surface can be represented in three-space as functions of u and w. A surface mapped into a u, w coordinate system is obtained by generating a parametric cubic in two directions. The general equation for the surface bi-cubic patch is represented as

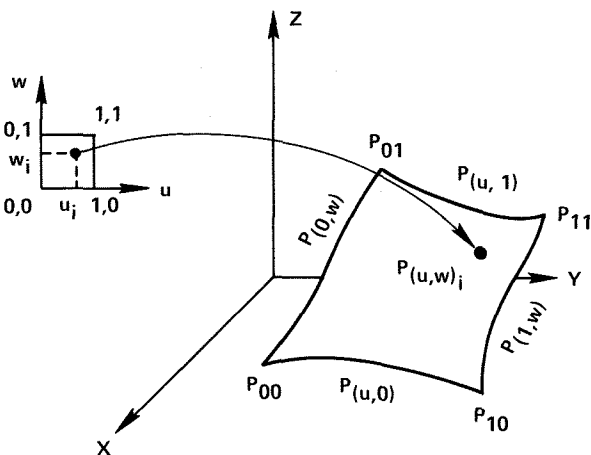


Fig. 3 The mapping of a surface from three-space to two-space.

$$P(u,w) = [u^3 \ u^2 \ u \ 1][M][B][M]^T \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix} \quad (6a)$$

remembering that P is expressed as

$$P(u,w) = \begin{cases} P_x(u,w) = x(u,w) \\ P_y(u,w) = y(u,w) \\ P_z(u,w) = z(u,w) \end{cases} \quad (6b)$$

where [M] was defined in Eq. (5) and

$$[B] = \begin{bmatrix} P_{00} & P_{01} & P_{00} \ w & P_{01} \ w \\ P_{10} & P_{11} & P_{10} \ w & P_{11} \ w \\ \hline P_{00} \ u & P_{01} \ u & P_{00} \ uw & P_{01} \ uw \\ P_{10} \ u & P_{11} \ u & P_{10} \ uw & P_{11} \ uw \end{bmatrix} \quad (6c)$$

which is called the boundary matrix. In the boundary matrix, $P_{nn}(u)$ and $P_{nn}(w)$ are the first derivatives with respect to u and w. The $P_{nn}(uw)$ is the second derivative with respect to u and w. When the expression is expanded for x, y, and z, a total of 48 boundary coefficients for a patch are obtained. These boundary coefficients are the geometric coefficients of the patch surface. The [B] matrix can be partitioned into four quadrants such that the upper left four variables are the corner points, the upper right quadrant is the derivative with respect to w (slope information in the w direction), the lower left quadrant is the derivative with respect to u (slope information in u direction), and the lower right quadrant is the derivative with respect to u and w (the internal patch or twist information).

The capability to define a surface patch allows the user to accurately define a three-dimensional geometry. The generation of the object requires the connecting of many patches to create the proper definition of the geometry. The blending of two patches requires both position and slope continuity along their common border. These constraints are attained through the manipulation of the boundary matrix elements. The technique for blending two patches is covered in detail in Peter's paper.¹

Surface and Geometry Development

The geometry definition method used by the Aircraft Aerodynamics Branch at Ames to mathematically model aerodynamic shapes is that of parametric bi-cubic patches and tension splines.⁸ The approach outlined in the following has been extracted from work done by R. Carmichael in the Aircraft Aerodynamics Branch.

Cross sections of the geometry are defined by points and then splined. A spline is a sequence of curves arranged so that they are continuous and have continuous derivatives at the boundary points. The generation of the cubic spline is fitted through N number of points. In addition to the coordinates of the points there are two additional degrees of freedom that may be resolved by specifying tangent vectors at each endpoint of the spline. Note that the user is not required to use the same number of points for the definition of each cross section of the geometry.

The behavior of spline functions in general is usually erratic in regions of high curvature.

To allow greater flexibility and more control over the curvature of the spline, a special class of cubic splines, called tension splines,⁸ is used. As illustrated in Fig. 4, the number of points used to define the geometry is arbitrary and the tangents at the endpoints of the spline can be specified. In addition, the tension at each point can also be specified.

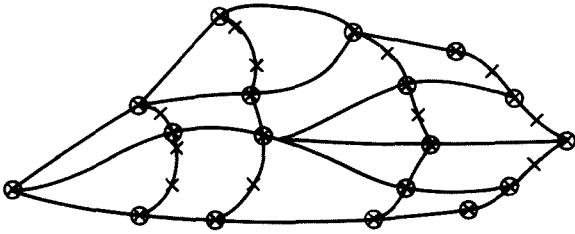


Fig. 4 A parametric bi-cubic surface using a tension spline fit.

Once the cross sections of the aerodynamic shape have been defined to the users satisfaction, a cubic spline is generated across the cross sections of the geometry. In conjunction with the spline fit, the derivative function is constrained to be continuous, which then enables the cross derivative to be computed at the points. The cross derivative provides twist information about the splines that will be used to develop the patches. Note that a single point from a cross section can be used to generate more than one cubic spline across the cross sections (e.g., the point that defines the nose tip).

The splines that have been fitted in both directions to define the geometry are then the patch boundaries. The cubic splines in both directions and the cross-derivative information at the points of intersection of the splines completely define the edges and the interior of the patch. A completed definition of a V/STOL fighter/attack aircraft, using this method, is shown in Fig. 5.

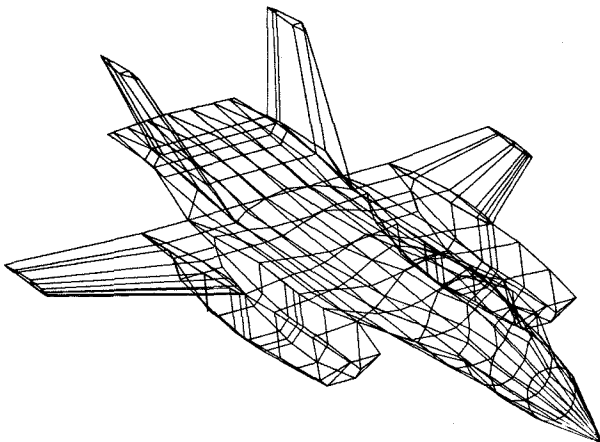


Fig. 5 A complete parametric bi-cubic definition of a V/STOL fighter/attack aircraft.

III. Geometry Verification* - Shaded-Surface Display

The mathematical model of the various geometries precisely describes a surface. The next process is to determine the "correctness" of the surface. An intuitive solution is a graphical display of the geometry. Advancements in computer graphics in the late 1960s and early 1970s led the Aircraft Aerodynamics Branch to investigate this approach as a solution for visual verification.

The initial efforts focused on vector displays created on a refresh cathode ray tube. Pictures of the three-dimensional aircraft geometries were generated as two-dimensional wire-mesh images. In the initial stages of the research, this provided a very useful tool for visualizing the aerodynamic geometries and detecting major flaws in the geometric definition. The method proved to be limited in detecting subtle flaws. This was resolved by using shaded-surface displays of the mathematical model.

By combining various techniques developed in shaded-surface theory, a new capability was developed to show three-dimensional geometric surfaces. The following is a brief discussion of the techniques that were implemented to make this visualization process possible; it is based on the work of Clark and others (Refs. 7, 9, 10).

Patch Subdivision

Most shaded-surface techniques use polygons to define the geometry, but the aircraft geometries modeled in the Aircraft Aerodynamics Branch at Ames are defined using smooth parametric bi-cubic patches. To accommodate the differences, the patch model must be redefined as a collection of polygons. This could be accomplished by initially defining the geometries with polygons. Using this approach would create a substantial loss in geometry definition. To retain the accuracy of the mathematical model, a patch subdivision method is used. Each patch is approximated by an arbitrary number of polygons based on a splitting criteria.

The subdivision method of dividing patches into polygons is controlled by a splitting criterion based on a "flatness" tolerance. The tolerance is defined as a nondimensional number ranging between 0.0 and 1.0 whose magnitude is compared with the magnitude of the derivative of the patch with respect to u and w (slope magnitude of patch in u and w directions). Tolerance values near zero mean a finer subdivision of the patch into polygons. The tolerance input capability allows the user to generate a coarse polygonal approximation of the patch surface for a quick view or a fine approximation for a more informative and realistic view.

Figure 6 is a patch surface in three-dimensional space that has been subdivided to an arbitrary "flatness" tolerance to yield a polygonal approximation. The splitting criteria,

*The word verification, in this context, refers to the process of locating unintended errors in geometry definition as opposed to errors in the accuracy of the geometry within a tolerance limit.

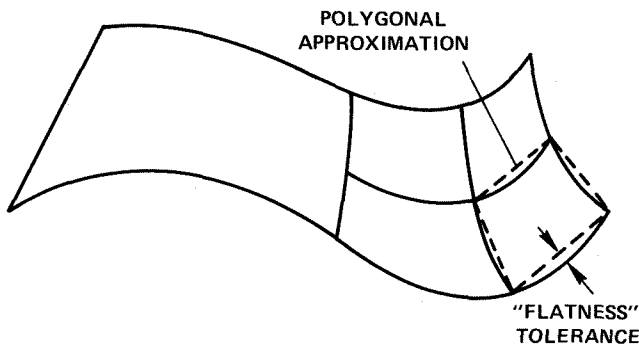


Fig. 6 A patch surface in three-space, subdivided and approximated by polygons.

directly extracted and implemented from Clark's paper,⁹ used in the subdivision process are the following:

- 1) Are both the $u = 1$ and the $u = 0$ boundary curves of the patch "flat"? If not, subdivide in the w direction, and recursively test the new subpatches generated. That is, for $u = 1$ curve, are all of the components of both $dfw(1,0)$ and $dfw(1,1)$ less in magnitude than the tolerance. Similarly, for the $u = 0$ curve.
- 2) Are both the $w = 1$ and the $w = 0$ boundary curves of the patch "flat"? If not, subdivide in the u direction, and recursively test the new subpatches generated.
- 3) Is the middle of the patch "flat"? If not, subdivide in the u direction (arbitrary), and recursively test the new subpatches generated. This test is done on the magnitudes of the components $dfuw(u,w)$, for $u,w = 0,1$.

The process of subdividing patches is continued until the entire parametric bi-cubic mathematical model has been approximated by polygons.

Figure 7 shows an enlarged and exaggerated view of a portion of the patch surface that has

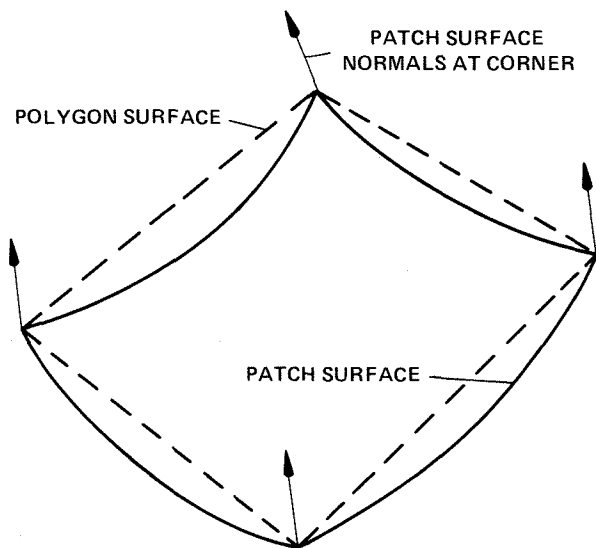


Fig. 7 A portion of a patch surface approximated by a polygon with the patch corner normals associated with the polygon vertices.

been subdivided and approximated by a polygon. Typically a polygon is considered flat, and the normal to the plane is obtained by crossing two vectors that lie in the plane. In the subdivision process, the normals for the polygon vertices are obtained by calculating the normals at the associated patch corners. The normals at the polygon vertices usually vary only slightly. The importance of the normals will become apparent in the subsequent discussion of the intensity calculations.

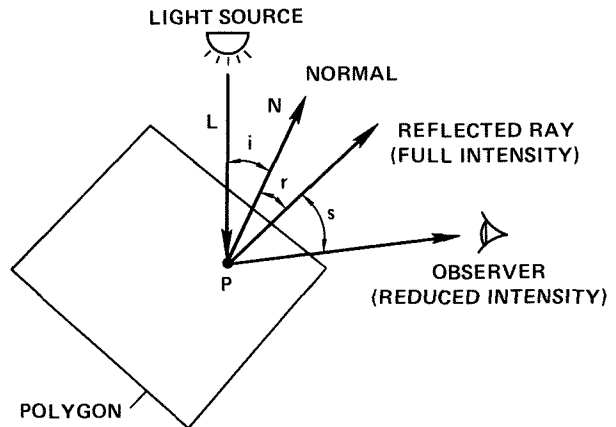
Shading Model and Patch Intensity

Once the polygons and their associated normals at the vertices have been generated from the parametric bi-cubic mathematical model, the shaded-surface picture can be rendered (generated). The rendering of a shaded image requires surface-intensity calculations. Figure 8 illustrates the basis for the empirical shading model, which has the following form:¹⁰

$$E_p = [R_p \cos(i) + W(i) (\cos(s))^n] I_p \quad (7)$$

where

- E_p = the intensity arriving at the eye
- R_p = the reflectance coefficient at P
- i = the angle of incidence
- $W(i)$ = the reflection coefficient of the material of the object (e.g., gold, silver, copper)
- s = the angle between the reflected ray and the observer
- n = the shininess coefficient 0.0 (dull finish) through 1.0 (highly metallic)
- I_p = diffuse reflection



SHADING MODEL EQUATION:

$$E_p = [R_p \cos(i) + W(i) (\cos(s))^n] I_p$$

Fig. 8 Shading model used for intensity calculations.

The shading model allows the user to create a color-shaded surface display with a great deal of realism because of the large variation in intensities. The possibilities are endless for creating various displays.

The intensities are calculated for each polygon and can be displayed directly based on Eq. (7).

The resulting image, if no further enhancements are performed, has a constant intensity for each polygon. The surface display is the faceted image, as shown in Fig. 9. The discrete separation of intensities from polygon to polygon is usually undesirable. A smooth shaded surface using an averaging technique can reduce the faceting effect.



Fig. 9 Surface display of an aircraft geometry with constant intensities per polygon creating a faceted image.

There are a variety of techniques available that generate a smooth shaded surface, but the approach chosen here is based on the work of Bui Tuong Phong.¹⁰ This approach does a linear averaging of the vertex normals to obtain the normals along the polygon edges.

For example, the surface normal at point E in Fig. 10 is a linear average of the normals at points A and D. Similarly, the normal at point F uses the normals at points B and C. The normal at point G or at any other point along a raster scan line is also a linear average of the normals at points E and F.

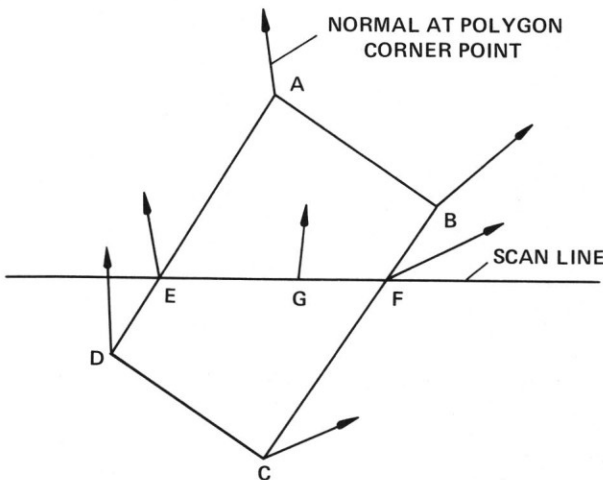


Fig. 10 A single polygon showing a linear averaging of the surface normals along a scan line.

This approach to obtaining the surface normals of a polygon will generate a smooth blend of intensities across the surface of the geometry. The blending operation continues across a scan line from polygon to polygon. This is illustrated in Fig. 11, where the normal on the scan line at point K is obtained by a linear averaging of the same point F from the previous polygon and point J of the next polygon. This results in smooth shading along the raster scan line and fairly accurate shading in the vertical direction. This shading method uses surface normals for the polygon that were obtained from the original surface patch. The result for a complete model is shown in Fig. 12.

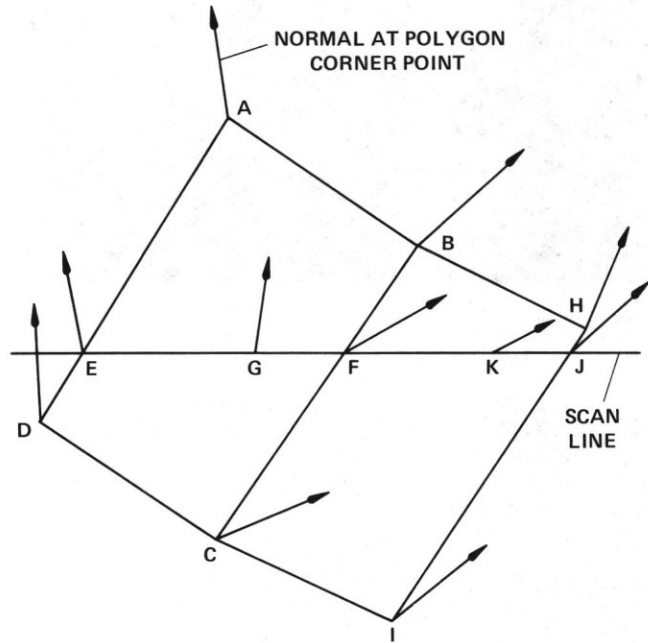


Fig. 11 Two polygons showing the linear averaging of surface normals along a scan line.



Fig. 12 A smooth shaded display of a complete aircraft geometry.

The practicality of these techniques was apparent in their first use. The aircraft geometry in Figure 13 was generated using the methods described. The flaws in the definition became much more obvious than in the wire-mesh-type of display. The shaded display reveals such flaws as the wrinkle in the fuselage that is very difficult to detect in a wire mesh.

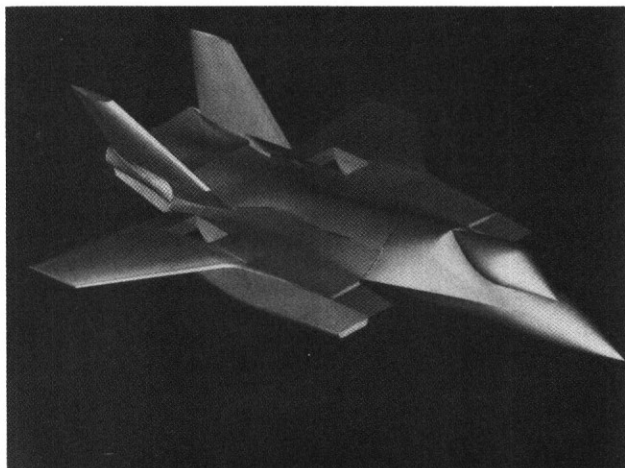


Fig. 13 A smooth shaded display of an aircraft geometry depicting major and subtle flaws.

The methods discussed up to this point have been used in the initial development and verification of aircraft geometries. The next section will show that with simple extensions, these methods can and are being used in other areas of aerodynamic analyses.

IV. Extensions: Data Display and Model Fabrication

The purpose of the research on mathematical modeling and verification of the aircraft geometries is to facilitate theoretical analyses and experimental testing. Advances in hardware, software, and their integration have made this feasible and practical.

Data Display

In the theoretical analyses of aerodynamic shapes, the parametric bi-cubic models can be used to develop the required inputs for the computer codes. The inputs are often the corner points of polygons of specified fineness.

One of the theoretical computer codes requires panel corner points of the geometry as input. The paneling of an aerodynamic shape is similar to the generation of a finite-element mesh used in structural analysis. The initial surface definition developed with parametric bi-cubics provides an accurate shape, which in turn allows accurate definition of the geometric panels generated from the patches. A computer program was developed to generate the panels from the patch definition for any desired panel density. These panels can be displayed, as shown in Fig. 14, for visual verification. The same view of the geometry with hidden lines¹¹ removed is illustrated in Fig. 15.

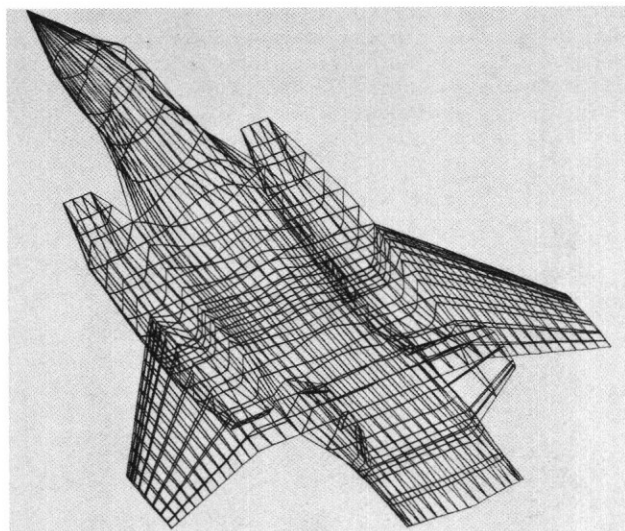


Fig. 14 An aircraft geometry paneled using the parametric bi-cubic definition of the model.

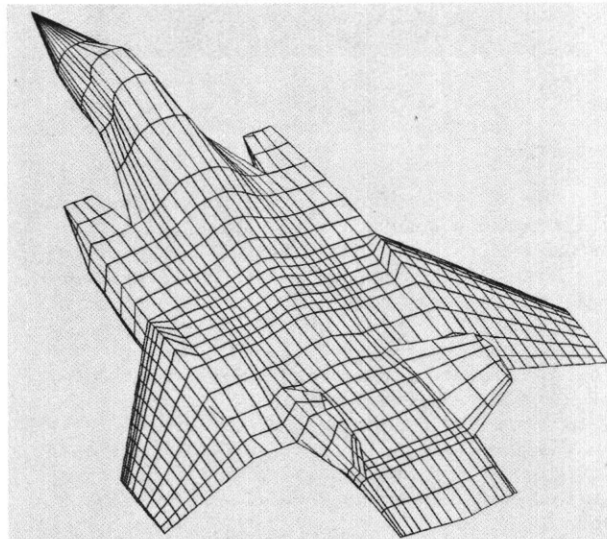
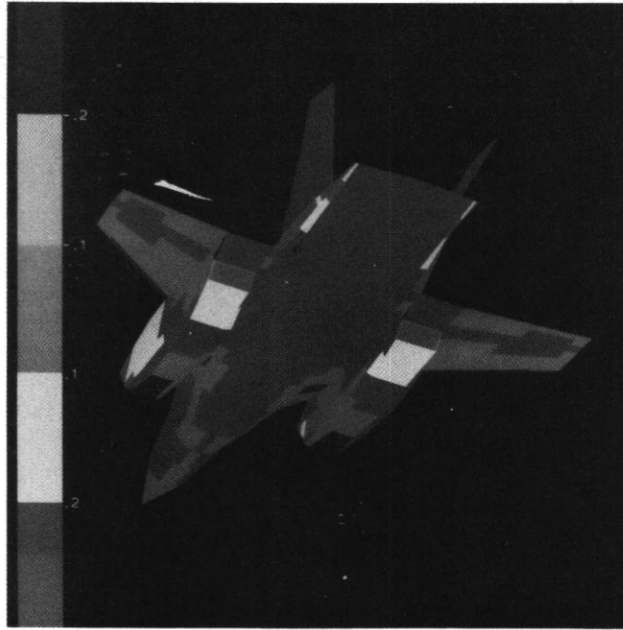
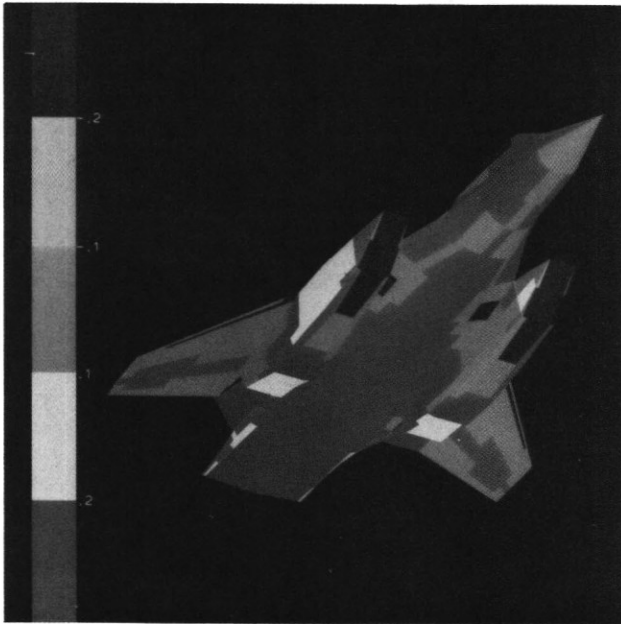


Fig. 15 The paneled aircraft geometry with the hidden lines removed.

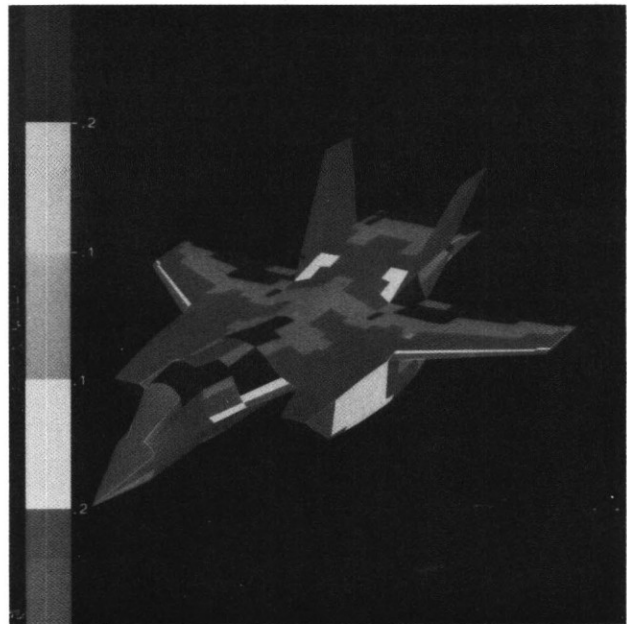
The panels generated from the patch definitions, along with various boundary conditions, are submitted to the theoretical computer codes for analysis. The aerodynamic results, such as the pressures, are generated in the program. The vast amount of results for an aerodynamic shape is a problem to review in list form. For pressure results, the technique described to generate a color-shaded surface display of the geometry can be extended to display the pressure results on the same geometry. This is done by a color mapping of the pressures in each panel directly on the aircraft (Figs. 16a-16c). The color is mapped from the pressures to the corresponding color bar (CPs from -0.2 to 0.2 are mapped to the range of colors in the spectrum between blue and red). This approach to data display reduces the time required for analysis and evaluation of the data.



a



b



c

Fig. 16 A color mapping of pressures, derived from aerodynamic analysis, on the aircraft geometry.

Wind-Tunnel Component Fabrication

The actual fabrication of components for testing in a wind tunnel is being studied at Ames Research Center. An important reason for pursuing this research is that a better correlation of the theoretical and the experimental results can be obtained if the same mathematical definition is used for inputs in both sides of the research.

The Aircraft Aerodynamics Branch and the Model and Instrument Machining Branch at Ames are generating test pieces of wind-tunnel models from the parametric bi-cubic mathematical models. Efforts are progressing to quickly develop the mathematical model and then to numerically control a machine to make the models. A test case involved a

feasibility study and resulted in an aerodynamic shape being machined. The results of the test case design, visual review, and machined component are shown in Fig. 17.

V. Conclusion

Five main conclusions can be drawn from this work. First, the use of parametric bi-cubic patches for the mathematical modeling of three-dimensional geometries provides an accurate definition of the geometries. Second, the color-shaded surface displays of the mathematical model provide a more complete verification of the geometry than the wire-frame displays. Third, the color-shading techniques can be extended to perform color mapping of aerodynamic data on the mathematical model of

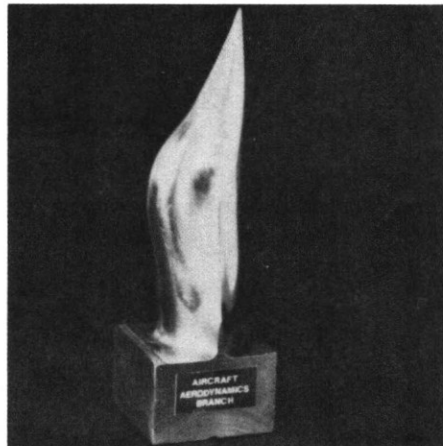
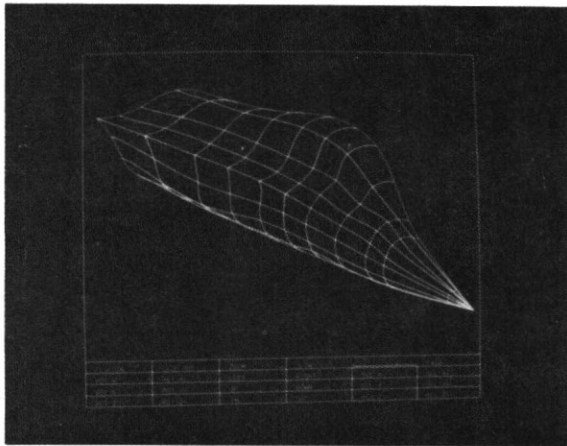
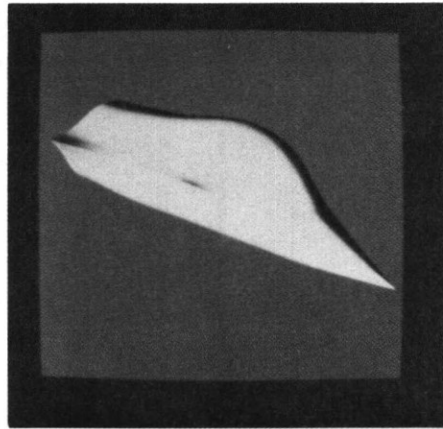
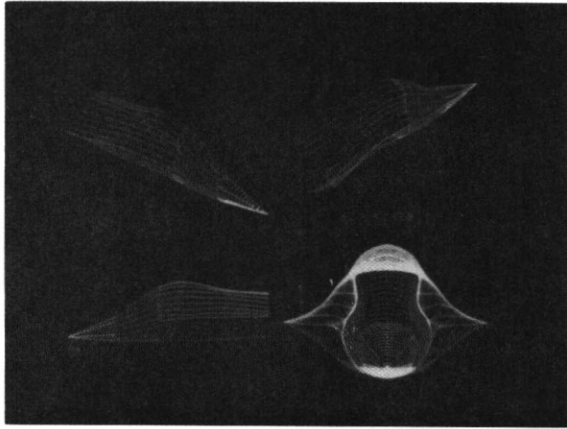


Fig. 17 An aircraft forebody used for the wind-tunnel component fabrication test case.

the aircraft geometry. Fourth, the parametric bi-cubic patches can be used directly for machining model components. Fifth, a better correlation of aerodynamic experimental and theoretical research results should be expected when a common mathematical model is used for the theoretical analysis as well as for constructing the wind-tunnel models.

Acknowledgments

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