ICAS-80-22.4 AIRCRAFT PERFORMANCE OPTIMIZATION BY FORCED SINGULAR PERTURBATION

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Abstract

Forced singular perturbation technique (FSPT), based on artificial insertion of a "small" parameter into the equations of motion, has been used to generate approximate feedback solutions in several aircraft performance optimization problems. This approach has some inherent limitations, not being exposed in previous works. The paper presents and discusses such limitations revealed by a recent investigation. In spite of the restrictions FSPT provides an attractive methodology for a large class of properly formulated problems. This potential is demonstrated by two examples of air combat performance optimization.

I. Introduction

Optimization of aircraft trajectories has been a challenging topic, being strongly motivated by the development requirements of new generations of high performance airplanes. The only rigorous method to deal with such problems (even if they are based on a simplified point-mass mathematical model of a lifting vehicle) is by solving a nonlinear two-point boundary value problem (NTPBVP) of high dimension involving constraints on the state and control variables. Numerical solutions can be obtained by several iterative algorithms all requiring an excessive amount of computation.

In the preliminary design phase of a new airplane a very large number of optimal performance problems has to be solved in order to allow an effective trade-off analysis. At the other end, the airborne application of optimal control laws has to be performed in "real time". For both purposes the exact iterative solution may not be practical and approximation methods, based on mathematical models of reduced order, are preferred. The idea to obtain a *nearly optimal* control law in a *feedback* form seems to be the attractive goal.

An approach which provides such a solution is the application of singular perturbation techniques (SPT). This method of approximation[1-7] has been conceived to be used in problems involving a small parameter multiplying the derivations of some state variables. If this parameter becomes zero the order of the dynamic system is reduced. The solution of the reduced order model is called the *zero-order outer solution*, or by analogy to fluid mechanics, the "free stream". This reduced order solution is unable to satisfy the initial (and/or terminal) conditions imposed in the original problem on those variables for which the dynamics is neglected in the "free stream" equations. Such a discrepancy is corrected by initial (and/or terminal) "boundary layers" or *inner solutions* allowing rapid changes of these variables using a stretched time scale. The *boundary layer solutions* has to satisfy the violated end conditions and match the *outer solution*. The uniformly valid additive composite of the "free stream" and the matched "boundary layers" presents the zero-order SPT approximation (obtained by taking $\varepsilon = 0$) of the original problem. If all variables of the problem can be expanded in a form of uniformly valid asymptotic power series of ε, the accuracy of the zero-order solution can be improved by taking into account higher order terms.

The dynamic behavior of a system in which state variables can be separated, due to their different time scales as "slow" and "fast" ones is very similar to the response of a singularly perturbed model. The "fast" variables reach their equilibrium in a very short time, and hence can be considered as controls of the "slow" dynamics in a reduced order system.

In atmospheric flight mechanics problems, the existence of a small parameter of physical significance is not always obvious. However, in many problems a time scale separation of variables is well known, either from analysis or experience. The best known example is the energy-state approximation of an airplane. It offers a reduced order mathematical model based on the observation that in many aircraft maneuvers the specific energy varies slowly compared to the other state variables as speed or altitude. Such a model was used first for optimal climb analysis[8] and applied later in other performance optimization problems[9-11]. This reduced order approximation has provided an insight leading to an improved understanding of high performance aircraft trajectory optimization. However, energy-state solutions have had only a limited value for direct airborne applications. As an inherent property of the reduced order model, discontinuities of the "fast" state variables may be required. Since such a model cannot satisfy a part of the initial and terminal conditions of the complete real problem, it seemed to be appealing to modify the solution by including "boundary layer" corrections. Therefore it was proposed[12-14] to insert artificially a "small" parameter ε as a multiplier of the differential equations of the "fast" variables, and to use the methodology of singular perturbations. This artificial, or "forced singular perturbation technique" (FSPT), was applied in the past with some success in several atmospheric flight mechanics optimization problems[15-22].

The successful results may almost have created the impression that application of FSPT in nonlinear optimal control problems involved in flight mechanics is a straight forward engineering approach, This impression is, unfortunately, not true.

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First, it has to be acknowledged that a general theory which is applicable for singularly perturbed NTPBVP is by no means complete. A recent survey[6] showed that the characteristics of singularly perturbed optimal control problems has been extensively investigated, but the efforts were mostly oriented towards linear quadratic problems. Only a few particular non-linear cases were investigated[23-25]. These works show that the existence of uniformly valid SPT solutions is based on a set of mathematical assumptions which can only be partially verified a priori.

Applicability of these results to flight mechanics problems has not yet been explored.

A second area of difficulties relates to the appropriate transformation of an original atmospheric flight mechanics problem to a forced (artificial) singularly perturbed one. Previous studies have provided guidelines only for very few cases (as for problems with singular arcs[18]).

The objective of this paper, summarizing the main results of a recently completed research effort[26], is to point out some inherent limitations of the method of singular perturbations in applications to constrained non-linear optimization and to indicate those problems in atmospheric flight mechanics which are well suited for the FSPT analysis.

In the next section the SPT analysis of non-linear autonomous optimal control problems is presented briefly. Special emphasis is given to the "multiple time scale" version, which has the potential to provide *feedback* solutions.

In section III the inherent limitations of the SPT methodology are described and analysed. These limitations include "ill posed" mathematical models for which SPT fails and cases where the technique is unable to provide *"true" feedback* solutions. Section IV addresses the proper formulation of aircraft optimization problems by a "forced" singularly perturbed mathematical model. In sections V and VI two examples of air combat performance optimization, solved by FSPT and yielding feedback controls, are presented.

II. Optimal Control of Autonomous Singularly Perturbed Nonlinear Dynamic Systems

Since aircraft performance optimization involves a set of autonomous nonlinear differential equations (see Appendix) we concentrate on this class of problems. An "ℓ" dimensional dynamic system with a small parameter $\varepsilon>0$, described by

$$\dot{z}_0 = F(z,u,\varepsilon) \qquad z(t_0) = z_0 \tag{1}$$

(u being an "m" dimensional control vector), has a singularly perturbed structure if the state vector z can be decomposed to subvectors x and y (of dimensions "ℓ-k" and " k" respectively) such that Eq. (1) can be separated into

$$\dot{x} = f(x,y,u,\varepsilon) \qquad x(t_0) = x_0 \tag{2}$$

$$\varepsilon\dot{y} = g(x,y,u,\varepsilon) \qquad y(t_0) = y_0 \tag{3}$$

Due to the smallness of ε, the comparison of the rate of change of the variables clearly indicates that x is the "slow" subvector of the problem and y is the "fast" one.

The optimization problem consists of finding the control vector $u^*(t,\varepsilon)$ which transfers the dynamic system (1), subject to a set of nondifferential constraints defined by

$$C(x,y,u,\varepsilon) = 0 \tag{4}$$

to a "q" dimensional ($q \leq \ell$) terminal manifold (at some time $t_f > t_0$) specified by

$$\psi[x,y,\varepsilon]_{t = t_f} = 0 \tag{5}$$

while minimizing a scalar cost function

$$J = \int_{t_0}^{t_f} L(x,y,u,\varepsilon)dt \tag{6}$$

Assuming that the functions f, g, C, ψ, L are continuous and differentiable in all of their arguments, solution of this optimal control problem can be stated. The first step is to define a scalar function (the variational Hamiltonian)

$$H(x,y,\lambda_x,\lambda_y,u,\varepsilon) \triangleq -L+\lambda_x^T f+\lambda_y^T g+\nu^T C \tag{7}$$

where λ_x, λ_y are the vectors of the costate (adjoint) variables and ν is a vector of multipliers. According the the Maximum Principle[27] the optimal control vector has to maximize H

$$u^*(t,\varepsilon) = \arg\max H(x,y,\lambda_x,\lambda_y,u,\varepsilon) \tag{8}$$

while the costate subvectors λ_x,λ_y has to satisfy the adjoint differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} - \lambda_x^T \frac{\partial f}{\partial x} - \lambda_y^T \frac{\partial g}{\partial x} - \nu^T \frac{\partial C}{\partial x} \tag{9}$$

$$\varepsilon\dot{\lambda}_y = -\frac{\partial H}{\partial y} = \frac{\partial L}{\partial y} - \lambda_x^T \frac{\partial f}{\partial y} - \lambda_y^T \frac{\partial g}{\partial y} - \nu^T \frac{\partial C}{\partial y} \tag{10}$$

The end values of the costate components are determined at $t = t_f$ by the "transversality conditions" [27], expressing the orthogonality of the adjoint vector to the tangent plane of the terminal manifold $\psi = 0$. Moreover, since the problem is autonomous, the Hamiltonian computed along an optimal trajectory remains unchanged.

$$H^* \triangleq H(x^*, y^*, \lambda_x^*, \lambda_y^*, u^*, \varepsilon) = \text{const.} \tag{11}$$

If the final time t_f is not prescribed this constant is zero.

In order to simplify the solution of such "2ℓ" dimensional nonlinear two point boundary value problem (NTPBVP) which is an iterative process, obviously uncompatible with the requirements of an airborn application, let us assume that all variables can be expressed by uniformly valid (for all $t_0 \leq t \leq t_f$) power series of ε.

$$x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \tag{12}$$

$$\lambda_x(t,\varepsilon) = \lambda_{x_0}(t) + \varepsilon\lambda_{x_1}(t) + \varepsilon^2_{x_2}(t) + \ldots \tag{13}$$

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots \tag{14}$$

$$\lambda_y(t,\varepsilon) = \lambda_{y_0}(t) + \varepsilon\lambda_{y_1}(t) + \varepsilon^2\lambda_{y_2}(t) + \ldots \tag{15}$$

$$u(t,\varepsilon) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots \tag{16}$$

Such expansions, *if they exist*, satisfy the conditions of asymptotic behaviour for $\varepsilon \to 0$[2,5], the series need not be convergent in the regular sense.
Taking $\varepsilon=0$ as a zero order approximation, the dynamic system (2) (3) becomes of reduced dimensions $(\ell - k)$.

$$\dot{x}^\circ = f(x^\circ, y^\circ, u^\circ, 0) \qquad x^\circ(t_0) = x_0 \tag{17}$$

$$0 = g\,(x^\circ, y^\circ, u^\circ, 0) \tag{18}$$

The respective adjoint equations are

$$\dot{\lambda}^\circ_x = -\frac{\partial H^\circ}{\partial x} \tag{19}$$

$$0 = -\frac{\partial H^\circ}{\partial y} \tag{20}$$

Solution of this reduced order problem, or as it is named by analogy to fluid mechanics, "the free stream" is called the *zero order outer solution* (its variables are annoted by the superscript°). It is a constrained optimization problem (see Eq. (18)), where the "fast" variable y° plays the role of a pseudo-control, as indicated by Eq. (20). Equation (18) allows the elimination of y° from the equations. Substituting

$$y^\circ = \phi(x^\circ, u^\circ) \tag{21}$$

the optimal control function of outer solution is determined by

$$\tilde{u}^\circ(t) = \arg\max\,(H^\circ(x^\circ, \lambda^\circ_x, u^\circ, 0) \tag{22}$$

The zero order outer solution can be a good approximation of the original control problem if ε is very small. However, it cannot satisfy the initial condition $y(t_0) = y_0$ or any terminal constraint involving the fast variable. This discrepancy is corrected by "boundary layers" or *inner solutions* (variables annoted by superscript i) using a stretched time scale

$$\tau = \frac{t - t_0}{\varepsilon} \tag{23}$$

in the initial boundary layer and a similar one of reversed direction

$$\delta = \frac{t_f - t}{\varepsilon} \tag{24}$$

in the terminal layer.

The method of solution for both boundary layers is analogous. For the sake of brevity only the initial boundary layer solution will be presented here. Substituting Eq. (23) into Eqs. (2), (3), (9) and (10) setting $\varepsilon = 0$ yields

$$\frac{dx^i}{d\tau} = \varepsilon f(x^i, y^i, u^i) = 0; \quad x^i(0) = x_0 \tag{25}$$

$$\frac{dy^i}{d\tau} = g(x^i, y^i, u^i) \quad ; \quad y^i(0) = y_0 \tag{26}$$

$$\frac{d\lambda^i_x}{d\tau} = -\varepsilon\frac{\partial H^i}{\partial_x i} = 0 \tag{27}$$

$$\frac{d\lambda^i_y}{d\tau} = -\frac{\partial H^i}{\partial_y i} \tag{28}$$

Consequently we obtain from Eqs. (25) and (27)

$$x^i(\tau) = \text{const} = x_0 \tag{29}$$

$$\lambda^i_x(\tau) = \text{const} = \lambda_{x_0}(x_0) \tag{30}$$

The optimal control of this zero order boundary layer is determined by

$$\tilde{u}^i(t) = \arg\max H^i\,(x_0, \lambda_{x_0}, y^i, \lambda^i_y, u^i\ 0) \tag{31}$$

In order to have a uniformly valid solution for $t_0 \le t \le t_f$, it is necessary that integration of Eq. (26) with the control function obtained by Eq. (31), satisfy

$$\lim_{\tau \to \infty} y^i(\tau) = \lim_{t \to t_0} y^\circ \tag{32}$$

i.e.,

$$y_0 + \lim_{\tau \to \infty} \int_0^\tau g\,(x_0, y^i, \tilde{u}^i)d\theta = \phi(x_0, \overset{\circ}{u}(x_0)) \tag{33}$$

Such "matching" (required also for all other variables) is possible if the following two conditions of Tihonov[1] are satisfied:

(1) The initial boundary layer solution is asymptotically stable.

(2) The initial condition y_0 is in the "domain of attraction" of the asymptotically stable equilibrium point $y^\circ_0 = \phi(x_0, u^\circ(x_0))$ of the solution.

It was shown [12] that asymptotic stability of the equilibrium point, determined by the outer solution, is guaranteed if

$$H^\circ_{uu} < 0 \tag{34}$$

$$H^\circ_{yy}\ g^{\circ 2}_u - 2\,H^\circ_{yu}\ g^\circ_y\ g^\circ_u + H^\circ_{uu}\ g^{\circ 2}_y < 0 \tag{35}$$

Stability conditions for a terminal boundary layer are similar but has to be defined in the *reversed* time direction (see Eq. (24). Consequently inequality (35) is reversed for a terminal boundary layer. For a linear dynamic system it means that for stable terminal boundary layer solutions *positive* eigenvalues are required.

If the conditions of boundary layer stability and matching are satisfied it is possible to construct an additive *composite* control of zero order

$$\tilde{u}^c_0(t) = \tilde{u}^\circ(t) + \tilde{u}^i\left(\frac{t}{\varepsilon}\right) - (CP)_u \tag{36}$$

where $(CP)_u$ is the common part in both control functions to be cancelled by the matching

$$(CP)_u = \tilde{u}^\circ(t_0) \tag{37}$$

This zero order uniformly valid "open loop" approximation of the optimal control function enables to obtain a continuous solution of the trajectory satisfying all boundary conditions. If the accuracy of such suboptimal solution is not satisfactory, improvement can be expected by taking into account first and higher order terms of the power series expansions (12)-(16). This technique, called "matched asymptotical expansions",[2,16] calls to expand also the functions f,g,C and L to power series of ε.

Substitution of the expansions into the original set of equations (2)-(10) yields, as zero order terms, the "free stream" and zero order "boundary layer" equations already solved above. Equating the terms which multiply equal higher order powers of ε provides a set of *linear* differential equations. From these equations the first and higher order terms can be obtained recursively. Formally there are separate expansions for the outer and inner solutions and their respective terms have to be matched.

Solving several lower dimension optimization problems (as the "free stream" and the "boundary layers") is no doubt a much simpler computational task than the solution of the original one. However, if there is more than one "fast" state variable in a boundary layer ($k\geq 2$) which cannot be properly linearized, the basic difficulties involved in the iterative solution of a NTPBVP cannot be avoided. Closed form solutions can be obtained either if the boundary layer problems are linear-quadractic[25] or in the case of a single "fast" variable ($k=1$).

For a one dimensional boundary layer λ_y^i can be eliminated from the Hamiltonian

$$\tilde{H}^i = -L(x^\circ,y^i,\tilde{u}^i) + \lambda_{x_0} f(x_0,y^i,\tilde{u}^i) + \lambda_y^i\, g(x_0,y^i,\tilde{u}^i) = \text{const} = \tilde{H}^\circ \tag{38}$$

Combining Eqs. (31) and (38) the optimal control law for the boundary layer is obtained in a feedback form

$$\tilde{u}^i = \tilde{u}^i[y^i,x_0,\lambda_{x_0}(x_0)] = \tilde{u}^i(y^i,x_0) \tag{39}$$

If there exists a closed form solution for the reduced order (free stream) problem, that is both the control vector $\tilde{u}^\circ$ and the adjoint variable λ_x° can be expressed in a feedback from

$$\tilde{u}^\circ = \tilde{u}^\circ(x^\circ) \tag{40}$$

$$\lambda_x^\circ = \lambda_x^\circ(x^\circ) \tag{41}$$

the zero order composite control can be written as

$$\tilde{u}_0^c = \tilde{u}^\circ(x^\circ) + \tilde{u}^i(y^i,x_0) - (CP)_u \tag{42}$$

Such formulation invites to express a zero order optimal control in a true feedback form[15]

$$\tilde{u}_0^c = \tilde{u}^i(y,x) \tag{43}$$

by replacing in the boundary layer control the frozen initial condition of the slow variable by its current value. It is easy to see that if the matching conditions are satisfied, the composite control is equal to the free stream control $\tilde{u}^\circ$ everywhere, but near to the initial condition. On the other hand, this expression allows the variations of the slow variable in boundary layer, which indeed occur, since ε is not really zero. Integration of the equations of motion (2) and (3) using this uniformly valid feedback control law yields an approximation of the optimal trajectory which is comparable to the *first order* approximation obtained by the "open loop" method of matched asymptotic expansions.

A special case occurs if the original dynamic system (1) is decomposed, due to appropriate time scale separation, to multiple boundary layers with a single state variable in each. Such a multiple boundary layer system is described by

$$\dot{x} = f(x,y_1,y_2 \,..\, y_k,u,\varepsilon)\ ;\ x(t_0) = x_0 \tag{44}$$

$$\varepsilon^j \dot{y} = g_j(x,y_1 \,..\, y_k,\, u,\, \varepsilon)\ ;\ y_j(t_0) = y_{j0} \tag{45}$$

$$j = 1,2 \,..\, k$$

The zero order composite control for such a case is given by the control function of the last boundary layer ($j=k$) expressed as feedback of *all* the state variables

$$\tilde{u}_0^c = \tilde{u}_0^k\ (x,y_1,y_2 \,..\, y_k) \tag{46}$$

Such a control law is easy to implement in airborne applications since the state variables can be directly measured.

III. Inherent Limitations of SPT

We start by summarizing the set of assumptions made, either explicitly or implicitly, in the previous section describing the technique of singular perturbation in autonomous nonlinear optimal control problems:

(i) Both the original optimal problem formulated by Eqs. (2)-(10) and its reduced order version determined by Eqs. (17)-(22) have unique solutions in the interval $t_0 \leq t \leq t_f$.

(ii) The functions f,g, C and L are differentiable with respect to their arguments.

(iii) The Tihonov conditions[1] are satisfied.

These hypotheses are *necessary* for the existence of uniformly valid expansions (12)-(16), which guarantee that the zero order composite SPT solution is indeed a reasonable approximation (of the order of $O(\varepsilon)$) to the original problem. In other words, the existence of uniformly valid asymptotic expansions indicates that the singular perturbation problem is "well posed".

In addition to the above listed necessary conditions, a set of further hypotheses are required to formally demonstrate the existence of a uniformly valid asymptotic solution. Such formal proofs were given for linear systems[28], for nonlinear systems of special structure[25] and for nonlinear systems with unconstrained control solutions[23]. In many problems of interest related to aircraft performance optimization the validity of the hypotheses can hardly be verified in advance. In some other cases one of the assumptions is obviously violated, but an SPT solution can, however, be obtained. It can be thus concluded that the results of mathematical investigations have only a limited value to guide the aircraft performance analyst in the use of SPT (or FSPT). When formulating an FSPT problem (by artificial insertion of ε) special care has to be taken to avoid "ill posed" mathematical models.

A. "Ill Posed" Singular Perturbation Problems

Such problems can be characterized by one of two different phenomena (or both):

(i) No satisfactory zero-order composite solution can be found.

(ii) Higher order terms of the asymptotic expansion cannot be determined.

In a recent paper[29] three types of autonomous nonlinear singularly perturbed optimal control problems of "ill posed" structures, encountered in atmospheric flight mechanics, were reported. They are presented here merely as examples of warning in furture analysis.

1) If any of the partial derivatives of the variational Hamiltonian becomes unbounded (or undetermined) for the reduced order solution the respective variable cannot be expressed by a uniformly valid asymptotic expansion. If this variable is the "fast state" y or the control u the condition for boundary layer stability stated in Eq. (36) cannot be satisfied. If $\partial H/\partial x$ is unbounded in the "free stream" the costate variable λ_x° cannot be determined. As a consequence the SPT solution fails.

Though similar observations are mentioned in previous works[2,23], their implication to flight mechanics problems has not been pointed out. This can, however, be a frequent case in aircraft performance optimization. In horizontal (or nearly horizontal) turning maneuvers at the rate of change of the azimuth angle is expressed by

$$\dot{\chi} = \frac{g}{v}\sqrt{n^2 - 1} \tag{47}$$

and the aerodynamic load factor n is the commonly used control variable. In cases where the "free stream" solution is a straight line i.e.

$$\tilde{n}^\circ = 1 \tag{48}$$

the partial derivative

$$\frac{\partial \dot{\chi}}{\partial n} = \frac{g}{v}\frac{n}{\sqrt{n^2 - 1}}\Bigg|_{\tilde{n}^\circ = 1} \tag{49}$$

becomes infinite.

Such difficulty is generally avoided by replacing the aerodynamic load factor "n" by the bank angle "μ" as a control variable. In nearly horizontal flight (see Fig. 1)

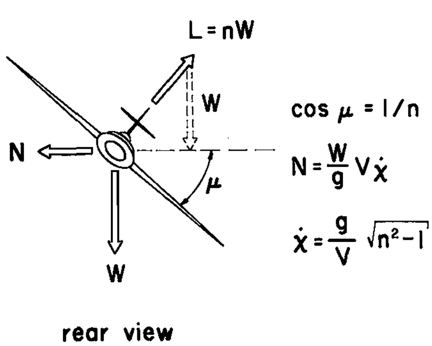

Fig. 1: Force equilibrium in horizontal turn.

$$\cos\mu = \frac{1}{n} \tag{50}$$

and consequently Eq. (47) can be written as

$$\dot{\chi} = \frac{g}{v}\,\mathrm{tg}\mu \tag{51}$$

for which the partial derivative

$$\frac{\partial \dot{\chi}}{d\mu} = \frac{g}{v}\frac{1}{\cos^2\mu} \tag{52}$$

has a finite value in the free stream ($\tilde{u}^\circ = 0$).

In the other two cases state variables of the same time scale (either the "slow" free stream variables or variables in one of the boundary layers) do not appear explicitly in the Hamiltonian. Such cases can be frequently encountered in aircraft performance optimization where the horizontal displacements (x,y) and sometimes also the azimuth angle (χ) do not influence the dynamics (see Appendix). In Ref. 26 the set of such possible problems is listed. The difficulties created by passive behaviour are demonstrated here (merely for the sake of simplicity) by a model of one dimensional free stream and a single fast variable (i.e.: ℓ=2, k=1).

2) <u>Autonomous System with a Passive Fast Variable.</u>

Such a system is characterized by

$$\dot{x} = f(x,u)\ ;\ x(t_0) = x_0\ ,\quad x(t_f) = x_f \tag{53}$$

$$\varepsilon\dot{y} = g(x,u);\ y(t_0) = y_0\ ,\ y(t_f) = y_f \neq y_0 \tag{54}$$

If the pay-off is also free from y

$$J = \int_{t_0}^{t_f} L(x,u)dt \tag{55}$$

then

$$\frac{\partial H}{\partial y} = 0 \tag{56}$$

In this case the zero order outer solution is identical to the solution of the original unreduced problem with free end conditions on y, the "fast" variable. The zero order inner solutions (boundary layers) are stationary and consequently no transition is possible between the different prescribed end conditions. Moreover, higher order terms cannot be obtained due to contradictory equations implying the non-existence of a uniformly valid asymptotic expansion. This non-existence is clearly indicated by the fact that the reduced order solution cannot determine the zero order term in the outer expansion of the fast variable. (For more details see Ref. 26 and 29). Due to these reasons the dynamic system (53),(54) cannot be analysed by SPT. For any nonzero value of ε the complete two-point boundary value problem has to be solved.

Note that in this case the inequality (35) cannot be satisfied. Comparison to a formal mathematical analysis indicate that one of the assumptions of the basic existence, theorem of the SPT solution (condition (i) of Theorem 4.1 in Ref. 23), is violated.

3) <u>Autonomous System with a Passive Slow Variable</u>

These types of dynamic systems are described by

$$\dot{x} = f(y,u)\ ;\ x(t_0) = x_0\ ,x(t_f) = x_f \neq x_0 \tag{57}$$

$$\varepsilon\dot{y} = g(y,u) \; ; \; y(t_0) = y_0 \; , y(t_f) = y_f \tag{58}$$

If the pay-off to be optimized is also independent of x, i.e.,

$$J = \int_{t_0}^{t_f} L(y,u)dt \tag{59}$$

the slow variable will not appear in the variational Hamiltonian and consequently

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \qquad \lambda_x = \text{const} \tag{60}$$

In this case the zero order outer solution is stationary, or in other words the rate of change of the slow variable is constant. The optimal values of the *original control* $\tilde{u}^\circ$ and the *pseudo control* y° are determined independently of the slow variable x°. The zero order boundary layer solutions are obtained in a feedback form using the constant value of $\lambda_x{}^\circ$ derived from the free stream.

$$\tilde{u}^i = \tilde{u}^i(y^i, \lambda_x{}^\circ) \tag{61}$$

Matching of the boundary layer solutions require that

$$y_0 + \int_0^{\tilde{\tau}} g[y^i, u^i(y^i)]d\tau = y^\circ \tag{62}$$

and similarly for the terminal boundary layer

$$y_f + \int_0^{\tilde{\delta}} g[y^i, u^i(y^i)]d\delta = y^\circ \tag{63}$$

These equations can be used to determine the durations $\tilde{\tau}$ and $\tilde{\delta}$ of the respective boundary layers. Variations of the slow variable during these periods are computed by

$$\Delta x^i(\tilde{\tau}) = \varepsilon \int_0^{\tilde{\tau}} f[y, \tilde{u}^i(y)]d\tau = \varepsilon \int_{y_0}^{y_0} \frac{f[y^i, \tilde{u}(y^i)]}{g[y^i, \tilde{u}(y^i)]} dy \tag{64}$$

$$\Delta x^i(\tilde{\delta}) = \varepsilon \int_0^{\tilde{\delta}} f[y, \tilde{u}^i(y^i)]d\delta = -\varepsilon \int_{y^\circ}^{y_f} \frac{f[y^i, \tilde{u}(y^i)]}{g[y^i, \tilde{u}(y^i)]} dy \tag{65}$$

The change of x in the free stream is given by

$$\Delta x^\circ = \int_{t_0+\frac{\tilde{\tau}}{\varepsilon}}^{t_f-\frac{\tilde{\delta}}{\varepsilon}} f(y^\circ, \tilde{u}^\circ)dt = f(y^\circ, \tilde{u}^\circ)\left[(t_f - t_0) - \frac{\tilde{\tau} + \tilde{\delta}}{\varepsilon}\right] \tag{66}$$

Thus the total variation of the slow variable is composed by 3 parts: (1) variation in the initial boundary layer, $\Delta x^i(\tilde{\tau})$, (2) variation in the free stream Δx°, (3) variation in the terminal boundary layer $\Delta x^i(\tilde{\delta})$. Note that the variations in the boundary layers do not depend on the end conditions due to the passive role of x in the equations. The duration of the free stream is therefore determined from

$$x_f - x_0 = \Delta x^i(\tilde{\tau}) + \Delta x^\circ + \Delta x^i(\tilde{\delta}) \tag{67}$$

If the sum of the variations in the boundary layers is larger than the value prescribed by the end conditions, i.e.

$$\Delta x^i(\tilde{\tau}) + \Delta x^i(\tilde{\delta}) > x_f - x_0 \tag{68}$$

this zero order SPT solution is not compatible. Due to the stationary free stream solution all higher order terms of power series expansions are indentically zero and therefore improvement cannot be expected.

It can be concluded that an autonomous system of a passive slow variable can be solved using the SPT approximation only for particular end conditions. For other cases, the method of "constrained matching"[30] was suggested. This method seems to lead to a reasonable approximation, but it requires an iterative procedure, which may not have computational advantage compared to the exact TPBVP solution.

B. Inherent Limitations of the SPT Feedback Solution

As it has been shown in the previous section, the most appealing feature of an SPT solution lies in its potential to provide *feedback* control laws. Whenever such feedback solutions, requiring only state variable measurements, cannot be obtained, the SPT approximation loses substantially from its attractiveness. It seems therefore important to convey two observations, relating to the on-line implementation of SPT solutions, which have not yet been pointed out in previous works. Feedback control implementation of SPT is inherently limited to:

(i) zero order approximation,

(ii) problems without a terminal boundary layer.

These two limitations will be briefly discussed in the sequel.

1. Computation of First and Higher Order Terms

For the sake of simplicity, one dimensional free stream and a single boundary layer with scalar control is assumed in this subsection. Since the zero order SPT solution can be obtained in a feedback form (43) the state variables x,y are assumed to be correct. They are indeed measured in a real-time implementation or obtained by integration of the state equations using (43). Only the costate variables λ_x, λ_y and the control u have to be expanded in asymptotic power series of ε as in (13), (15) and (16).

For the first order term of Eq. (3) in the outer expansion yields

$$\varepsilon\dot{y}^\circ = g(x^\circ, y^\circ(x), u_0^\circ(x) + \varepsilon u_1^\circ) = g^\circ + g_u(\varepsilon u_1^\circ) \tag{69}$$

Since $y^\circ(x)$ is known from Eqs. (22) and (40) the corrective term for the control is directly obtained

$$u_1^\circ\;(x,y) = \frac{f(x,y,u_0^\circ(x))}{g_u(x,y,u_0^\circ(x))} \cdot \frac{dy^\circ(x)}{dx} \tag{70}$$

The first order term in the Hamiltonian of the outer solution is

$$H_1^\circ = (L_u + \lambda_{x_0}^\circ(x)f_u + \lambda_{y_0}^\circ(x)g_u)u_1^\circ + \lambda_{x_1}^\circ f + \lambda_{y_1}^\circ g \tag{71}$$

where all functions and their partial derivatives are computed for the argument $(x,y,u^\circ(x))$. Optimality requires that $\frac{\partial H_1^\circ}{\partial u}=0$ and also $H_1^\circ = 0$. Consequently the relationship between the costate correction terms is given by

$$\lambda^\circ_{y_1}/\lambda^\circ_{x_1} = -g/f \tag{72}$$

Substituting Eqs. (71) and (72) into the first order term of Eq. (11)

$$\dot{\lambda}^\circ_y(x) = \frac{d\lambda^\circ_{y1}(x)}{dx} f = -\frac{\partial H_1^\circ}{\partial y} \tag{73}$$

gives a closed from feedback expression for the costate corrections:

$$\lambda^\circ_{x_1} = f\,\frac{\frac{g}{g_u}\frac{d\lambda^\circ_{y0}}{dx}g^\circ_u - \frac{dy^\circ}{dx}(L_{uy} - \lambda^\circ_{x0}f_{uy} - \lambda^\circ_{y0}g_{uy})}{fg_y - gf_y} = \lambda^\circ_{x_1}(x,y) \tag{74}$$

while $\lambda^\circ_{y_1}(x,y)$ is obtained from Eq. (72).

In the inner expansion we have from Eq. (28) the following first order term

$$\frac{d\lambda^i_{x1}}{d\tau} = -\frac{\partial H^i_0}{\partial x} = L_x - \lambda^i_{x0}f_x - \lambda^i_{y0}g_x \tag{75}$$

All terms of the right hand side are known from the zero order solution thus Eq. (75) can be directly integrated

$$\lambda^i_{x_1}(\tau) = \lambda^i_{x_1}(0) + \int_0^\tau (L_x - \lambda^i_{x0}f_x - \lambda^i_{y0}g_x)d\tau \tag{76}$$

The unknown constant of integration $\lambda^i_{x_1}(0)$ has to be determined by matching $\lambda^i_{x_1}$ to λ°_{x1} of the outer solution given by Eq. (74), i.e.

$$\lambda^i_{x_1}(0) = \lambda^\circ_{x_1}(x,y) - \int_0^{\tilde\tau}(L_x - \lambda^i_{x0}f_x - \lambda^i_{y0}g_x)d\tau \tag{77}$$

where $\tilde\tau$ is the duration of the initial zero order boundary layer determined by

$$y_0 + \int_0^{\tilde\tau} g(x,y)dt = y^\circ(x) \tag{78}$$

Once $\lambda^i_{x0}(0)$ is known, the initial boundary layer solution can be corrected to the first order by simply replacing $\lambda^i_{x_0}(x_0)$ by

$$\lambda^i_x(x,y^i) = \lambda^i_{x_0}(0) + \varepsilon\lambda^i_{x_1}(x,y) \tag{79}$$

in Eqs. (38) and (39). Consequently the unformly valid composite control can be corrected to the first order and expressed also in a feedback form.

However, it is obvious that the computational process involving Eq. (76) can be carried out only after the zero order trajectory is known. The integration of $\lambda^i_{x1}(\tau)$ and the matching defined by Eq. (77) are clearly *off-line* computations. Consequently the first order SPT approximation cannot be considered as a "true" *(on-line)* feedback solution.

Computation of the second and higher order terms follow a similar pattern requiring additional iterations.

2. Terminal Boundary Layers

The existence of a uniformly valid SPT solution for fixed end points requires asymptotic stability of the terminal boundary layer solution in the reversed stretched time scale of Eq. (22). Such a solution is defined in a "open-loop" form

$$\tilde{u}^c(t) = \tilde{u}^i\left(\frac{t-t_0}{\varepsilon}\right) + \tilde{u}^\circ(t) + \tilde{u}^i\left(\frac{t_f-t}{\varepsilon}\right) - (CP)_u \tag{80}$$

Though formally the solution can be transformed to a state feedback expression, its implementation as an *on-line* computational process presents serious difficulties. The source of these difficulties, which were observed in some studies [10,22] but have not been analysed, is the very fact of asymptotic stability of the terminal boundary layer in the reverse time scale.

The stability of the initial boundary layer guarantees that a trajectory starting from *any* initial condition in the domain of influence will reach the reduced order solution. The feedback solution in the form of Eq. (43) is uniformly valid for the initial boundary layer and the "free stream". Such a formulation is not sensitive to disturbances in the state variables. Moreover, due to the asymptotic stability the exact duration of the boundary layer characterized by $\tilde\tau$ (see Eq. (78)) is not critical.

All these nice features do not exist in the terminal boundary layer. For an on-line feedback implementation, intended to reproduce the "open-loop" solution, the uniformly valid initial boundary layer and free stream control function has to be "switched" to the form of the terminal boundary layer control at a precise instant δ, determined by the matching of the fast variable (see Eq.(64)). The exact timing is critical in order to satisfy the prescribed end conditions. Being in the "free stream" no state feedback information can be used to determine the conditions for the boundary layer initiation. This can be done only by an iterative *off-line* integration of the equations of motion until the prescribed end conditions are met. Moreover, even if it is correctly started, the terminal bounday layer trajectory computed in the real (forward) time direction is unstable with respect to disturbances in the state variables. This inherent instability can be suppressed only by modifying the SPT control law. In summary, an implementable feedback control near to the terminal manifold has nothing in common with SPT.

Computation of a terminal boundary layer is not required if the "fast" variables are free at the terminal manifold. Thus feedback implementation of SPT analysis should be limited to such problems; moreover, one has to be satisfied with a zero order approximation.

IV. Formulation of Aircraft Performance Optimization Problems by FSPT.

In the equations of motion of atmospheric flight mechanics (see Appendix) it is hard to find genuine small parameters of physical significance leading to formulate aircraft performance optimization problems with a singularly perturbed structure. Moreover, in many cases time scale separation of the variables is well known either from experience and/or analysis. For such cases the following type of transormation can be suggested. Given the original dynamic system

$$\begin{aligned} \dot{x} &= f(x,y,u) \quad , \quad x(t_0) = x_0 \\ \dot{y} &= g(x,y,u) \quad , \quad y(t_0) = y_0 \end{aligned} \tag{81}$$

where y is known to be the obvious "fast" variable of the problem. The first step is to rewrite the equations (81) in a non-dimensional form by setting

$$\begin{aligned} \hat{x} &= x/x_{ref} \\ \hat{y} &= y/y_{ref} \end{aligned} \tag{82}$$

Thus the rate of change of normalized variables can be compared. The reference values have to be chosen in such a manner that the known time scale separation is preserved and made evident, i.e.

$$\left| \frac{d\hat{x}}{d\hat{y}} \right| = \left| \frac{f/x_{ref}}{g/y_{ref}} \right| \leq \varepsilon_g \tag{83}$$

where $\varepsilon_g < 0$ is a *genuine* small parameter.

Now it is easy to define a normalized time scale

$$\hat{t} = t/t_{ref} \tag{84}$$

such that the resulting set of differential equations

$$\begin{aligned} \frac{d\hat{x}}{d\hat{t}} &= \hat{f}(\hat{x},\hat{y},u) \ ; \quad \hat{x}(\hat{t}_0) = x_0/x_{ref} \\ \varepsilon_g \frac{d\hat{y}}{d\hat{t}} &= \hat{g}(\hat{x},\hat{y},u) \ ; \quad \hat{y}(\hat{t}_0) = y_0/y_{ref} \end{aligned} \tag{85}$$

has a mathematically true singularly perturbed structure. It has to be remembered that such an apparently arbitrary transformation has a practical significance only if a *genuine* time scale separation of variables exists.

The complication involved in such a transformation (accompanied some times also by a loss of physical insight) is, however, not necessary. In previous works[12-17] it was suggested that a small parameter ε should be artifically introduced to multiply the time derivative of the "fast" variable

$$\begin{aligned} \dot{x} &= f(x,y,u) \ ; \quad x(t_0) = x_0 \\ \varepsilon\dot{y} &= g(x,y,u) \ ; \quad y(t_0) = y_0 \end{aligned} \tag{86}$$

In this"forced" singularly perturbed formulation, $\varepsilon = 1$. The approximation obtained by taking $\varepsilon = 0$ can be regarded as a first phase of a "continuation" process[22,31]. Moreover, it can be demonstrated[32] that the zero order SPT feedback controls for the two dynamic systems (85) and (86) are equivalent and consequently the respective suboptimal trajectories are identical. Since we have shown in the previous section that only the zero order SPT approximation can be obtained in a true feedback form, the use of "forced" singular perturbation technique (FSPT) for on-line real time applications is fully justified if the problem exhibits a genuine time scale separation.

In order to avoid an unsuccessful analysis, the following rules have to be observed, however, at the formulation of a forced singularly perturbed model for a non-linear optimization problem:

(i) "Ill posed" mathematical models have to be avoided.

(ii) The fast variable should be free on the terminal manifold (to avoid terminal boundary layer).

(iii) The reduced order problem has to exhibit a reasonable similarity to the original one.

Engineering judgement based on some knowledge of the exact optimal solution is essential for further guidance. It is particularly indispensable for the proper formulation of a multiple time scale forced singular perturbation model.

Successful application of FSPT for aircraft performance optimization is demonstrated in the next sections by two examples:

(a) An air to air interception in the horizontal plane.

(b) A time optimal turning maneuver in the vertical plane (half "loop").

V. Air to Air Interception in the Horizontal Plane

In this problem a fighter airplane equipped with an air to air missile has to intercept in minimum time an evading target flying at a constant speed V_E to a given direction. The relative motion between the two airplanes is described (see Fig. 2) in polar coordinates by

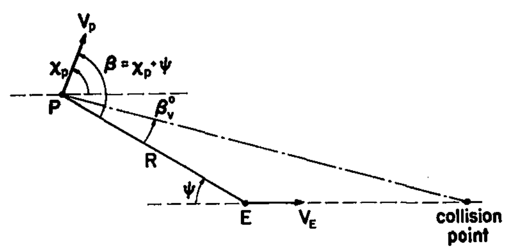

Fig. 2: Geometry of Pursuit.

$$\dot{R} = V_E \cos\psi - V_p \cos(\chi_p + \psi) \ , \quad R(t_0) = R_0 \tag{87}$$

$$\dot{\psi} = \frac{1}{R}[V_p \sin(\chi_p + \psi) - V_E \sin\psi] \ , \quad \psi(t_0) = \psi_0 \tag{88}$$

The dynamics of the pursuer airplane in a horizontal plane at a given altitude (h) is determined by (see Eqs. (A23) with $\gamma = 0$ and (51))

$$\dot{V}_p = \frac{g}{W}[\eta T_{max}(h,V_p) - D_0(h,V_p) - \sec^2\mu D_i(h,V_p)] \; ; \; V_p(t_0) = V_{p_0} \tag{89}$$

$$\dot{\chi}_p = \frac{g}{V_p} \operatorname{tg}\mu \qquad ; \; \chi_p(t_0) = \chi_{p_0} \tag{90}$$

where $D_0(h,V_p)$ and $D_i(h,V_p)$ are zero lift drag and the induced drag in straight and level flights respectively.

The objective of the pursuer is to minimize the time of capture t_f defined by

$$R(t_f) \triangleq d \tag{91}$$

where "d" is the missile firing range using the set of optimal controls η^*, μ^* subject to the constraints

$$0 \leq \eta < 1 \quad ; \quad -\mu_{max} \leq \mu \leq \mu_{max} \tag{92}$$

The variational Hamiltonian of this problem is therefore written as

$$\begin{aligned} H = -1 &+ \lambda_R[V_E\cos\psi - V_p\cos(\chi_p + \psi)] - \\ &- \lambda_\psi[V_E\sin\psi - V_p\sin(\chi_p + \psi)]\frac{1}{R} + \\ &+ \lambda_V\frac{g}{W}[\eta T_{max} - D_0 - \sec^2\mu D_i] + \lambda_\chi\frac{g}{V_p}\operatorname{tg}\mu \end{aligned} \tag{93}$$

The necessary conditions of optimality include the set of adjoint differential equations

$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = \frac{\lambda_\psi}{R^2}[V_p\sin(\chi_p + \psi) - V_E\sin\psi] \tag{94}$$

$$\begin{aligned} \dot{\lambda}_\psi = -\frac{\partial H}{\partial \psi} &= -\lambda_R[V_p\sin(\chi_p + \psi) - V_E\sin\psi] + \\ &+ \lambda_\psi\frac{1}{R}[V_p\cos(\chi_p + \psi) - V_E\cos\psi] \; ; \; \lambda_\psi(t_f) = 0 \end{aligned} \tag{95}$$

$$\begin{aligned} \dot{\lambda}_V = -\frac{\partial H}{\partial V_p} &= \lambda_R\cos(\chi_p + \psi) - \frac{\lambda_\psi}{R}\sin(\chi_p + \psi) - \\ &- \lambda_V\frac{g}{W}[\eta\frac{\partial T_{max}}{\partial V_p} - \frac{\partial D_0}{\partial V_p} - \sec^2\mu\frac{\partial D_i}{\partial V_p}] + \\ &+ \lambda_\chi\frac{g}{V_p^2}\operatorname{tg}\mu \; ; \qquad \lambda_V(t_f) = 0 \end{aligned} \tag{96}$$

$$\begin{aligned} \dot{\lambda}_\chi = -\frac{\partial H}{\partial \chi_p} &= -\lambda_R V_p\sin(\chi_p + \psi) - \frac{\lambda_\psi}{R}V_p\cos(\chi_p + \psi) \; ; \\ &; \; \lambda_\chi(t_f) = 0 \end{aligned} \tag{97}$$

The optimal control function η^*, μ^* have to maximize the Hamiltonian

$$\eta^*, \mu^* = \underset{\substack{0 \leq \eta \leq 1 \\ \mu < \mu_{max}}}{\arg\max} H \tag{98}$$

yielding

$$\eta^* = \frac{1}{2}[1 + \operatorname{sign}\lambda_V] \quad ; \quad \lambda_V \neq 0 \tag{99}$$

$$\operatorname{tg}\mu^* = \frac{\lambda_\chi}{\lambda_V}\;\frac{W}{2V_pD_i} \tag{100}$$

Moreover, since the problem is autonomous and the final time is not prescribed

$$H^* \triangleq H(\eta^*, \mu^* \ldots) = 0 \tag{101}$$

An approximate feedback solution of this NTPBVP is attempted by transforming the original set of differential equations to a multiple boundary layer forced singular perturbation system. Similar problems[19-21] were analysed by FSPT in the past based on slightly different formulation. The present analysis is a part of an ongoing effort[33] concentrated to express the optimal control function in a *direct* state feedback form. Our transformation is based on the following observations indicating time scale separation of the variables:

(i) If the initial distance of separation R_0 is large compared to the radius of turn of the pursuer, the rate of change of the line of sight ψ is much slower than the direction change of the aircraft.

(ii) The longitudinal acceleration of an airplane is much smaller than the lateral accelerations experienced in turning maneuvers.

Consequently the differential equations describing pursuer dynamics (89) and (90) are replaced by

$$\varepsilon\dot{V}_p = \frac{g}{W}(\eta T_{max} - D_0 - \sec^2\mu D_i) \; ; \; V_p(t_0) = V_{p_0} \tag{102}$$

$$\varepsilon^2\dot{\chi}_p = \frac{g}{V_p}\operatorname{tg}\mu \qquad ; \; \chi_p(t_0) = \chi_{p_0} \tag{103}$$

while Eqs. (87) and (88) remain unchanged.

Similarly the costate equation (96) and (97) become

$$\varepsilon\dot{\lambda}_V = -\frac{\partial H}{\partial V_p} \qquad \lambda_V(t_f) = 0 \tag{104}$$

$$\varepsilon^2\dot{\lambda}_\chi = -\frac{\partial H}{\partial \chi_p} \qquad \lambda_\chi(t_f) = 0 \tag{105}$$

By setting $\varepsilon = 0$ the reduced order problem is obtained. Its solution is rather simple,

$$\tilde{\mu}^\circ = 0 \tag{106}$$

$$\tilde{\eta}^\circ T_{max} - D - D_i = 0 \tag{107}$$

Moreover, Eqs. (104) and (105) indicate that in this outer solution V_p and χ_p play the roles of control variables maximizing the reduced Hamiltonian defined by

$$H^\circ \triangleq -1 + \lambda_R^\circ\dot{R}^\circ + \lambda_\psi^\circ\dot{\psi}^\circ \tag{108}$$

subject to the constraints (106) and 107)

Consequently we obtain for $(\tilde{\eta}^\circ = 1)$

$$V_p^\circ = \max\arg\{T_{max} - D_0 - D_i = 0\} \tag{109}$$

and

$$\operatorname{tg}(\chi^\circ_p + \psi^\circ) = -\frac{\lambda_\psi^\circ}{R^\circ\lambda_R^\circ} \tag{110}$$

The optimal direction χ_p° is the well known "collision course" defined by

$$V_p^\circ \sin\beta^\circ = V_E \sin\psi^\circ \tag{111}$$

where, for the sake of simplification, we define (see Fig. 2)

$$\beta \triangleq \chi_p + \psi \tag{112}$$

The costate variables λ°_R and λ°_ψ are expressed in a feedback form

$$\lambda_R^\circ = \frac{-1}{V_p^\circ \cos\beta^\circ - V_E \cos\psi^\circ} \tag{113}$$

$$\lambda_\psi^\circ = \frac{R\, tg\beta^\circ}{V_p^\circ \cos\beta^\circ - V_E \cos\psi^\circ} \tag{114}$$

Substituting (111) into (88) reconfirms that $\dot{\psi}^\circ=0$ as required bv the constant bearing collision course.

The first zero order initial boundary layer equations are obtained by using the stretched time scale

$$\tau_i \triangleq \frac{t}{\varepsilon} \tag{115}$$

and setting again $\varepsilon = 0$. These equations yield

$$\frac{dR^i}{d\tau_i} = \frac{d\psi^i}{d\tau_i} = \frac{d\lambda_R^i}{d\tau_i} = \frac{d\lambda_\psi^i}{d\tau_i} = \tilde{\mu}^i = \frac{\partial H^i}{\partial \chi_p^i} = 0 \tag{116}$$

$$\frac{dV_p^i}{d\tau_i} = \frac{g}{W}(\eta T_{max} - D_0 - D_i), \quad V_p(0) = V_{p_0} \tag{117}$$

$$\frac{d\lambda_V^i}{d\tau_i} = \frac{\partial H^i}{\partial V_p^i} \tag{118}$$

where H^i is defined as

$$H^i \triangleq -1+\lambda^\circ_{R_0}(V_E\cos\psi_0 - V_p^i\cos\beta^i) + \lambda^\circ_{\psi_0}(V_p^i \sin\beta^i - V_E \sin\psi_0)\frac{1}{R_0} + \lambda_V^i \frac{g}{W}[\eta T_{max} - D_0 - D_i] \tag{119}$$

Maximization of H^i with respect to the controls η, χ_p^i leads to

$$\tilde{\eta}^i = \frac{1}{2}[1 + \text{sign}\,\lambda_V^i], \; \lambda_V^i \neq 0 \tag{120}$$

$$\tilde{\beta}^i = \tilde{\chi}_p^i + \psi_0^\circ = tg^{-1}\left(\frac{-\lambda^\circ_{\psi 0}}{R^\circ \lambda^\circ_{R0}}\right) = \tilde{\beta}_0^\circ \tag{121}$$

where λ_V^i can be eliminated from (119) (since optimality and matching require that $H^i = 0$) yielding

$$\lambda_V^i = \frac{W/g}{\cos\beta_0^\circ(V_p^\circ\cos\beta_0^\circ - V_E\cos\psi_0^\circ)} \cdot \frac{V_p^\circ - V_p}{T_{max} - D_0 - D_i} \tag{122}$$

Due to (109) $\lambda_V^i > 0$ and consequently $\tilde{\eta}^i = 1$, indicating that full thrust is optimal. The matching requirement

$$\lim_{\tau_i \to \infty} V_p^i = V_{p_0} + \lim_{\tau_i \to \infty} \int_0^{\tau_i} \frac{g}{W}(T_{max} - D_0 - D_i)d\tau_i = V_p^\circ \tag{123}$$

is automatically satisfied.

The second initial boundary layer deals with turning dynamics using another stretched time scale defined by

$$\tau_j \triangleq \frac{t_i}{\varepsilon} = \frac{t}{\varepsilon^2} \tag{124}$$

and setting $\varepsilon = 0$ we obtain

$$\frac{dR^j}{d\tau_j} = \frac{d\psi^j}{d\tau_j} = \frac{dV_p^j}{d\tau_j} = \frac{d\lambda_R^j}{d\tau_j} = \frac{d\lambda_\psi^j}{d\tau_j} = \frac{d\lambda_V^j}{d\tau_j} = 0 \tag{125}$$

$$\frac{d\chi_p^j}{d\tau_j} = \frac{g}{V_p^j} tg\mu \qquad \chi_p^j(0) = \chi_{p0} \tag{126}$$

$$\frac{d\lambda_x^j}{d\tau_j} = -\frac{\partial H^j}{\partial \chi_p} \tag{127}$$

where H^j is defined by

$$H^j \triangleq -1 + \lambda^\circ_{R0}[V_E\cos\psi_0 - V^i_{p0}\cos\beta^j] + \lambda^\circ_{\psi 0}\frac{1}{R_0}(V^i_{p0}\sin\beta^j - V_E\sin\psi_0) + \lambda^i_{V0}\frac{g}{W}(T_{max} - D_0 - \sec^2\mu D_i) + \lambda_\chi^j \frac{g}{V_{p_0}} tg\mu \tag{128}$$

The necessary conditions of optimality

$$\frac{\partial H^j}{\partial \mu} = 0 \wedge tg\tilde{\mu}^j = \frac{\lambda_0^j}{\lambda_V^i}\frac{W}{2V^i_{p0}D_i} \tag{129}$$

and $H^j(\tilde{\mu}^j) = 0$ enables us to express the optimal bank angle $\tilde{\mu}^j$, using (122), in a feedback form

$$tg^2\tilde{\mu}^j = \frac{T_{max} - D_0 - D_i}{D_i} \frac{V^i_{p0}}{V_p^\circ - V^i_{p0}} [1 - \cos(\beta^j - \beta^\circ)] \tag{130}$$

The sign of $tg\mu$ has to be such that the resulting turn tends to decrease the difference $(\beta^j - \beta^\circ) = (\chi_p^j - \chi_p^\circ)$; thus

$$\text{sign}(tg\tilde{\mu}^j) = \text{sign}\left(\frac{d\chi_p^j}{d\tau_j}\right) = \text{sign}(\beta^\circ - \beta^j) \tag{131}$$

The uniformly valid composite control function is obtained by replacing the frozen initial values of the state variables in (130) by their current (measured) value

$$tg\tilde{\mu}^c = \left\{\frac{V_p}{D_i} \cdot \left(\frac{T_{max} - D_0 - D_i}{V_p^\circ - V_p}\right)[1-\cos(\beta - \beta_V^\circ)]\right\}^{\frac{1}{2}} \cdot \text{sign}(\beta_V^\circ - \beta) \tag{132}$$

where β_V° is the instantaneous collision direction defined by

$$\beta_V^\circ = \sin^{-1}(V_E\sin\psi/V_p) \tag{133}$$

Results of a numerical example are shown in Figs. 3 - 5.

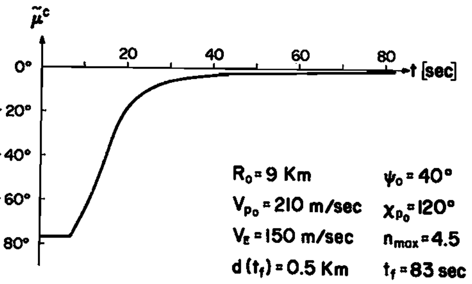

Fig. 3: Optimal FSPT bank angle.

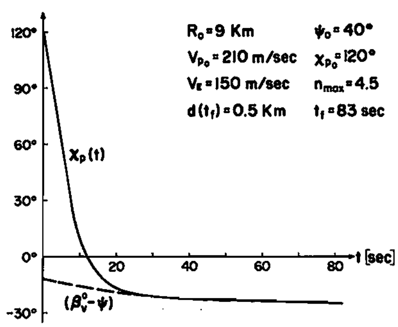

Fig. 4: Time history of pursuer direction.

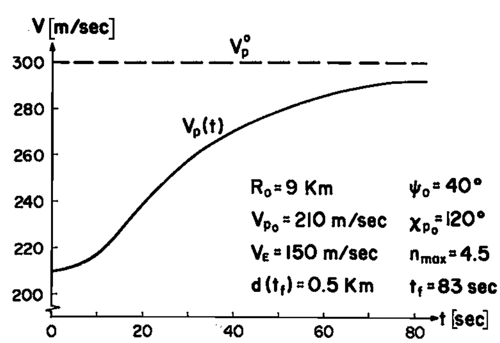

Fig. 5: Time histroy of pursuer velocity.

The following remarks, relating to the feedback control law of (132), are of interest:

(i) Comparing Eqs. (129) to (100) it is clearly seen that the FSPT and the original optimal control expressions are formally identical. The only difference is that the costate variables in Eq. (129) are approximated using FSPT analysis.

(ii) When V_p tends towards V_p° the FSPT control remains finite since at $V_p^\circ = V_p^\circ$ Eq. (109) holds. At the limit

$$\lim_{V_p \to V_p^\circ} \left\{ \frac{T_{max} - D_0 - D_i}{V_p^\circ - V_p} \right\} = \tag{134}$$

$$= - \lim_{V_p \to V_p^\circ} \frac{\partial}{\partial V_p} (T_{max} - D_0 - D_i) > 0$$

(iii) Near to the collision course Eq. (132) can be expressed as

$$\mathrm{tg}\tilde{\mu}^c \simeq - k(V_p)(\beta - \beta_V^\circ) \tag{135}$$

and consequently

$$\dot{\chi}_p = - \frac{g}{V_p} k(V_p)(\beta - \beta_V^\circ) \tag{136}$$

In such a condition the rate of turn of the line of sight is given approximately

$$\dot{\psi} \simeq \frac{V_p}{R} \cos\beta_V^\circ \, (\beta - \beta_V^\circ) \tag{137}$$

Combining Eqs. (136) and (137) we obtain

$$\dot{\chi}_p = - N(V_p, R)\dot{\psi} \tag{138}$$

which is a *proportional navigation* control law with variable gain $N(V_p,R)$.

VI. Minimum Time "Half-Loop".

In this vertical turning maneuver, which is frequently used in air to air combat, the pilot wants to change rapidly his heading by nearly 180° while gaining altitude. It is a pull up maneuver initiated from level flight. In order to preserve agility the maneuver has to be performed without excessive loss of specific energy. The optimal control problem can be thus formulated for a minimum time maneuver between prescribed specific energy levels.

The variables for a maneuver in the vertical plane are defined in Fig. 6.

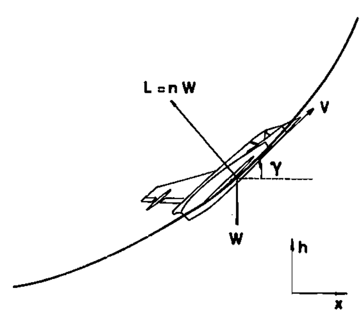

Fig. 6: Variables in Vertical Motion.

The equation of motion for the half-loop maneuver using the specific energy

$$E \triangleq h + V^2/2g \tag{139}$$

as a state variable (rather than the velocity) can be written as [see Eqs. (A-24)(A-25)(A-5) and (A-7) with $\chi = 0$]

$$\dot{E} = \frac{V}{W}[\eta T_{max} - D_0 - n^2 D_i]; \quad E(t_0)=E_0 \quad E(t_f)=E_f \tag{140}$$

$$\dot{\gamma} = \frac{g}{V}(n - \cos\gamma) \quad ; \quad \gamma(t_0)=0 \quad \gamma(t_f)=\pi \tag{141}$$

$$\dot{h} = V\sin\gamma \quad ; \quad h(t_0)=h_0 \tag{142}$$

$$\dot{X} = V\cos\gamma \quad ; \quad X(t_0)=0 \tag{143}$$

The variational Hamiltonian for this problem is

$$H = -1 + \lambda_E V \frac{\eta T_{max} - D_0 - D_i}{W} + \lambda_\gamma \frac{g}{V}(n-\cos\gamma) + \lambda_h V\sin\gamma + \lambda_x V\cos\gamma + \text{constraints} \tag{144}$$

The necessary conditions for optimality require that the costate variables satisfy the adjoint set of differential equations.

$$\left.\begin{aligned} \dot{\lambda}_E &= -\partial H/\partial E\big|_h \\ \dot{\lambda}_\gamma &= -\partial H/\partial \gamma \\ \dot{\lambda}_n &= -\partial H/\partial h\big|_E \\ \dot{\lambda}_x &= -\partial H/\partial X = 0 \end{aligned}\right\} \tag{145}$$

with the boundary conditions

$$\lambda_h(t_f) = \lambda_x(t_f) = 0 \tag{146}$$

The optimal control functions $\eta^*(t)$, $n^*(t)$ have to maximize the Hamiltonian

$$\eta^*, n^* = \arg\max_{\eta,n} H(E, \gamma, h, \lambda_E, \lambda_\gamma, \lambda_h, \eta, n) \tag{147}$$

yielding for the thrust control

$$\eta^* = \tfrac{1}{2}[\text{sign}\, \lambda_E + 1] \; ; \; \lambda_E \neq 0 \tag{148}$$

The optimal unconstrained load factor is obtained

$$n^* \triangleq \frac{\lambda_\gamma}{2\lambda_E} \frac{g}{V^2} \frac{W}{D_i} \tag{149}$$

This problem was investigated in previous works[11,26,34] by several methods including FSPT analysis. In this example the recent results of the FSPT analysis[26] are briefly presented.

The investigations of optimal turning maneuvers have shown, that if a substantial energy loss is permitted by the prescribed value of E_f, the major part (including the terminal phase) of the manuever is performed at an almost constant specific energy level. A rapid loss of specific energy occurs at the initial phase of the pull-up. Based on this observation it is proposed to select, for the time optimal problem defined in this section the specific energy as the "fast" variable of the problem.

The mathematical formulation of the forced singular perturbation problem leads for this case to replace Eq. (140) by

$$\varepsilon\dot{E} = (\eta T_{max} - D_0 - n^2 D_i)\frac{V}{W} \tag{150}$$

while the other state equations remain unchanged. If ε tends toward zero an equality constraint of "thrust is equal to drag" has to be satisfied.

The variational Hamiltonian of the forced singular perturbation problem and the original one are identical, as well as the adjoint equations for λ_γ, λ_h and λ_x. The only modified costate equation is

$$\varepsilon\dot{\lambda}_E = -\frac{\partial H}{\partial E}\bigg|_h \tag{151}$$

which requires, for $\varepsilon \to 0$, to maximize the Hamiltonian as a function of the specific energy. It can be interpreted as to choose, for any given set of "γ" and "h", the value of specific energy (or speed since the altitude is given) that maximizes the vertical turning rate, subject to the constraint $T = D$ (an analogue of the best sustained horizontal turn). This approach is called FSPT 1.

Since such reduced solutions cannot satisfy the prescribed end conditions, both initial and terminal boundary layer solutions are required. In order to avoid difficulties created in a terminal boundary layer (as mentioned in section 3B) an alternative formulation (FSPT2) was also suggested[34]. The terminal boundary layer is not required if the reduced order solution is considered to be flown at the prescribed final specific energy level E_f. In this "suboptimal" formulation Eq. (151) is not used and λ_E° is regarded as a simple constraint multiplier.

For both versions the FSPT solution is obtained in a feedback form and the respective results can be compared as in Ref. 26.

In the reduced order solution the controls $\tilde{\eta}^\circ$, $\tilde{n}^\circ$ are obtained directly from the constraints:

a) structural limit load factor (thrust is less than maximum:

$$\tilde{n}^\circ = n_{max} \qquad \tilde{\eta}^\circ = \frac{D_0 + n_{max}^2 D_i}{T_{max}} \tag{152}$$

b) maximum available thrust equals drag:

$$\tilde{n}^\circ = \left(\frac{T_{max} - D_0}{D_i}\right)^{\frac{1}{2}} \qquad \tilde{\eta}^\circ = 1 \tag{153}$$

c) aerodynamic lift limit (thrust is less than maximum):

$$\tilde{n}^\circ = n(C_{L_{max}}) \triangleq n_L \qquad \tilde{\eta}^\circ = \frac{D_0 + n_L^2 D_i}{T_{max}} \tag{154}$$

In the first version (FSPT 1) the optimal specific energy E° is determined by Eq. (151) with $\varepsilon = 0$.

Since the controls are known, free stream trajectories can be computed directly without requiring the solution of a NTPBVP. For each version the adjoint variables are integrated backwards along these trajectoris and stored as a function of γ (the genuine independent variable of the problem). The terminal value of λ°_γ is determined from $H^\circ = 0$ yielding

$$\lambda^\circ_\gamma(t_f) = \frac{V^\circ_p}{g} \; \frac{1}{\tilde{n}^\circ_f + 1} \qquad (155)$$

The initial boundary layer problem obtained by the stretching transformation has only a single state variable E^i. Consequently λ^i_E can be eliminated from the Hamiltonian and the optimal unconstrained load-factor can be determined in a feedback from the following quadratic equation:

$$(\tilde{n}^i)^2 - 2n^i\left[\cos\gamma + \frac{V}{g\lambda^\circ_\gamma}(1-\lambda^\circ_h V\sin\gamma\right] + \left(\frac{\eta T_{max} - D_0}{D_i}\right) = 0 \qquad (156)$$

It can be shown that this quadratic has two positive real roots. The larger root corresponds to negative specific energy rate and it is the relevant one to problems with $(E_0 > E^\circ)$. When this solution does not violate the control constraints λ^i_E will be positive and as a consequence full thrust $(\tilde{\eta}^i = 1)$ has to be used. For constrained load-factor "n_c" λ^i_E can be found from the Hamiltonian

$$\lambda^i_E(\tilde{\eta}^i T_{max} - D_0 - n^i_c{}^2)\frac{V}{W} = 1 - \lambda^\circ_\gamma \frac{g}{V}(\tilde{n}^i_c - \cos\gamma) - \lambda^\circ_h V\sin\gamma \qquad (157)$$

The term multiplying λ^i_E is the specific energy rate, assumed to be negative even with full thrust, to satisfy $E_0 > E^\circ$. Therefore the sign of λ^i_E which determines the value of $\tilde{\eta}^i$ is opposite to the sign of the right side in (157). For $E_0 < E^\circ$ always full thrust $(\tilde{\eta}^i = 1)$ is required.

The terminal boundary layer, which has to complete the solution of the FSPT 1 version, is formally similar. However, it cannot be implemented in feedback form for forward integration of the equations of motion (see section 3B). For such a purpose an iterative procedure was used requiring an increased computational effort.

Solutions of both FSPT versions are compared to the exact optimal results in Figs. 7 and 8.

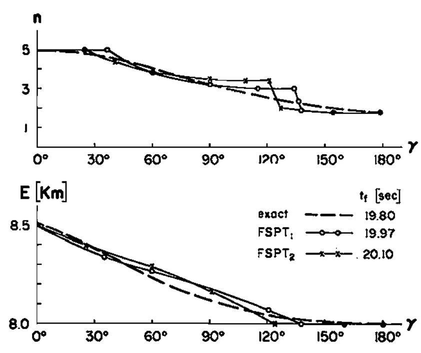

Fig. 7: Optimal Load Factor and Specific Energy vs. Flight Path Angle.

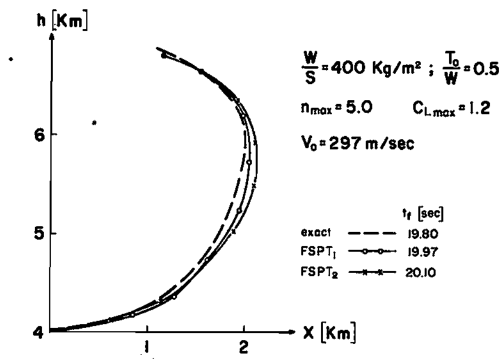

Fig. 8: Optimal "Half-Loop" Trajectories.

VII. Concluding Remarks

In this paper the challenges and the pitfalls of FSPT application in aircraft performance optimization are discussed. The attention is focused on the multiple time scale version of the singular perturbation methodology, the only one possessing the potential to provide feedback control strategies. The importance of such a solution cannot be overemphasized for real-time on-line airborne implementations, as well as for qualitative tradeoff analysis at the preliminary design phase of a new airplane.

This appealing approach has, however, certain limitations which have not been pointed out in previous works. The paper issues a warning on these difficulties, in order to avoid frustrating disappointments in future analysis. It is shown that the attractive freature of true feedback control is limited, as a zero order approximation only, to non-linear optimization problems without terminal boundary layers.

The paper also attempts to provide some guidelines for the appropriate formulation of aircraft performance optimization problems for FSPT analysis. It indicates that the FSPT approach is justified if

(i) the time scale separation of the variables is correctly assessed, and

(ii) ill-posed mathematical models are avoided.

The examples given in this paper demonstrate that the FSPT methodology (being, by the way, equally applicable to solve non-linear zero-sum differential games [35]) is a useful and productive tool of analysis. It is strongly believed that properly applied FSPT may lead to an enhanced insight in aircraft performance optimization and consequently to improved designs and operations.

Appendix

Mathematical Model of Atmospheric Flight Mechanics

The motion of a point mass lifting vehicle over a flat non-rotating earth, assuming symmetrical flight, is governed by the following set of non-linear ordinary differential equations:

$$\dot{V} = g\left[\frac{\eta T_{max}(h,V)\cos(\alpha+{}_{T}) - D}{W} - \sin\gamma\right] \quad \text{(A1)}$$

$$\dot{\gamma} = \frac{g}{V}\left[\frac{\eta T_{max}(h,V)\ \sin(\alpha+\varepsilon_T) + L}{W}\cos\mu - \cos\gamma\right] \quad \text{(A2)}$$

$$\dot{\chi} = \frac{g}{V}\ \frac{\eta T_{max}(h,V)\ \sin(\alpha+\varepsilon_T) + L}{W\cos\gamma}\sin\mu \quad \text{(A3)}$$

$$\dot{W} = -c(h,V,T) \quad \text{(A4)}$$

$$\dot{X} = V\cos\gamma\cos\chi \quad \text{(A5)}$$

$$\dot{Y} = V\cos\gamma\sin\chi \quad \text{(A6)}$$

$$\dot{h} = V\sin\gamma \quad \text{(A7)}$$

where

$$L = \frac{1}{2}\rho(h)\ V^2 S\ C_L(\alpha,M) \quad \text{(A8)}$$

$$D = \frac{1}{2}\rho(h)\ V^2 S C_D(M,C_L) \quad \text{(A9)}$$

For a parabolic drag polar

$$C_D(M,C_L) = C_{D_0}(M) + K(M)\ C_L^2(\alpha,M) \quad \text{(A10)}$$

The aerodynamic load factor is defined by

$$n \triangleq \frac{L}{W} = \frac{\frac{1}{2}(h)\ V^2\ C_L(\alpha,M)}{\frac{W}{S}} \quad \text{(A11)}$$

and the drag force can be expressed as

$$D = D_0 + n^2 D_i = D(h,V,n) \quad \text{(A12)}$$

where D_0 is the zero lift drag and D_i the induced drage in level flight

$$D_i = \frac{K\ W^2}{\frac{1}{2}\rho(h)\ V^2\ S} \quad \text{(A13)}$$

In many cases the specific energy of the aircraft, defined by

$$E \triangleq h + \frac{V^2}{2g} \quad \text{(A14)}$$

is used as a state variable and Eq.(A1) is replaced by

$$\dot{E} = \frac{\eta T_{max}\ \cos(\alpha+\varepsilon_T) - D}{W}\ V \quad \text{(A15)}$$

The control variables for the point mass equations are the throttle parameter "η", the angle of attack α, the thrust deflection relative to the body axis ε_T, and the bank angle "μ". In Eqs.(A1) - (A7) the aerodynamic load factor "n" or the lift coefficient C_L may be used as alternative control variables instead of the angle of attack α.

As any airplane maneuver should take place in the "dynamic flight envelope" the following constraints have to be satisfied.

(1) State Constraints

Minimum altitude limit

$$h > 0 \quad \text{(A16)}$$

Maximum dynamic pressure limit

$$q = \frac{1}{2}\rho(h)\ V^2 \le q_{max} \quad \text{(A17)}$$

Maximum Mach number limit

$$V \le a(h)\ M_{max} \quad \text{(A18)}$$

Loft ceiling "h_L" limit, expressed by

$$\frac{1}{2}\rho(h_L)V^2 \ge \frac{W/S}{C_{L_{max}}(M)} \quad \text{(A19)}$$

(2) Control Constraints

Assuming that "n" replaces α as control variable the control constraints can be expressed as

$$0 \le \eta \le 1 \quad \text{(A20)}$$

$$|n| \le n_{max} \quad \text{(A21)}$$

$$|n| \le \frac{\frac{1}{2}(h)\ V^2 C_{L_{max}}(M)}{W/S} = n_L(h,M) \quad \text{(A22)}$$

Note that the problem has no explicit dependence on the time.

In many performance optimization problems small angles of attack, constant weight and fixed thrust direction can be assumed, yielding a simplified set of equations replacing (A1)-(A3)

$$\dot{V} = g\left[\frac{\eta T_{max} - D_0 - n^2 D_i}{W} - \sin\gamma\right] \quad \text{(A23)}$$

or alternatively

$$\dot{E} = \frac{\eta T_{max} - D_0 - n^2 D_i}{W} V \quad \text{(A24)}$$

$$\dot{\gamma} = \frac{g}{V}(n\cos\mu - \cos\gamma) \quad \text{(A25)}$$

$$\dot{\chi} = \frac{g\ n\sin\mu}{V\cos\gamma} \quad \text{(A26)}$$

which, however, remain non-linear.

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