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QUASI – HOMOGENEOUS APPROXIMATIONS FOR WINGS WITH CURVED SUBSONIC LEADING EDGES AT SUPERSONIC SPEEDS

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Abstract

In this paper quasi homogeneous approximations are constructed for the flow around delta like supersonic wings with curved subsonic leading edges. If the leading edges are given by algebraïc curves and the boundary conditions are of polynomial form, the boundary value problems are reduced to completely algebraïc problems. The solutions can be expressed in terms of known simple functions.
The parameters defining the leading edges appear in the same way as those defining the boundary conditions. The results may be useful to acquire a qualitative insight in the possibilities of a large class of planforms.

I. Introduction

1.1 Preliminary considerations

From the view-point of the designer of supersonic aircraft it is essential to acquire a qualitative insight into the aerodynamic properties of wings which corresponds to the requirements to be satisfied. Conversely these requirements should be formulated in such a way that the design problem will be "well posed" in the mathematical sense and will permit a unique solution. Moreover the requirements should permit a solution that can be realized physically. During the design process, the linearized form of potential theory for supersonic wings is a powerful tool. Due to the linearity of the equations, the superposition principle applies and a systematic approach is possible.
In the last phases of the design process, numerical techniques may be useful for the study of complicated geometries through the use of high-speed digital computers. In the earlier phases of the design however one is less interested in numerical values. One needs a qualitative insight which may take the form of a parameter study. Obviously, the planform of the wing deserves a high priority. Because of the theoretical difficulties, one must generally assume that the planform is fixed and one studies the effects of variations in the boundary conditions at the wing surface.

This situation is not very satisfactory. In the approximations proposed in this paper the parameters defining the planform and those defining the boundary conditions at the wing surface appear in the same way and can be given the same priority. In fact, planforms with curved, subsonic leading edges of algebraïc form and boundary conditions at the wing surface of polynomial form permit a reduction of the boundary value problems to purely algebraïc problems. The number of algebraïc problems, which finally result, is related to the order of approximation desired.

1.2 The governing equation

In linearized steady supersonic wing theory the perturbation potential must be a solution of

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \qquad (1.1)$$

with $\beta^2 = M^2 - 1$, $M = \frac{U}{a}$ (a is the velocity of sound).

The trirectangular system $(x, y, z.)$ is fixed to a convenient time averaged position of the wing and moves with a constant speed U in the negative x-direction with respect to the atmosphere.

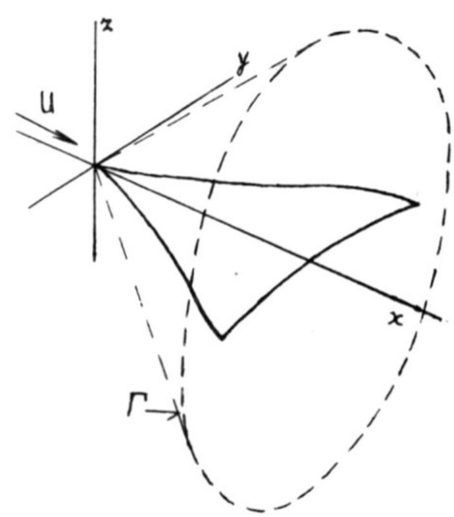


fig. 1

The assumptions that lead to equation (1.1) are discussed in many textbooks and are not repeated here.
If the (x,y,z) system has its origin at the apex of the wing the envelope of the disturbances is given by the Mach cone, Γ,

$$x^2-\beta^2(y^2+z^2)=0 \qquad (1.2)$$

The equation (1.1) is a second order linear partial differential equation of hyperbolic type with constant coefficients. This equation occurs in mathematical physics and has been studied extensively. If one interprets x as a time variable one establishes the analogy with the two dimensional wave equation. On the other hand by putting $x'=\frac{i\beta}{x}$ one establishes a formal analogy with the three dimensional Laplace equation $(i^2=-1)$.
The initial value and boundary value problems however are specific for supersonic wing theory.

1.3 The boundary conditions

The boundary condition at the wing surface follows from the requirement that the flow must be tangential to the wing surface. The angle between the normal to the wing-surface and the z-axis is small. Near round leading edges this assumption is violated but the region in which this happens will, in general, be relatively small. Moreover, if the perturbation velocity is small with respect to U, the boundary condition can be put in the form

$$w=\varphi_z\approx -U\alpha(x,y), \qquad (1.3)$$

$\alpha(x,y)$ being the local angle of incidence.
If the wing surface is given by $z=g(x,y)$ one has

$$g_x\approx -\alpha(x,y) \qquad (1.4)$$

The plane $z=0$ can be taken close to the wing surface so that the boundary condition can be applied at the projection, S, of the wing planform.

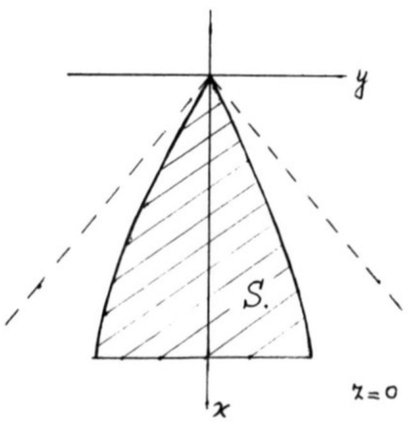


fig. 2

At the envelope of the disturbances, generated by the wing, , the boundary condition is $\varphi=0$.

1.4 The four types of problems

The pressure distribibution is related to the perturbation velocity potential by the linearized Bernoulli equation:

$$p'=-\rho U\varphi_x, \qquad (1.5)$$

ρ being the density of the air.
Equations (1.3) and (1.5) suggest two types of problems. If the wing geometry is given one knows φ_z at the wing surface. The problem of finding the corresponding pressure distribution on the wing is called the direct problem. If the pressure distribution is prescribed and one is asked to find the geometry of the wing generating this pressure distribution, one calls it the inverse problem. The perturbation potential $\varphi(x,y,z)$ of a flow around a planar wing which lies near the plane $z=0$ can be considered as the sum of an even and an odd part:

$$\varphi=\varphi^{(e)}+\varphi^{(o)} \qquad (1.6a)$$

with

$$\left.\begin{aligned} +\ \varphi^{(e)}(x,y,z)&=\varphi^{(e)}(x,y,-z)\\ -\ \varphi^{(o)}(x,y,z)&=\varphi^{(o)}(x,y,-z)\end{aligned}\right\} (1.6b)$$

The first is even in z and is associated with obstacles symmetric with respect to z; it is referred to as the thickness case. The second is odd in z and is associated with wings without thickness; it is referred to as the lifting case.
It follows that there are four types of problems:

(i) The direct thickness problem (D.T.P.).
(ii) The direct lifting problem (D.L.P.).
(iii) The inverse thickness problem(I.T.P.).
(iv) The inverse lifting problem (I.L.P.).

1.5 Outline

If the leading edges are straight and form the same angle with the x-axis, the homogeneous flow theory of P.Germain and M. Fenain leads to a systematic treatment for a very large class of problems. It is possible to handle the solutions efficiently and therefore they are very useful for the aircraft designer. It is clearly attractive to exploit the possibilities of homogeneous flow theory to the utmost.
If the planform of the wing is delta like and the leading edges are only slightly curved the solutions will differ little form the homogeneous flow solutions. It will be shown that for quasi-conical planforms it is possible to construct quasi-homogeneous approximations which can be calculated analytically in a workable scheme. This approach seems to be the most natural one for a large class of wing planforms of practical interest, say of the Concorde type. Before proceeding to the description of this method it is necessary to explain some important features of homogeneous flow theory.

II An outline of homogeneous flow theory.

P. Germain [1] generalized Busemann's conical flow theory into the theory of homogeneous flows. The problem is reduced to the construction of analytic functions. Especially M. Fenain carried out many calculations which were synthesised in [2] in a very elegant form. The solutions can be expressed in terms of functions that can be determined once and for all and in terms of coefficients that are uniquely related to the boundary conditions at the wing surface. The problem is thus reduced to a purely algebraic one.

2.1 Definition of homogeneous flows.

Germain defined a homogeneous flow of order n as a flow in which the perturbation potential $\varphi_n(x_1, x_2, x_3.)$ is a homogeneous function of degree n in the variables x_1, x_2, x_3. This potential satisfies the equation

$$f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^n f(x_1, x_2, x_3). \quad (2.1)$$

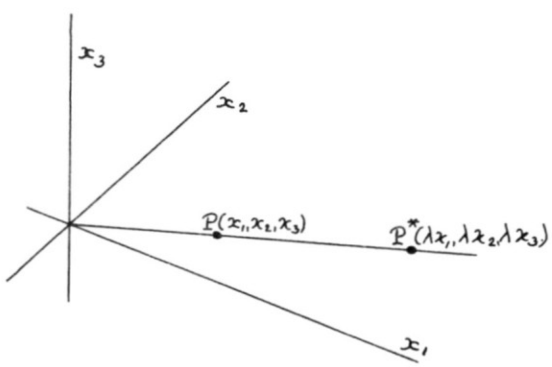


fig. 3.

Differentiating (2.1) with respect to λ and putting $\lambda = 1$ one obtains Euler's relation:

$$x_1 f_{x_1} + x_2 f_{x_2} + x_3 f_{x_3} = n f. \quad (2.2)$$

Equations (2.1) and (2.2) are equivalent, they can be deduced from each other.

2.2 The n^{th} derivatives.

For natural numbers n, it follows that all n^{th} derivatives with respect to x_1, x_2 and x_3 are homogeneous functions of order zero. They are constants on straight lines through the origin. If φ_n is a solution of

$$\beta^2 \varphi_{n x_1 x_1} - \varphi_{n x_2 x_2} - \varphi_{n x_3 x_3} = 0, \quad (2.3)$$

the n^{th} derivatives

$$\varphi_n^{(n-p-q,p,q)} = \frac{\partial^n \varphi_n}{\partial x_1^{n-p-q} \partial x_2^p \partial x_3^q}, \quad (2.4)$$
$$(0 \leq p+q \leq n)$$

are also solutions of (2.3).

The boundary conditons are specified at $x_3 = 0$ and one is primarily interested in the solutions at the wing surface. The most important n^{th} derivatives are:

$$\begin{cases} u_{nq} = \varphi_n^{(n-q,q,0)} = \dfrac{\partial^{n-1} u_n}{\partial x_1^{n-1-q} \partial x_2^q}, \\ w_{ns} = \varphi_n^{(n-1-s,s,1)} = \dfrac{\partial^{n-1} w_n}{\partial x_1^{n-1-s} \partial x_2^s}, \end{cases} \quad (2.5)$$

with $u_n = \dfrac{\partial \varphi_n}{\partial x_1}$ and $w_n = \dfrac{\partial \varphi_n}{\partial x_3}$.

2.3 Reduction to the construction of analytic functions.

It is usual to introduce the coordinates r, χ and θ by

$$\begin{cases} x_1 = \beta r \chi, & \chi > 0, \\ x_2 = r\cos\theta, & -\pi \leq \theta \leq \pi. \\ x_3 = r \sin\theta. \end{cases} \quad (2.6)$$

In this way a one-one relationship is established between a pair of values of χ, θ and a straight line through the origin. The Mach cone Γ, corresponds to $\chi = 1$. Only the interior of Γ, $\chi > 1$, is considered. The n^{th} derivatives (2.4) depend on χ and θ only and satisfy

$$(\chi^2 - 1) f_{\chi\chi} + \chi f_\chi + f_{\theta\theta} = 0. \quad (2.7)$$

With $\chi = \cosh\psi$ one obtains

$$f_{\psi\psi} + f_{\theta\theta} = 0, \quad (2.8)$$

and the problem is reduced to the construction of harmonic functions. Analytic functions can be introduced. Two conformal mappings are of special interest:

(i) $Z = X + iY = e^{-\psi} e^{i\theta} = \rho e^{i\theta}$. (2.9)

The Z plane can be used with advantage for the construction of the form of the solutions.

(ii) $\tilde{z} = \tilde{x} + i\tilde{y} = 2Z(1+Z^2)^{-1}$ (2.10)

The $\tilde{z}$ plane is more profitable for practical calculations.

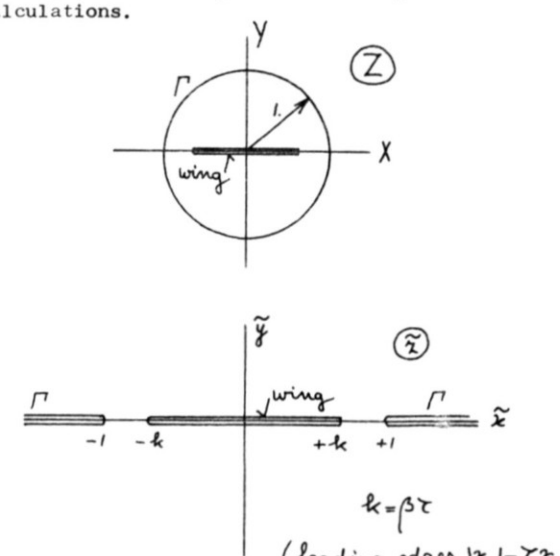


fig. 4

2.4 The boundary conditions.

(i) On Γ one has $\varphi_n^{(n-p-q,p,q)} = 0$, and especially $u_{nq} = 0$, $\omega_{ns} = 0$.
(ii) At the wingsurface, all ω_{ns} are known in a D.P. In an I.P. all u_{nq} will be known.
(iii) Outside of the wing ($x_3 = 0$) one has in a L.P.: $u_{nq} = 0$ and in a T.P.: $\omega_{ns} = 0$.

The D.T.P. and the I.L.P. are relatively simple because the boundary conditions are known for the same functions at all points of the boundary. The D.L.P. and the I.T.P. are more difficult; they are of mixed type. In order to solve these problems the compatibility conditions are needed.

2.5 The compatibility relations.
The n^{th} derivatives $\varphi_n^{(n-p-q,p,q)}$ are harmonic functions which can be considered as the real parts of analytic functions $\Phi_n^{(n-p-q,p,q)}$ say of Z or $\tilde{z}$. All $\varphi_n^{(n-p-q,p,q)}$ are derivatives of the same φ_n and are therefore not independent. In fact, one can prove the equivalence of the expressions,

$$d\Phi_n^{(n,0,0)} = (-1)^p \left(\frac{\tilde{z}}{\beta}\right)^{p+q} \left(\frac{i}{\sqrt{1-\tilde{z}^2}}\right)^q d\Phi_n^{(n-p-q,p,q)} \qquad (2.11)$$

which implies that knowing one of the n^{th} derivatives, all the other derivatives follow with (2.11).
With $\begin{cases} u_{nq} = R.P.\, U_{nq}, \\ \omega_{ns} = R.P.\, W_{ns}, \end{cases}$ one may write

$$\left(-\frac{\tilde{z}}{\beta}\right)^q \frac{dU_{nq}}{d\tilde{z}} = \left(-\frac{\tilde{z}}{\beta}\right)^{s+1} \left(\frac{-i}{\sqrt{1-\tilde{z}^2}}\right) \frac{dW_{ns}}{d\tilde{z}}, \qquad (2.12)$$

which relates the pressure distribution to the upwashfield.

2.6 Euler's relation.
From $n\varphi_n = x_1\varphi_{n x_1} + x_2\varphi_{n x_2} + x_3\varphi_{n x_3}$, one easily deduces:

$$n!\,\varphi_n = \left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)^n \varphi_n$$

$$= \left\{x_1\varphi_n^{(1,0,0)} + x_2\varphi_n^{(0,1,0)} + x_3\varphi_n^{(0,0,1)}\right\}^n, \qquad (2.13)$$

with the notation

$$\left\{\varphi_n^{(1,0,0)}\right\}^q \left\{\varphi_n^{(0,1,0)}\right\}^r \left\{\varphi_n^{(0,0,1)}\right\}^s = \varphi_n^{(q,r,s)}.$$

φ_n can be obtained from the n^{th} derivatives without carrying out any integrations.
For $x_3 = 0$, the relations (2.13) simplify and by comparison with (2.5) one obtains:

$$\begin{cases} u_n = \displaystyle\sum_{q=0}^{n-1} \frac{x_1^{\,n-1-q}\, x_2^{\,q}}{(n-1-q)!\,q!}\, u_{nq} \\ \omega_n = \displaystyle\sum_{s=0}^{n-1} \frac{x_1^{\,n-1-s}\, x_2^{\,s}}{(n-1-s)!\,s!}\, \omega_{ns} \end{cases} \qquad (2.14)$$

2.7 Elementary flows.
Homogeneous flows are called "elementary" if the first derivatives of φ_n for which the boundary conditions are specified at the wing surface, are homogeneous polynomials, of degree (n-1), in x_1 and x_2. This implies that for elementary flows:
(i) in a D.P. the ω_{ns} are constants on the wing.
(ii) in an I.P. the u_{nq} are constants on the wing.
Problems with boundary conditions that are odd in x_2 only make sense in cases without thickness. They are referred to as D.$\tilde{L}$.P.
The coefficients introduced into the boundary conditions are easily related to the u_{nq} or the ω_{ns}.
For instance, in a D.L.P. with

$$\omega_n = \sum_{s=0}^{n-1} c^*_{n-1-s,s}\, x_1^{\,n-1-s} \left|\frac{x_2}{\tau}\right|^s, \qquad (2.15a)$$

one obtains upon comparison with (2.14):

$$\omega_{ns} = \frac{(n-1-s)!\, s!\, |x_2|^s}{\tau^s\, x_2^s}\, c^*_{n-1-s,s}. \qquad (2.15b)$$

There are n coefficients $c^*_{n-1-s,s}$.

2.8 The form of the solutions
From the compatibility relations it is clear that in order to determine the form of the solutions it is sufficient to operate on one n^{th} derivative of φ_n only, say W_{n0}.
In a D.L.P. for instance one obtains the result (with 2.12)

$$\frac{dW_{ns}}{d\tilde{z}} = \frac{2i}{\pi}(-1)^{s-1}\left(\frac{\beta}{\tilde{z}}\right)^s \sqrt{\frac{1-\tilde{z}^2}{k^2-\tilde{z}^2}} \sum_{p=1}^{n} \frac{\lambda^*_{np}\, k^{2p}}{(\tilde{z}^2-k^2)^p}. \qquad (2.16a)$$

There are n real coefficients λ^*_{np} that can be used to satisfy the boundary conditions (2.15).
The strongest singularity ($p = n$) that is admitted at the leading edge is a square root singularity in the first derivatives of φ_n.
All W_{ns} are imaginary on Γ. With the compatibility relations (2.12) one verifies that all u_{nq} are zero outside of the wing ($\tilde{y} = 0$).
For an I.T.P. one obtains:

$$\frac{dU_{nq}}{d\tilde{z}} = \frac{2i\tau}{\pi}(-1)^q\left(\frac{\beta}{\tilde{z}}\right)^q \sqrt{\frac{k^2-\tilde{z}^2}{1-\tilde{z}^2}} \sum_{p=1}^{n+1} \frac{l_{np}\, k^{2p-2}}{(\tilde{z}^2-k^2)^p} \qquad (2.16b)$$

In the D.T.P. and the I.L.P. one admits logarithmic singularities in the first derivatives of φ_n and obtains

(D.T.P.) $$\frac{dW_{ns}}{d\tilde{z}} = \frac{2i}{\pi}(-1)^{s-1}\left(\frac{\beta}{\tilde{z}}\right)^s \sum_{p=1}^{n} \frac{\lambda_{np}\, k^{2p-1}}{(k^2-\tilde{z}^2)^p} \qquad (2.16c)$$

(I.L.P.) $$\frac{dU_{nq}}{d\tilde{z}} = \frac{2i\tau}{\pi}\left(\frac{-\beta}{\tilde{z}}\right)^q \sum_{p=1}^{n} \frac{\nu^*_{np}\, k^{2p-1}}{(\tilde{z}^2-k^2)^p} \qquad (2.16d)$$

One can adjoin:

(D.$\tilde{L}$.P.) $$\frac{dW_{ns}}{d\tilde{z}} = -\frac{2i}{\pi}\frac{1}{\tilde{z}}\left(\frac{-\beta}{\tilde{z}}\right)^s \sqrt{\frac{1-\tilde{z}^2}{k^2-\tilde{z}^2}} \sum_{p=1}^{n} \frac{\tilde{\lambda}^*\, k^{2p+1}}{(\tilde{z}^2-k^2)^p} \qquad (2.16e)$$

The relations between the coefficients introduced in the boundary conditions and those in (2.16) can now be determined:
In the D.T.P. and the I.L.P. the relations can be given explicitly (see section 2.10).
In a D.L.P. and an I.L.P., the problems of mixed type, the situation is more complicated. One must solve n equations with n unknown coefficients in the D.L.P., and $(n+1)$ equations with $(n+1)$ unknowns in the I.T.P.

2.9 The expression of the solutions.

The coefficients, introduced in the preceding section uniquely determine the solutions. In the D.P. for instance, the $\frac{dU_{nq}}{d\tilde{z}}$ are found from the compatibility relations. The functions U_{nq} can be determined by integration from Γ to a point $\tilde{z}$. Evaluation of all U_{nq} involves the calculation of n^2 integrals which is very laborious.
In a D.L.P., for instance, this difficulty can be circumvented by the introduction of

$$Q_n(\tilde{z}) = \frac{-1}{\tau}\sum_{q=0}^{n-1}\frac{U_{nq}}{(n-1-q)!\,q!}\left(\frac{\tilde{z}}{\beta}\right)^q. \quad (2.17)$$

At the image of the wing one has

$$u_n^+ = -\tau x_1^{n-1} Q_n(\tilde{z}).$$

From (2.19) one obtains

$$\frac{dQ_n}{d\tilde{z}} = \frac{-1}{\tau}\sum_{q=0}^{n-2}\frac{U_{n,q+1}\tilde{z}^q}{(n-2)!\,q!\,\beta^{q+1}}, \quad (2.18a)$$

and by successive differentiations:

$$\frac{d^nQ_n}{d\tilde{z}^n} = \frac{-1}{\tau\beta^{n-1}}\frac{dU_{n,n-1}}{d\tilde{z}} = \frac{-1}{\tau}\left(\frac{-1}{\tilde{z}}\right)^{n-1}\frac{dU_{no}}{d\tilde{z}}. \quad (2.18b)$$

From the compatibility relations one obtains

$$\frac{dU_{no}}{d\tilde{z}} = \frac{2}{\pi}\left(\frac{\beta}{\tilde{z}}\right)^{-1}\sum_{p=1}^{n}\frac{(-1)^p k^{2p}\lambda^*_{np}}{(k^2-\tilde{z}^2)^{p+1/2}}. \quad (2.19)$$

From (2.18b) and (2.19) one obtains

$$\frac{d^nQ_n}{d\tilde{z}^n} = \frac{2}{\pi}\sum_{p=1}^{n}\frac{(-1)^{p+n}k^{2p-1}\lambda^*_{np}}{\tilde{z}^{n-2}(k^2-\tilde{z}^2)^{p+1/2}}. \quad (2.20a)$$

By putting

$$\frac{d^nQ_n}{d\tilde{z}^n} = \frac{2}{\pi}\sum_{p=1}^{n}\lambda^*_{np}\frac{d^nQ_{np}}{d\tilde{z}^n}, \quad (2.20b)$$

one finds that $Q_{np}(\tilde{z})$ must satisfy

$$\frac{d^nQ_{np}}{d\tilde{z}^n} = \frac{(-1)^{p+n}k^{2p-1}}{\tilde{z}^{n-2}(k^2-\tilde{z}^2)^{p+1/2}}. \quad (2.21)$$

The solution at the wing surface then becomes

$$u_n^+ = -\frac{2\tau}{\pi}x_1^{n-1}\sum_{p=1}^{n}\lambda^*_{np}Q_{np}(\tilde{z}), \quad (2.22a)$$

$$\text{or } \varphi_n^+ = -\frac{2\tau}{\pi}x_1^{n}\sum_{p=1}^{n}\lambda^*_{np}Q_{n+1,p}(\tilde{z}). \quad (2.22b)$$

The terms in the solutions have thus been arranged with respect to the coefficients which are connected to the boundary conditions.
From equation (2.21) one can deduce a number of recurrence relations so that only three simple integrations need to be carried out. Thus the problem is reduced to a completely algebraic one. The same can be shown for all other elementary problems by arguments analogous to those used in this section. Moreover, Fenain showed in [2] that the results can be unified by the introduction of the function $K_{np}^{(r,s,t)}$ with $\rho = \frac{\tilde{z}}{k} = \xi + i\eta$, which satisfies the equation:

$$\frac{d^nK_{np}^{(r,s,t)}}{d\rho^n} = \frac{(-1)^{n+p}\rho^{r-n}}{(1-\rho^2)^{p+s}(1-k^2\rho^2)^t} \quad (2.23)$$

To avoid cunfusion of the functions to be used in the different problems, the following notations are used:

$$\left\{\begin{array}{llll}
\text{D.T.P.} & Q_{np} = F_{np}(\rho,k) = K_{np}^{2,0,1/2} & & \\
\text{D.L.P.} & Q_{np} = F^*_{np}(\rho) = K_{np}^{2,1/2,0} & & \\
\text{I.L.P.} & S_{np} = H^*_{np}(\rho,k) = -K_{np}^{0,0,-1/2} & & (2.24)\\
\text{I.T.P.} & S_{np} = H_{np}(\rho) = -K_{np}^{0,-1/2,0} & & \\
\text{D.}\tilde{\text{L}}\text{.P.} & Q_{np} = \tilde{F}^*_{np}(\rho) = K^{1,1/2,0} & &
\end{array}\right.$$

Equation (2.23) makes it possible to deduce a number of recurrence relations which allows the evaluation of all the required functions once and for all up to any desired degree of homogeneity. Only a small number of integrations has to be carried out to provide a starting point.

2.10 Summary of the results.

D.T.P. Boundary conditions:

$$w_n^+ = \sum_{s=0}^{n-1}c_{n-1-s,s}\,x_1^{n-1-s}\left|\frac{x_2}{\tau}\right|^s. \quad (2.25a)$$

Solution:

$$u_n = -\frac{2\tau}{\pi}x_1^{n-1}\sum_{p=1}^{n}\lambda^{np}F_{np}. \quad (2.25b)$$

with
$$\lambda_{np} = \sum_{s=0}^{n-1}c_{n-1-s,s}\sum_{t=0}^{p}\frac{(-1)^{p-t}(p-1)!\,(n+2t-2)!}{(2t+s-1)(p-t)!\,(t-1)!\,(2t-2)!}; \quad (2.25c)$$

D.L.P. Boundary conditions:

$$w_n = \sum_{s=0}^{n-1}c^*_{n-1-s,s}\,x_1^{n-1-s}\left|\frac{x_2}{\tau}\right|^s. \quad (2.26a)$$

Solution:

$$u_n^+ = -\frac{2\tau}{\pi}x_1^{n-1}\sum_{p=1}^{n}\lambda^*_{np}F^*_{np}. \quad (2.26b)$$

The coefficients λ^*_{np} follow from

$$\sum_{p=1}^{n}(-1)^{p-1}\alpha_p^s\lambda^*_{np} = (n-1-s)!\,s!\,c^*_{n-1-s,s}. \quad (0\le s\le n-1) \quad (2.26c)$$

D.$\tilde{\text{L}}$.P. This problem is formally equivalent to the D.L.P. if one replaces $c^*_{n-1-s,s}$ by $\tilde{c}^*_{n-1-s,s}$, λ^*_{np} by $\tilde{\lambda}^*_{np}$, F^*_{np} by $\tilde{F}^*_{np}$ and α_p^s by α_p^{s+1}.

I.T.P. Boundary conditions:

$$\varphi_n = -\tau\sum_{t=0}^{n}\alpha_{n-t,t}\,x_1^{n-t}\left|\frac{x_2}{\tau}\right|^t. \quad (2.27a)$$

Solution:

$$w_n = \frac{2}{\pi}x_1^{n-1}\sum_{p=1}^{n}\ell_{np}H_{np}. \quad (2.27b)$$

The coefficients ℓ_{np} follow from

$$\sum_{p=1}^{n+1}(-1)^{p-1}\ell_{np}\,\bar{\alpha}_p^t = (n-t)!\,t!\,\alpha_{n-t,t} \quad (2.27c)$$
$$(0\le t\le n)$$

I.L.P. Boundary conditions:

$$u_n^+ = -\tau \sum_{q=0}^{n-1} a^*_{n-1-q,q}\, x_1^{n-1-q} \left|\frac{x_2}{\tau}\right|^q. \qquad (2.28a)$$

Solution:

$$w_n = \frac{2}{\pi} x_1^{n-1} \sum_{p=1}^{n} \nu^*_{np} H^*_{np}, \qquad (2.28b)$$

with

$$\nu^*_{np} = \sum_{q=0}^{n-1} a^*_{n-1-q,q} \sum_{t=1}^{p} \frac{(-1)^{p-t}(p-1)!(n+2t-2)!}{(2t+q-1)(p-t)!(t-1)!(2t-2)!}. \qquad (2.28d)$$

The coefficients α_p^s and $\bar{\alpha}_p^t$ are given in [2]

III Wings with curved subsonic leading edges

3.1 Introductory remarks

If the leading edges of the wing are curved, as in the case of gothic and ogee planforms, the flow cannot in general be represented as a superposition of homogeneous flows. In the previous outline of homogeneous flow theory it was seen that discontinuities and singularities occur at the leading edges. It is impossible therefore, to write the solution in a form $\varphi = {}^{(0)}\varphi + \varepsilon\,{}^{(1)}\varphi + O(\varepsilon^2)$ (in x,y,z) with ${}^{(0)}\varphi$ as the solution for a delta wing with straight leading edges, in some way "close" to the actual leading edges, and ε as a small parameter representing a measure for the "deviation from straight". A solution of that type would posses singularities and discontinuities not on but only near the leading edges. A possibility to circumvent these difficulties is the introduction of a transformation shifting all points of the curved leading edges to straight lines through the origin. Moreover it will be required that the Mach cone through the origing remains a straight circular cone. The boundaries where boundary conditions can be imposed are then of precisely the same conical nature as those occurring in homogeneous flow theory. In the next sections it will be shown that the transformed differential equation and the transformed boundary conditions can be satisfied in terms of functions which are solutions of homogeneous flow problems.

3.2 Straightening the leading edges

The wing planforms considered will be symmetric with respect to the x-axis and the subsonic leading edges are given by

$$F(x,|y|) = 0, \quad (z=0), \qquad (3.1)$$

F being a rational function.
In [5] transformations in the form of general three dimensional power series are introduced as follows:

$$\begin{cases} x_1 = x + \sum\limits_{i+j+k=2} a_{ijk}\, x^i y^j z^k, \\ x_2 = y + \sum\limits_{i+j+k=2} b_{ijk}\, x^i y^j z^k, \\ x_3 = z + \sum\limits_{i+j+k=2} c_{ijk}\, x^i y^j z^k. \end{cases} \qquad (3.2)$$

In [6] these transformations are discussed systematically. The leading edges can be transformed into straight lines in the $(x, x_2 x_3)$ space $|x_2| = \tau x_1, (x_3 = 0)$, without deforming the Mach cone Γ

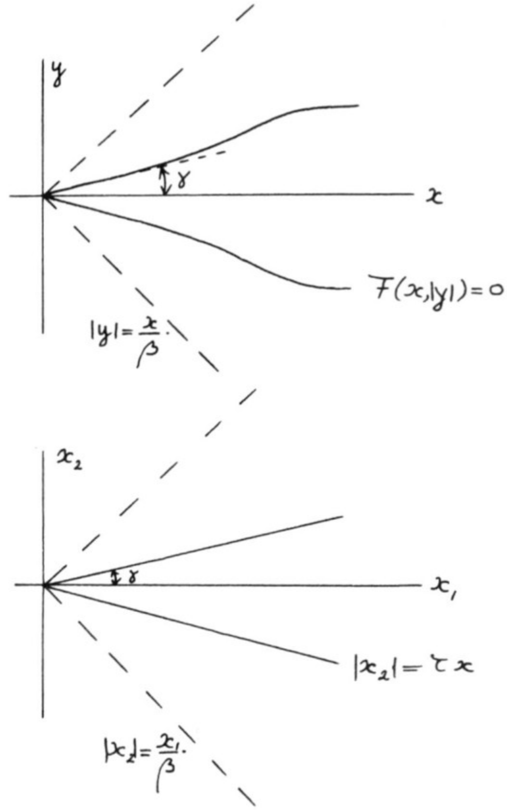


fig. 5

The fact that the transformations (3.2) are of polynomial form is important since boundary conditions at the wing surface, which are of polynomial form in the (x, y) coördinates lead to boundary conditions of polynomial form in the x_1, x_2 coordinates. In this case one may hope to construct solutions that can be related to elementary homogeneous flow solutions.
The requirements concerning the straightening of the leading edges, without deforming the Mach cone, reduce the number of arbitrary coefficients in (3.2) but do not uniquely determine any one of them.
The requirement to retain, in some, degree the symmetry of the flow field reduces the number of arbitrary coefficients still further but not sufficiently to determine them uniquely. In [6] it is shown, that, in fact it is sufficient to stretch the x coördinate only. An example is the transformation:

$$\begin{cases} x_1 = x + \dfrac{\{x^2 - \beta^2(y^2+z^2)\}\, f(x)}{\tau[x^2 - \beta^2\{\tau x + f(x)\}^2]} \\ x_2 = y, \quad x_3 = z. \end{cases} \qquad (3.4)$$

The formulae (3.4) straighten leading edges of the form

$$|y| = \tau x + f(x), \quad (z = 0)$$

With $f(x) = \sum_{i=2} a_i x^i$, (3.4) can be written in the form (3.2) for

$$(1 - \beta^2\tau^2)x^2 > \beta^2 f(x)\{2\tau x + f(x)\} \qquad (3.4^*)$$

Other possibilities and relations between the leading edges and the transformations are discussed in [6].

3.3 The transformed equation

If the x-coordinate alone is stretched the transformed differential equation becomes :

$$\tilde{\varphi}_{x_1x_1}\left\{\beta^2\left(\frac{\partial x_1}{\partial x}\right)^2 - \left(\frac{\partial x_1}{\partial y}\right)^2 - \left(\frac{\partial x_1}{\partial z}\right)^2\right\} - \tilde{\varphi}_{x_2x_2} - \tilde{\varphi}_{x_3x_3} -$$

$$-2\tilde{\varphi}_{x_1x_2}\frac{\partial x_1}{\partial y} - 2\tilde{\varphi}_{x_1x_3}\frac{\partial x_1}{\partial z} + \tilde{\varphi}_{x_1}\left\{\beta^2\frac{\partial^2 x_1}{\partial x^2} - \frac{\partial^2 x_1}{\partial y^2} - \frac{\partial^2 x_1}{\partial z^2}\right\} = 0. \qquad (3.5)$$

3.4 Solutions of the transformed equation (i)

This equation admits solutions involving sums of solutions of

$$\beta^2\varphi_{x_1x_1} - \varphi_{x_2x_2} - \varphi_{x_3x_3} = 0, \qquad (3.6)$$

multiplied by power series in x_1 and $\beta^2(x_2^2 + x_3^2) = \beta^2\rho^2$:

$$\tilde{\varphi} = \sum_n \sum_{p=0} \frac{\partial^p \varphi_n}{\partial x_1^p}(x_1^2 - \beta^2\rho^2)^p \sum_{i+2j} \alpha_{i,2j}^{(p,n)} x_1^i (\beta^2\rho^2)^j, \qquad (3.7)$$

in which φ_n is a solution of (3.6).
Substitution of (3.7) into (3.5) permits a unique determination of the coefficients $\alpha_{i,2j}^{(p,n)}$
and the homogeneous flow solutions $\varphi(x_1, x_2, x_3)$ can be used to satisfy the boundary conditions at the wing surface. The procedure is rather laborious, but the coefficients can be determined once and for all. The solution involves the solution of systems of equations which beyond $i + 2j + p = 4$ become too complicated for analytical treatment. In the case of hyperbolic leading edges, however, only coefficients $\alpha_{i,2j}^{(0,n)}$
arise and these can be determined analytically for arbitrary values of n, i and j.
In this case the leading edges are given by

$$\left(\beta^2 y^2 + \frac{|y|}{a\tau}\right) - \left(x^2 + \frac{x}{a}\right) = 0, \quad (z = 0),$$

The corresponding transformation is:

$$\begin{cases} x_1 = x + a\{x^2 - \beta^2\rho^2\}, \\ x_2 = y, \quad x_3 = z. \end{cases}$$

The solution is:

$$\tilde{\varphi} = \sum_n \varphi_n \sum_{i+2j=0} \alpha_{i,2j}^{(0,n)} x_1^i (\beta^2\rho^2)^j,$$

$$\text{with } \alpha_{i,2j}^{(0,n)} =$$

$$= (-a)^{i+2j}(-1)^j(2n+1)\frac{(2n+2i+4j-1)!(n+i+j)!}{2^{2j-1}\,i!\,j!\,(n+i+2j-1)!(2n+i+2j+1)!}$$

The boundary value problems for φ_n are formally identical to the problems of homogeneous flow theory.

3.5 Solutions of the transformed equation (ii)

Another approach to construct solutions of the transformed differential equation (3.5) consists of the following argument:
Consider a function $\psi_n(x, y, z)$, homogeneous of degree n in x , y and z , which is a solution of

$$\beta^2\psi_{nxx} - \psi_{nyy} - \psi_{nzz} = 0. \qquad (3.8)$$

The new coordinates (x_1, x_2, x_3) can be introduced by an inverse transformation of the form

$$\begin{cases} x = x_1 + (x_1^2 - \beta^2\rho^2)\,G(x_1, \beta^2\rho^2), \\ y = x_2, \quad z = x_3. \end{cases} \qquad (3.9)$$

$$G = \sum g_{i,2j}\, x_1^i (\beta^2\rho^2)^j$$

Without actually calculating the transformed differential equation one obtains solutions of the transformed equation by introducing x_1, x_2 and x_3 into $\psi_n(x, y, z)$.
Expanding $\psi_n(x, y, z)$ in terms of the new coordinates gives:

$$\psi_n(x, y, z) = \psi_n\{x_1 + (x_1^2 - \beta^2\rho^2)G(x_1, \beta^2\rho^2), x_2, x_3\}$$

$$= \psi_n(x_1, x_2, x_3) + (x_1^2 - \beta^2\rho^2)G(x_1, \beta^2\rho^2)\psi_{nx_1} + \cdots$$

$$= \sum_{i=0}^{N}(x_1^2 - \beta^2\rho^2)^i\{G(x_1, \beta^2\rho^2)\}^i \frac{1}{i!}\frac{\partial^i \psi_n}{\partial x_1^i} + R_{N+1}. \qquad (3.10)$$

Because $\psi_n(x, y, z)$ is a solution of (3.8), $\psi_n(x_1, x_2, x_3)$ is a solution of

$$\beta^2\psi_{nx_1x_1} - \psi_{nx_2x_2} - \psi_{nx_3x_3} = 0 \qquad (3.11)$$

The derivatives $\dfrac{\partial^p \psi_n}{\partial x_1^p}$ are also solutions of (3.11). The solutions (3.10) can be shown to be equivalent to (3.7).
If the transformation (3.9) straightens a class of leading edges, the solutions (3.10) can be used to solve the transformed problem. If a homogeneous function $\psi_n(x_1, x_2, x_3)$ admits a singularity in a point (x_1, x_2, x_3) then the same type of singularity occurs for all points $(\lambda x_1, \lambda x_2, \lambda x_3)$.
Thus the solutions (3.10) can be used to locate the singularities in the solutions at the transformed leading edges.

It is easily verified that the inverse transformation (3.9) straightens leading edges of the form:

$$x - \frac{|y\beta|}{k} - \left(\frac{\beta^2 y^2}{k^2} - \beta^2 y^2\right) G\left\{\frac{|y\beta|}{k}, \beta^2 y^2\right\} = 0. \qquad (3.12a)$$

The expression (3.12) can be put in the form

$$\frac{|y\beta|}{kx} = 1 - \frac{\beta^2 y^2 (1-\beta^2\tau^2)}{k^2 x} G\left(\frac{|y\beta|}{k}, \beta^2 y^2\right). \qquad (3.12b)$$

With

$$\frac{\beta^2 y^2 (1-\beta^2\tau^2)}{k^2 x} G\left(\frac{|y\beta|}{k}, \beta^2 y^2\right) \ll 1,$$

the "deviation from straight" can be considered to be small.
On the other hand, the expansion (3.10) suggests that for rapid convergence $\frac{x_1^2 - \beta^2 \rho^2}{x_1} G(x_1, \beta^2\rho^2)$ should not become too large with respect to unity. Near the leading edges the second criterium coïncides with the first.
These criteria can be formulated in dimensionless form but it seems to be sufficient, in practical applications, to truncate the solutions at $x = 1$, say, and take the coefficients $g_{i,2j}$ sufficiently small.
If these coefficients are taken small of $O(\varepsilon)$ one may put:

$$\tilde{\Psi}_n = \Psi_n(x_1, x_2, x_3) + (x_1^2 - \beta^2\rho^2) G(x_1, \beta^2\rho^2) \Psi_{n x_1} + O(\varepsilon^2) \qquad (3.13)$$

If the leading edges are only slightly curved, (3.13) will be sufficient. The expression (3.10) makes it possible however to calculate higher order terms.

3.6 The boundary conditions

If, in a D.P. one has in the physical space at the wing surface $\omega_n^+(x, y)$, the transformed boundary condition becomes

$$\omega_n^+\{x_1 + (x_1^2 - \beta^2 x_2^2) G(x_1, \beta^2 x_2^2), x_2\} =$$

$$= \sum_{i=0} (x_1^2 - \beta^2 x_2^2)^i \{G(x_1, \beta^2 x_2^2)\}^i \frac{1}{i!} \frac{\partial^i \omega_n^+}{\partial x_1^i}.$$

Thus the solution (3.10) satisfies the transformed boundary conditions at the wing surface if φ_n satisfies the boundary condition.

$$\frac{\partial \varphi_n^+}{\partial x_3}(x_1, x_2) = \omega_n^+(x_1, x_2). \qquad (3.14)$$

One simply replaces the variables x and y in the boundary conditions in the physical space by the variables x_1 and x_2 to obtain the boundary conditions for $\varphi_n(x_1, x_2, x_3)$ at the transformed wing surface. The same is true for inverse problems.
A difficulty in the solutions (3.10) is the occurrance of too strong singularities. The problems that arise for the functions $\varphi_n(x_1, x_2, x_3)$ differ from the ones in homogeneous flow theory only by the fact that the too strong singularities must be compensated for.

The too strong singularities can be partly removed as follows:
If Ψ_n is a solution of (3.11), homogeneous of degree n in x_1, x_2, x_3, a solution of (3.11), homogeneous of degree (n+1), is obtained by

$$\Psi_{n+1} = -(2n+1) x_1 \Psi_{n x_1} + (x_1^2 - \beta^2\rho^2) \Psi_{n x_1 x_1} \qquad (3.15)$$

The singularity in Ψ_{n+1} is stronger than in Ψ_n if there is one. (3.15) is a recurrence relation and can be used to generate solutions that can be substituted into (3.10). They can be multiplied by constants, chosen in such a way that the strongest singularities are removed. In general, however, too strong singularities remain present and must be removed by another technique, based one a detailed study of the behaviour of the functions $K_{np}^{r,s,t}$ near the leading edges, defined in equation (2.23).
The boundary conditions at the Mach cone present no difficulties and are automatically satisfied if the $\Psi_n(x_1, x_2, x_3)$ are homogeneous flow solutions (the n^{th} derivatives being zero on Γ).

3.7 Concluding remarks

In the preceding sections it was seen that the transformed equation admits solutions in which the homogeneous flow solutions occur in a simple way. Orders of approximation can be associated with the number of terms included.
The terms in the solutions can be arranged with respect to ascending degrees of homogeneïty and solved successively by the same techniques as those applied in Fenain's theory. Up to a certain order of approximation, the problem can be reduced to a completely algebraïc one. In these problems the parameters defining the leading edges can be made to occur in the same way as those defining the boundary conditions.

IV Applications

4.1 Slightly curved leading edges

Leading edges of the form

$$|y| = \tau x + \varepsilon f(x), (z = 0), \qquad (4.1)$$

with $f(x) = \sum_{i=2} a_i x^i$ are straightened by the transformation

$$\begin{cases} x_1 = x + \varepsilon \dfrac{\{x^2 - \beta^2\rho^2\} f(x)}{\tau[x^2 - \beta^2\{\tau x + \varepsilon f(x)\}^2]}, \\ x_2 = y, \ x_3 = z, \end{cases} \qquad (4.2)$$

into $|x_2| = \tau x_1, \ (x_3 = 0).$ (4.3)

For $0 \leq x \leq 1$, with fixed coefficients a_i, the leading edges will be slightly curved for sufficiently small ε.
The inverse of (4.2) is, including terms of $O(\varepsilon)$:

$$\begin{cases} x = x_1 - \varepsilon \dfrac{\{x_1^2 - \beta^2\rho^2\} f(x_1)}{\tau x_1^2 (1-\beta^2\tau^2)} + O(\varepsilon^2), \\ y = x_2, \ z = x_3. \end{cases} \qquad (4.4)$$

The solution of the transformed problem can be put in the form: $\varphi = {}^{(0)}\varphi + \varepsilon\,{}^{(1)}\varphi.$ (4.5)

As an example, consider a D.L.P.
The boundary conditions can be put in the form

$$\omega_j = \sum_{s=0}^{j-1} c^*_{j-1-s,s}\, x^{j-1-s} \left|\frac{y}{\tau}\right|^s, \qquad (4.6)$$

and the leading edges in the form:

$$|y| = \tau x + \varepsilon a_i x^i \qquad (4.7)$$

${}^{(0)}\varphi$ will be homogeneous of degree j and we write ${}^{(0)}\varphi_j$

${}^{(1)}\varphi$ will be homogeneous of degree $(i+j-1)$ and we write ${}^{(1)}\varphi_{i+j-1}$

For ${}^{(0)}\varphi_j$ one has to solve the system

$$\sum_{p=1}^{j} (-1)^{p-1} \alpha_p^s\, {}^{(0)}\lambda^*_{jp} = (j-1-s)!\, s!\, c^*_{j-1-s,s} \qquad (4.8)$$

$$(0 \le s \le j-1)$$

The solution can be expressed as

$${}^{(0)}\varphi_j = -\frac{2\tau}{\pi} x_1^j \sum_{p=1}^{j} {}^{(0)}\lambda^*_{jp}\, F^*_{j+1,p}\,. \qquad (4.9)$$

The part of ${}^{(1)}\varphi_{i+j-1}$ generated by (3.10) can be considered as a particular solution of the terms of $O(\varepsilon)$ in the transformed equation ($x_3 = 0$).

$${}^{(1)}\varphi^{(part)}_{i+j-1} = \frac{2a_i}{\pi(1-\beta^2\tau^2)}(x_1^2-\beta^2 x_2^2)\,x_1^{i+j-3} \sum_{p=1}^{j} {}^{(0)}\lambda^*_{jp}\, F^*_{jp} \qquad (4.10)$$

The behaviour of F^*_{jj} near the leading edges is dominated by

$$(2j-1)!!\, F^*_{jj} \approx \frac{1}{\sqrt{1-\xi^2}}\,. \qquad (4.11)$$

To compensate for this too strong singularity ${}^{(1)}\varphi^{(hom)}_{i+j-1}$ must include a term with a square root singularity of opposite strenght at the leading edges. This is accomplished by taking

$${}^{(1)}\varphi^{(hom)}_{i+j-1} = -\frac{2\tau}{\pi} x_1^{i+j-1} \sum_{p=1}^{i+j} {}^{(1)}\lambda^*_{i+j-1,p}\, F^*_{i+j,p}, \qquad (4.12)$$

with

$${}^{(1)}\lambda^*_{i+j-1,i+j} = {}^{(0)}\lambda^*_{jj}\, a_i \frac{(2i+2j-1)!!}{\tau(2j-1)!!}\,. \qquad (4.13)$$

Now ${}^{(1)}\varphi^{(hom)}_{i+j-1}$ can be determined by solving the $(i+j-1)$ equations for the $(i+j-1)$ coefficients ${}^{(1)}\lambda^*_{i+j-1,p}$:

$$\sum_{p=1}^{i+j-1} (-1)^{p-1}\, {}^{(1)}\lambda^*_{i+j-1,p}\, \alpha_p^s = (-1)^{i+j} \alpha^s_{i+j}\, a_i\, {}^{(0)}\lambda^*_{jj} \frac{(2i+2j-1)!!}{\tau(2j-1)!!} \qquad (4.14)$$

The solution is given by

$$\varphi = -\frac{2\tau}{\pi} x_1^j \sum_{p=1}^{j} {}^{(0)}\lambda^*_{jp}\, F^*_{j+1,p} + \varepsilon \cdot \frac{2a_i(x_1^2-\beta^2x_2^2)\,x_1^{i+j-3}}{\pi(1-\beta^2\tau^2)} \sum_{p=1}^{j} {}^{(0)}\lambda^*_{jp}\, F^*_{jp} - \varepsilon \frac{2\tau}{\pi} x_1^{i+j-1} \sum_{p=1}^{i+j} {}^{(1)}\lambda^*_{i+j-1,p}\, F^*_{i+j,p}\,. \qquad (4.15)$$

The total solution can be obtained by summing over i and j.
If one wants to transform back to the physical coordinates one has:

$$\sqrt{\tau^2 x_1^2 - x_2^2} = \sqrt{\{\tau x + \varepsilon f(x)\}^2 - y^2}\,\left\{1 + \varepsilon f(x) \frac{\tau\beta^2}{x(1-\beta^2\tau^2)} + O(\varepsilon^2)\right\}. \qquad (4.16)$$

It is to be noticed that the singularities are located at the leading edges in the physical space.

The flat plate solution.

For the flat plate solution, or the incidence dependent part of the solution, one finds:

$$\varphi = -\frac{2\tau}{\pi} x_1\, {}^{(0)}\lambda^*_{11}\, F^*_{21} + \varepsilon \cdot \frac{2a_i(x_1^2-\beta^2x_2^2)\,x_1^{i-2}}{\pi(1-\beta^2\tau^2)}\, {}^{(0)}\lambda^*_{11}\, F^*_{11} - \varepsilon \cdot \frac{2\tau}{\pi} x_1^i \sum_{p=1}^{i+1} {}^{(1)}\lambda^*_{i,p}\, F^*_{i+1,p} \qquad (4.17)$$

Wings at ideal angle of attack

Solutions without square root singularities in the velocity field can be obtained for $j \geq 2$, by taking ${}^{(0)}\lambda^*_{jj} = 0$ in (4.8). From (4.13) one has ${}^{(1)}\lambda^*_{i+j-1,i+j} = 0$. By analogy with (4.13) one must put:

$${}^{(1)}\lambda^*_{i+j-1,i+j-1} = {}^{(0)}\lambda^*_{j,j-1}\, a_i \frac{(2i+2j-3)!!}{\tau(2j-3)!!} \qquad (4.18)$$

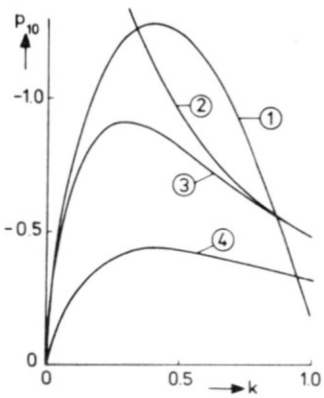
The solutions that can be constructed in this way may be interesting for the design of wings under cruising conditions.

For all other types of boundary value problems similar expressions can be derived [6].

4.2 Comparison with other results.

4.2.1 Consider a flat plate solution with leading edges.
$$|y| = \tau x + \varepsilon a_2 x^2. \qquad (4.19)$$
The solution can be expressed as follows $(w_1 = -U\alpha)$
$$\varphi(x,y,0^+) = -w_1\sqrt{(\tau x + \varepsilon a_2 x^2)^2 - y^2}\left\{\frac{1}{E'} + \varepsilon a_2 \beta p_{10} x + O(\varepsilon^2)\right\} \qquad (4.20)$$
with
$$p_{10} = \frac{2E'(k^3 - 2k) + K'(3k - k^3)}{(k^2-1)E'\{(1-2k^2)E' + k^2 K'\}}, \qquad (4.21)$$
with $k = \beta\tau$.

E' and K' have modulus $\sqrt{1-k^2}$. The factor p has been plotted and compared with other results:



① not so slender body theory
② first reflection integral
③ present theory
④ carafoli's result

fig. 6

In the limit $k \to 1$, the leading edges are nearly sonic. The factor p_{10}(4.21) approaches the corresponding factor extracted from the first reflexion integral, as it should. The slender body theory predicts $p_{10} = 0$ in the limit $k \to 0$. The not-so-slender body theory predicts values close to p_{10}(4.21), for small values of k. Carafoli's result is valid for k=0 only.

4.2.2 Wing at ideal angle of attack.
The formulae derived for the D.L.P. can be used to construct wings which sustain an interesting form of pressure distribution. In these cases the integral representation of the upwashfield permits an easy numerical evaluation. $(\beta = 1)$
Wings with a planform $|y| \leq 0{,}8x - 0{,}4x^2$, $(0 < x \leq 1)$ are considered. The pressure distribution is prescribed in the form
$$\varphi_x = (D_{00} + D_{10}x)\sqrt{\{0{,}8x - 0{,}4x^2\}^2 - y^2}. \qquad (4.22)$$
The corresponding φ_z is calculated in the form
$$\varphi_z = \lambda_1 x + \lambda_2|y| + \lambda_3 x^2 + \lambda_4 x|y| + \lambda_5 y^2 + \cdots \qquad (4.23)$$

The analytic expressions for the coefficients have been calculated including those associated with the third and fourth degree terms.
For $x = 0.4$; $x = 0.6$; $x = 0.8$; $x = 1$; φ_z is plotted for two cases
$$\begin{cases} (i)\ D_{00} = 1,\ D_{10} = 0, \\ (ii)\ D_{00} = 1,\ D_{10} = -1. \end{cases} \qquad (4.24)$$

The deviation of straight is not small and so for $x > 0.6\ldots$ requirement (3.4*) is not satisfied. Solutions in the form (3.7) are used.

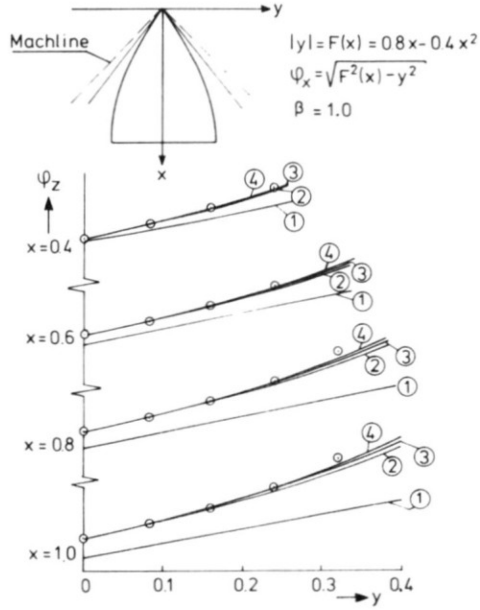


fig. 7

① $\lambda_1 x + \lambda_2 |y|$
② $+ \lambda_3 x^2 + \lambda_4 y^2$
③ $+ \lambda_5 x^3 + \lambda_6 xy^2 + \lambda_7 |y|^3$
④ $+ \lambda_8 x^4 + \lambda_9 x^2y^2 + \lambda_{10} y^4$
⊙ numerical approximation

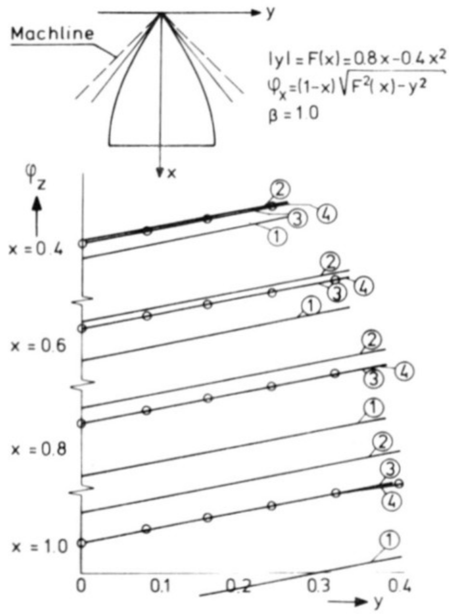


fig. 8

① $\lambda_1 x + \lambda_2 |y|$
② $+ \lambda_3 x^2 + \lambda_4 y^2$
③ $+ \lambda_5 x^3 + \lambda_6 xy^2 + \lambda_7 |y|^3$
④ $+ \lambda_8 x^4 + \lambda_9 x^2y^2 + \lambda_{10} y^4$
⊙ numerical approximation

In this case the deviation of straight is not small. Near $y=0$ there is good convergence. Near the tips in case (i), more terms are required. In case (ii) the convergence is very rapid. This may be due to the fact that there is little loading of the wing near the tips. The accuracy of the numerical approximation was found to be good, except near the leading edges, by comparison with exact homogeneous flow solutions. The accuracy and the speed of convergence does not only depend on the leading edges but also on the boundary conditions.

4.3. Another possibility which should be mentioned is the following. In (3.10) one may take into account only two coefficients, say g_{00} and g_{30}, and calculate the solutions including terms of say $O(\varepsilon^2)$. This leads to a workable scheme for a large class of planforms [6]. It can be expected to be sufficiently accurate to obtain some qualitative insight. The calculation of the terms of $O(\varepsilon^2)$ can be carried out along the same lines that led to the terms of $O(\varepsilon)$.

References.

[1] P. Germain, General theory of conical flow and its application to supersonic aerodynamics, NACA TM 1354.

[2] M. Fenain, La théorie des écoulements à potentiel homogène et ses applications au calcul des ailes en régime supersonique, Progress in Aeronautical Sciences Vol I, Pergamon Press 1961.

[3] E. Carafoli, Extension of conical motions to quasi-conical motions, Ev, Roum. Sci. Techn. Appl. Tome 10, no 3, Bucarest 1965.

[4] E. Carafoli, D. Matescu and A. Nastase, Wing theory in supersonic flow, Pergamon Press 1969.

[5] R. Coene, On a class of quasi-conical supersonic wings with curved subsonic leading edges, Journal of Engineering Math. Vol 4 (1970) 155-167.

[6] R. Coene, Quasi homogeneous approximations for wings with curved subsonic leading edges at supersonic speeds. Thesis to be published.

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