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Abstract

The influence of near-field flow on sonic boom has been studied for an incident triangular wing with constant lift distribution. Confined in its scope to the front shock of the wing, the present analysis reveals nevertheless features of flow important for sonic boom prediction which cannot be adequately described by the theory of Whitham in its prevalent form. Moreover, the present analysis points to limitations of the usual approximation procedure in sonic boom study and indicates some practical possibilities to influence the sonic boom in addition.

I. Introduction

Sonic boom has become a field of intensive research with the impending advent of supersonic civil transport. On the one hand, the effects of gradients of air density and air temperature and the effects of unsteady flight conditions are under concentrated stuy at present. On the other hand, however, the influence of near-field flow on the boom intensity and also the distinction between near- and far-field flows still seem to deserve more attention. So far, the theory of Whitham has formed the basis of most analytical investigations of sonic boom. Just as the asymptotic approximation of the shock front in the usual procedure of Whitham's theory has to be replaced in many cases by non-asymptotic approximation in view of the relative proximity of aircraft to ground, so may the approximation of the entire flow field by its far-field values and the equivalence of a wing to an axisymmetrical body in the theory of Whitham under circumstances also need modifications. The aim of the present study is to check, through revealing examples, the validity of equivalence of a wing to an axisymmetrical body and the accuracy of the widely employed far-field approximation in line with Whitham's theory in its present form by application of a more comprehensive analytical method.

II. Nomenclature

x, y, z	= Cartesian coordinates after the Prandtl-Glauert transformation with $x = X$, $y = Y \cot\alpha_\infty$ and $z = Z \cot\alpha_\infty$, X, Y, Z denoting the untransformed Cartesian coordinates, x and X being measured along the direction of free stream
x_o, y_o, z_o; x_1, y_1, z_1: see Eq. (1)	
ξ, η, ζ	= characteristic coordinates, ξ and η defined in (3)
M_∞	= free-stream Mach number
α_∞	= free-stream Mach angle
C_∞	= sound speed in free stream
u, v, w	= $\frac{U}{C_\infty}$, $\frac{V \tan\alpha_\infty}{C_\infty}$, $\frac{W \tan\alpha_\infty}{C_\infty}$ respectively, U, V, W denoting the velocity components of flow in (X,Y,Z) in the X-, Y-, and Z-directions respectively
c	= $\frac{C}{C_\infty}$, C being the local sound speed
ε	= angle of incidence
$\bar{\sigma}$	= $\tan\omega \tan\alpha_\infty$ (Fig. 4)
k	= $\frac{1}{2}\left[(\gamma + 1)\tan^2\alpha_\infty + (\gamma - 1)\right]$ (Eqs. (2), (5), (6))
γ	= ratio of specific heats
ϕ	= perturbation velocity potential (Eqs. (4), (11))
α	= local Mach angle

θ = local stream angle, i.e., angle between directions of free-stream and local stream line

s = $\frac{\Delta p}{p}$, shock strength (Eq.(10))

$K_1(\xi), K_2(\eta)$ = integration functions (Eq.(2))

$I(x_o', z_o')$ = a function proportional to the intensity of vortex distribution (Eq.(11))

x_o', z_o' = coordinates for vortex distribution

$\hat{x}_o$: see Fig. 3

$\hat{z}_o$: Eq. (12)

C_L = lift coefficient

H = height of cruising flight

W or A = total weight of aircraft plus loading or total lift of the wing

L_W = maximum chord length of the triangular wing

F = wing area

ψ : angle shown in Fig. 5

III. Analytical method and basic equations

The analytical method of characteristics employed in the present study has been developed(1)(2)(3)* in complement to the perturbation theory for two-dimensional problems based on the method of characteristics presented by Lin[4]. The essential features of the present method consist in the introduction of a coordinate system (e.g.(ξ,η,ζ)) composed of characteristic surfaces and in the expansion of the space variables as well as the physical quantities of flow in an ascending power series of some small parameters (e.g. angle of incidence, wing thickness, etc.) in this coordinate system of characteristics. Thus, with the series written in an implicit form, one has:

$$
\begin{aligned}
x(\xi,\eta,\zeta) &= x_o + x_1 + x_2 + \ldots\\
y(\xi,\eta,\zeta) &= y_o + y_1 + y_2 + \ldots\\
z(\xi,\eta,\zeta) &= z_o + z_1 + z_2 + \ldots\\
u(\xi,\eta,\zeta) &= M_\infty + u_1 + u_2 + \ldots\\
v(\xi,\eta,\zeta) &= \quad\;\; v_1 + v_2 + \ldots\\
w(\xi,\eta,\zeta) &= \quad\;\; w_1 + w_2 + \ldots\\
c(\xi,\eta,\zeta) &= 1 + c_1 + c_2 + \ldots
\end{aligned}
\tag{1}
$$

The original coordinate system (x,y,z) is depicted in Fig. 1. The x,z-plane stands for the projection plane of the wing, while the plane z = 0 represents the plane of symmetry vertical to the projection plane. Flow and shock front in the plane z = 0 will be studied in the present analysis. In Fig. 1 the triangular wing in question is shown to be at an angle of incidence ε to the free stream along the direction of the x-axis.

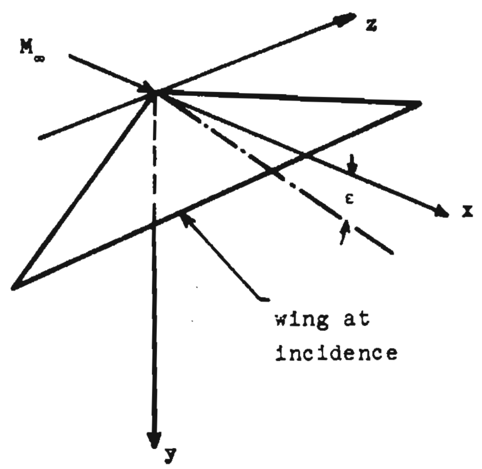

Fig. 1

As the present investigation is confined to a first approximation of the shock wave in the plane z = 0 due to the incident

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* For details of derivation of the analytical method of characteristics, the interested readers are referred to the cited literature (also [5]).

triangular wing, the basic equations involved are:

$$x_1 - y_1 = -\frac{M_\infty}{2}\int_{\bar\eta=\xi}^{\bar\eta=\eta}(v_1 - k\,u_1)\,d\bar\eta + K_1(\xi)$$
$$x_1 + y_1 = \frac{M_\infty}{2}\int_{\bar\xi=0}^{\bar\xi=\xi}(v_1 + k\,u_1)\,d\bar\xi + K_2(\eta) \qquad (2)$$

with

$$\xi = x_0 - y_0$$
$$\eta = x_0 + y_0 \qquad (3)$$

and $k = \frac{1}{2}\left[(\gamma+1)\tan^2\alpha_\infty + (\gamma-1)\right]$, $K_1(\xi)$ and $K_2(\eta)$ being the integration constants.

From previous investigations of shock waves (e.g.[6][7]), it can be shown that the values of $x_1 + y_1$ near the shock front are much smaller than those of $x_1 - y_1$ and are of a higher order than the present approximation. This makes possible the approximation: $x_1 + y_1 \approx 0$, and $x_1 - y_1 \approx 2x_1$.

The functions u_1 and v_1 in the integrands of (2) can be determined from the following equations:

$$\frac{\partial^2\phi}{\partial x_0^{\,2}} - \frac{\partial^2\phi}{\partial y_0^{\,2}} - \frac{\partial^2\phi}{\partial z_0^{\,2}} = 0 \qquad (4)$$

and

$$u_1 = \frac{\partial\phi}{\partial x_0}; \quad v_1 = \frac{\partial\phi}{\partial y_0}.$$

Formally, the equations in (4) are exactly the same as the corresponding equations for the linear acoustic approximation, except that the independent variables become now x_0, y_0, z_0 instead of x, y and z. Since the former are related to the latter in the characteristic coordinates, the first approximation by the present method couples therefore a first-order perturbation of the physical quantities of flow (velocities, pressure, etc.) with a first-order perturbation of the characteristics which denote the surfaces of propagation of disturbances in these flow quantities.

The equations in (2) can further be rewritten in the following manner:

$$\xi = x_0 - y_0 = (x - y) - (x_1 - y_1)$$
$$= x - y + \frac{M_\infty}{2}\int_{\bar\eta=\xi}^{\bar\eta=\eta}(v_1 - k\,u_1)\,d\bar\eta + K_1(\xi),$$
$$\eta = x_0 + y_0 = (x + y) - (x_1 + y_1) \qquad (5)$$
$$= x + y - \frac{M_\infty}{2}\int_{\bar\xi=0}^{\bar\xi=\xi}(v_1 + k\,u_1)\,d\bar\xi + K_2(\eta).$$

Expressed in words, the equations in (5) signify that the change of inclination of a characteristic curve is represented by a summation of the local changes of inclination along the characteristic in question. Generally, the integration functions $K_1(\xi)$ and $K_2(\eta)$, determined by the boundary conditions, may be set equal to zero, as will also be done here.

If the first equation for ξ in (5) is now recast into the following form with omission of $K_1(\xi)$:

$$\xi = x - y + M_\infty\int_{\bar y_0=0}^{\bar y_0=y_0}(v_1 - ku_1)\,d\bar y_0\,, \qquad (6)$$

it changes then into the fundamental equation of Whitham's theory except for the upper limit of integration, $-y_0$ instead of y. However, the replacement of y_0 by y is permissible in the present approximation. Thus, it becomes obvious that the theory of Whitham is closely related to the first approximation of the present theory, while in the usual application of Whitham's theory further simplifying approximations are made for the functions u_1 and v_1. This will, however, not be done here so that an adequate assessment of the influence of near-field flow may be ensured.

For the present investigation, the basic equations consist, therefore, of (2) (or (5) or (6)), (3) and (4).

IV. Determination of shock wave

As the present study is limited to the front shock of the wing, the shock wave in this case separates the region of free stream from the field of flow disturbed by the wing. The local inclination of the shock wave is expressed by:

$$\frac{dy}{dx} = \frac{\frac{\partial y}{\partial \xi}\cdot\frac{d\xi}{d\eta}+\frac{\partial y}{\partial \eta}}{\frac{\partial x}{\partial \xi}\cdot\frac{d\xi}{d\eta}+\frac{\partial x}{\partial \eta}} = \frac{\left(\frac{\partial y_o}{\partial \xi}+\frac{\partial y_1}{\partial \xi}\right)\frac{d\xi}{d\eta} + \left(\frac{\partial y_o}{\partial \eta}+\frac{\partial y_1}{\partial \eta}\right)}{\left(\frac{\partial x_o}{\partial \xi}+\frac{\partial x_1}{\partial \xi}\right)\frac{d\xi}{d\eta} + \left(\frac{\partial x_o}{\partial \eta}+\frac{\partial x_1}{\partial \eta}\right)}$$

$$= \frac{1}{2}\left[\tan\alpha_\infty + \tan(\alpha + \theta)\right] \tag{7}$$

The last expression in (7) signifies simply the angle property of the shock (also called the Pfriem-relation for a shock) according to which a weak shock bisects at each point the characteristics on both sides of the shock meeting at the point.

On account of the relations:

$$\frac{\partial y_1}{\partial \eta} = \frac{\partial x_1}{\partial \eta} + \frac{M_\infty}{2}(v_1 - k\,u_1)$$
$$\frac{\partial y_1}{\partial \xi} = -\frac{\partial x}{\partial \xi} + \frac{M_\infty}{2}(v_1 + k\,u_1)\ , \tag{8}$$

(7) can be reduced to:

$$\frac{d\xi}{d\eta} \approx \frac{\frac{M_\infty}{4}(v_1-k\,u_1)}{1 + 2\frac{\partial x_1}{\partial \xi}} \approx \frac{\frac{M_\infty}{4}(v_1-k\,u_1)}{1 + \frac{\partial(x_1-y_1)}{\partial \xi}}\ . \tag{9}$$

Here the procedure has been adopted to neglect terms with $\partial x_1/\partial\eta$ as against those with $\partial x_1/\partial\xi$, - a procedure which can also be justified a posteriori.

By solution of (9) with application of the basic equations in III, the relationship between ξ and η on the shock can be determined. Further, to the present order of approximation, one may replace $\eta(\eta = x_o + y_o)$ by $\xi + 2y_o$, and, furthermore, by $\xi + 2y$, and obtains thus a relationship between ξ and y on the shock wave. This also parallels Whitham's theory in its prevalent form with again the exception that now full velocity disturbances will be taken into account in (9).

The shock strength as expressed by the ratio of the pressure jump across the shock to the pressure in front of it can be shown to be:

$$s = \frac{\Delta p}{p} = \frac{2\gamma}{\gamma+1}\left(\frac{\cot\alpha_\infty}{M_\infty}\right)\ (v_1 - k\,u_1) \tag{10}$$

with v_1 and u_1 represented by functions of ξ and y. Thus the shock strength at a given distance y from the wing can be determined from (10) together with the relationship between ξ and y for the shock.

V. Statement of the problem

To investigate the influence of near-field flow on sonic boom, it would be desirable to choose a problem for which particularly aggravating influence of the near-field flow should be expected. From this viewpoint, the problem of flow due to the fuselage may be disregarded in the first place, for this itself constitutes normally a slender axi-symmetrical body in an analytical sense. Further, in order to make the most of an analytical study, numerical integration has to be avoided as much as possible. On the ground of these considerations, the problem of an incident triangular plate with supersonic leading edges and constant loading has been selected for the present analysis. The known results for a flat triangular plate with supersonic leading edges in the region of conical flow will be drawn upon for comparison.

In the present study only the front shock in the plane of symmetry vertical to the projection plane of the wing (Fig. 1) will be considered. The problem corresponds roughly to the study of influence of a wing with relatively large aspect ratio on the sonic boom due to the front shock during steady cruising flight at the maximum altitude without regard to the variation of air density and air temperature with the altitude. The latter effects can be considered separately (e.g.[8]).

VI. Various regions of flow

As the velocity perturbations u_1 and v_1 in the integrands of (2) (or (5) or (6)) are determined by (4), which are simply the acoustic equations in the coordinate system of (x_o, y_o, z_o), the known methods of solution for such equations can now be of great avail. For calculation of the flow at some distance from the wing, as is the case at present, the method of vortex distribution seems to be the most suitable. According to this method, the intensity of vortex distribution in the projection plane of the wing vanishes behind the trailing edge. This will greatly simplify the analytical treatment.

To solve the acoustic equations in (4) with the corresponding boundary conditions, the shape of the wing in the coordinate system of (x_o, y_o, z_o), which is linked up with the coordinate system of characteristic (ξ, η, ζ) by linear relationships, has to be known first. From previous investigations of delta wings[7], it follows that in the coordinate system of (x_o, y_o, z_o) the planform of the wing in the projection plane (x_o, z_o-plane) will also be a triangle with the same apex angle as the actual wing, provided that the leading edges are not nearly sonical. The effect of slight deviation of the trailing edge in the x_o, z_o-plane from a straight line may be safely neglected in the present approximation. Thus the wing form remains unchanged in the coordinate system (x_o, y_o, z_o).

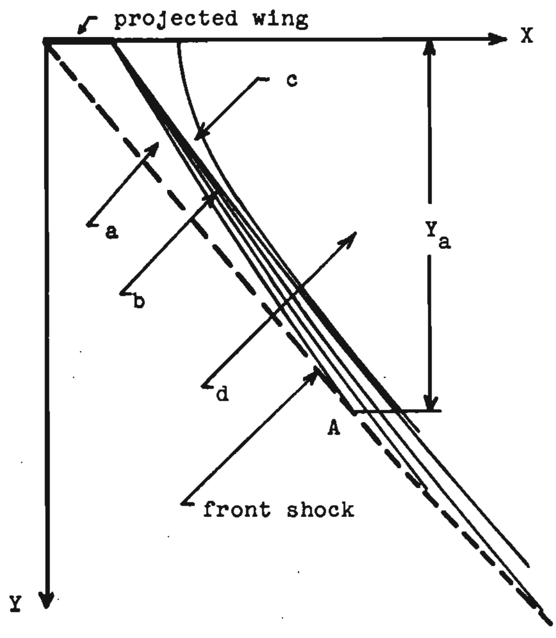
As is known with the method of vortex distribution, the potential of velocity perturbation at a point $P(x_o, y_o, 0)$ is given by:

$$\phi = \frac{1}{2\pi} \iint \frac{y_o(x_o - x_o') \, I(x_o', z_o')}{(z_o'^2 + y_o^2)\sqrt{(x_o - x_o')^2 - (z_o'^2 + y_o^2)}} \, dx_o' \, dz_o' \tag{11}$$

with $u_1 = \frac{\partial \phi}{\partial x_o}$ and $v_1 = \frac{\partial \phi}{\partial y_o}$.

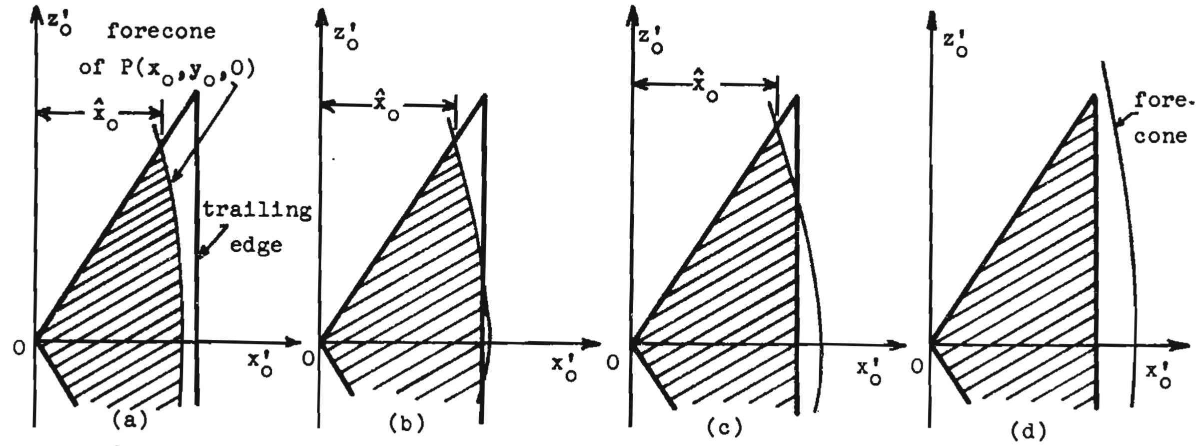
In (11), the x_o', z_o'-plane represents the plane of vortex distribution, while $I(x_o', z_o')$ denotes a function proportional to the vortex intensity. Here $I(x_o', z_o') = 2\Delta u = 4u_{1up}$, where u_{1up} stands for the velocity component u_1 on the upper (suction) side of the wing. Because of symmetry of the problem, the limits of integration in (11) are to be assigned by considering only one half of the wing.

In the field of disturbed flow due to a triangular plate at incidence, four different regions in the plane $z = 0$ ($z_o = 0$) can be ascertained from analytical considerations. They (Regions a-d) are shown qualitatively in Fig. 2



a: region of conical flow
b: region characterized by the trailing edge expansion
c: an intermediate region
d: region of influence by the whole wing

Fig. 2



shaded area = effective area of vortex distribution for $I(x\delta, z\delta)$ in (11)

Fig. 3

These regions of flow arise as a result of differences in limits of integration for the velocity potential in (11). The four different cases corresponding to those depicted in Fig. 2 are shown in Fig. 3 (a-d).

Case a: In this case, the curve of intersection of the forecone of the point $P(x_0, y_0, 0)$ with the x_0', z_0'-plane lies ahead of, or just touches (from the wing side) the trailing edge. Here ϕ is composed of two expressions with double integrals as in (11). The integration limits involved are:

$$0 \leqslant x_0' \leqslant \hat{x}_0 \ ; \quad 0 \leqslant z_0' \leqslant \bar{\sigma} x_0'$$

and $\hat{x}_0 \leqslant x_0' \leqslant x_0 - y_0; \quad 0 \leqslant z_0' \leqslant \hat{z}_0$
respectively,
with $$\hat{x}_0 = \frac{-x_0 + \sqrt{x_0^2 + (\bar{\sigma}^2 - 1)(x_0^2 - y_0^2)}}{\bar{\sigma}^2 - 1}$$

and $\hat{z}_0 = \sqrt{(x_0 - x_0')^2 - y_0^2}$.

In case a, the front shock is not yet, or just about to be, affected by the trailing edge, and the flow is conical.

Case b: In this case, the curve of intersection of the forecone of $P(x_0, y_0, 0)$ with the x_0', z_0'-plane just touches the trailing edge from the wake side. ϕ comprises also two expressions with double integrals as in (11). The integration limits are:

$$0 \leqslant x_0' \leqslant \hat{x}_0 \ ; \quad 0 \leqslant z_0' \leqslant \bar{\sigma} x_0'$$

and $\hat{x}_0 \leqslant x_0' \leqslant 1 \ ; \ 0 \leqslant z_0' \leqslant \hat{z}_0$ respectively.
Here a limiting process $x_0 - y_0 \to 1$ is to be taken on the results of integration, since $x_0' = 1$ has been assumed to represent the trailing edge. A jump in the values of u_1 and v_1 will occur if one determines these components from case a and from case b by taking the same limit $x_0 - y_0 \to 1$. The limiting process corresponds to approaching the trailing edge from the wing side or from the wake side, and the jump arises out of a generalized Prandtl-Meyer expansion at the sharp trailing edge (cf. Fig.2, Region b).

Case c: In case c, the curve of intersection of the forecone of $P(x_0, y_0, 0)$ with the x_0', z_0'-plane still lies ahead of the tips of trailing edge. The limits of integration for this case are the same as for case b. The values of u_1 and v_1 will be continuous and join smoothly to those for case b.

Case d: Here the curve of intersection of the forecone of $P(x_0, y_0, 0)$ with the x_0', z_0'-plane lies entirely outside the wing. The flow in the plane $z = 0$ (or $z_0 = 0$) is now affected by the entire wing. In this case, ϕ comprises only one expression with the following integration limits:

$$0 \leqslant x_0' \leqslant 1 \; ; \quad 0 \leqslant z_0' \leqslant \bar{\sigma} x_0' \, .$$

From the analytical point of view, the choice of a wing in the form of a triangular plate with constant lift distribution is desirable, because then in (11) $I(x_0', z_0')$, the function for the vortex intensity will simply become a constant. After repeated trials, the best procedure for the determination of u_1 and v_1 from (11) is found to be the following: The velocity component u_1 is to be determined first from (11). By use of the ϕ-integral in (11), this is accomplished by first integrating with respect to z_0', then by differentiating under the integral sign with respect to x_0 by proper application of the Leibnitz rule, and finally by integrating with respect to x_0'. After obtaining u_1, one can determine $\phi(x_0, y_0, 0)$ by the formula:

$$\phi = \int_{\bar{x}_0 = y_0}^{\bar{x}_0 = x_0} u \, d\bar{x}_0 ,$$

which amounts to an integration with respect to x_0 between the Mach cone from apex of the wing and $P(x_0, y_0, 0)$. With ϕ determined in this way, one obtains v_1 then by simple differentiation. The results can be checked by the condition of irrotationality:

$$\frac{\partial u_1}{\partial y_0} = \frac{\partial v_1}{\partial x_0} \, .$$

VII. Estimated region of interest for sonic boom

The relative extent of the various regions of flow as shown in Fig. 2 should certainly be interesting, though a precise knowledge of this has to await the final results. Nevertheless, it can be safely postulated that the extent of each region of flow will depend on the free-stream speed or free-stream Mach number, on the aspect ratio of wing, on its area, and on the angle of incidence. With a fixed speed of flight and a fixed aspect ratio, the angle of incidence of the wing should then be the governing factor for the relative distribution of various regions of flow. It would not, therefore, be out of place to make at first here a rough estimation of the angle of incidence (i.e. the angle of incidence of wing to the fuselage in level flight) which one might encounter with a supersonic aircraft.

To simplify the estimation here, a delta wing in the form of a flat triangular plate is now taken and the following symbols are introduced.

H = Height of cruising flight,
W = Total weight of aircraft plus loading
A = Total lift of the wing,
F = Area of the wing,
L_W = Maximum chord length of the wing
$\bar{\sigma}$ = $\tan\omega \, \tan\alpha_\infty$ (Fig. 4)

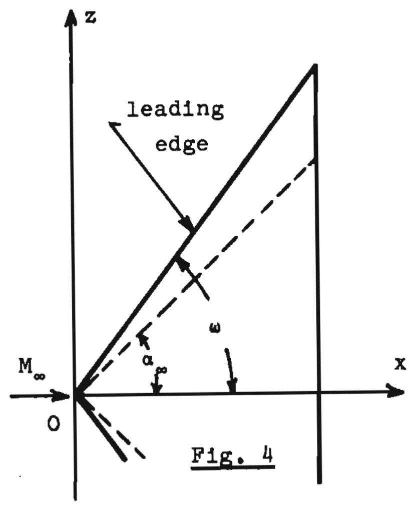


Fig. 4

For a flat triangular plate with supersonic leading edges, the angle of incidence is known to be expressed by:

$$\varepsilon = \frac{\sqrt{M_\infty^2 - 1}}{4} C_L$$

and
$$C_L = \frac{A \text{ (or } W)}{\frac{\rho_\infty U_\infty^2}{2} \cdot F} = \frac{2 A \sqrt{M_\infty^2 - 1}}{\gamma p_\infty M_\infty^2 L_W^2 \bar{\sigma}} \qquad (13)$$

C_L being here the lift coefficient. $\rho_\infty U_\infty^2/2$ the dynamic pressure, F and U_∞ being the wing area and speed of flight respectively, and p_∞, ρ_∞ and U_∞ referring to conditions at H.

Hence, $\varepsilon = \frac{A(M^2-1)}{2\gamma p_\infty M_\infty^2 L_w^2 \bar{\sigma}} = \frac{1}{2\gamma p_\infty}\left(1-\frac{1}{M_\infty^2}\right)\frac{A'}{\bar{\sigma} L_w^2}$ (14)

Assuming now as examples for estimation: $M_\infty = 3$, $H = 20$ km, $W = A = 200$ tons (metric) and

$L_w = 100$ m, $\bar{\sigma} = 1{,}2$ $(H = 200\ L_w)$ (15)

or

$L_w = 50$ m, $\bar{\sigma} = 4{,}8$ $(H = 400\ L_w)$ (16)

or

$L_w = 50$ m, $\bar{\sigma} = 1{,}2$ $(H = 400\ L_w)$, (17)

further setting $\gamma = 1{,}4$ and taking the air density at 20 km altitude to be one-tenth of its value at sea level or, due to temperature effect, $p_\infty \approx \frac{1}{12} \times 1$ kp (kilopound)/cm² $= 0{,}83 \times 10^3$ kp/m², one finds from (14)

for (15): $\varepsilon \approx 0{,}0064$,

for (16): $\varepsilon \approx 0{,}0064$,

and for (17): $\varepsilon \approx 0{,}0256$.

The first two values of ε are very small indeed. For (15) and (16), the wing area is kept constant, while for (17) a reduction of the wing area to one fourth of its value in (15) or (16) and a fourfold increase of the average wing loading compared with (15) or (16) are implied.

With the calculated values of ε above, one can readily estimate the extent of region a (Fig. 2) of conical flow. From previous work(7), for not excessively large $\bar{\sigma}$, the distance Y_a (Fig. 2) may be given by:

$$Y_a \approx \left(\frac{4\ \pi^2 \cot^6\alpha_\infty}{9\ M_\infty^8(\gamma+1)^2(\varepsilon\ \bar{\sigma})^2} - 1\right)L_w \tan\alpha_\infty \quad (18)$$

This gives for (15): $Y_a \approx 365\ L_w$;
for (16): $Y_a \approx 90\ L_w$; for (17): $Y_a \approx 90\ L_w$. The value of Y_a here for (16) should however, be treated with reservation, because $\bar{\sigma}$ for (16) is relatively large already. But it can still serve as a useful guide.

Evidently, the extent of region a (Fig.2) is strongly dependent on the product of $\varepsilon\bar{\sigma}$. With the altitude and the speed of flight kept fixed, ε is simply proportional to the reciprocal of wing area or of $\bar{\sigma}\ L_w^2$. Thus Y_a is proportional to L_w^4.

It is noteworthy that for (15), the region of conical flow would be solely responsible for the boom intensity on the ground in the plane $z = 0$. For (16) and (17), flow beyond region a should then be taken into account in a study of sonic boom. If one now considers region a and probably also region b (Fig. 2) to be near-field regions, then in many cases only near-field flows will be involved in the study of sonic boom due to the front shock.

VIII. A special feature in shock determination for triangular wing with supersonic leading edges

In the shock determination for a triangular wing in the symmetry plane $z = 0(z_0=0)$, or in its neighbourhood, a special feature of the problem should yet be taken into consideration. In his pioneering investigations of sonic boom by approximative methods, Whitham(8) (also Walkden(9)) deduced for thin wings with supersonic leading edges a set of planes in which the flow for shock determination should be considered. As they are, such deduced planes do not, however, suit for the study of shock wave in neighbourhood of the plane $z = 0$ (or $z_0 = 0$) for a triangular wing with supersonic leading edges. Analytically, a triangular wing has a discontinuous leading edge, and the apex might be regarded as a singular point on the leading edge. This could be perhaps best explained with the aid of Fig. 5.

In Fig. 5, the area bounded by OABB'A'O represents the influence zone in a plane x_0 = constant of a flat incident triangular plate (here projected as AA' on the z_0-axis) with its apex at $x_0 = y_0 = z_0 = 0$. The coordinate system considered is again (x_0, y_0, z_0). Alternatively, y_0 and z_0

in Fig. 5 may be considered to be conical coordinates y_o/x_o and z_o/x_o with x_o put equal to one. AB and A'B' stand for the plane wave front, while BB' depicts the curved front. B and B' denote here the points of tangency. In the plane $x_o = 0$, BB' would shrink into the apex.

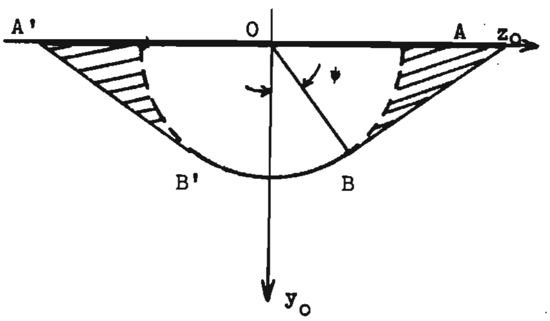


Fig. 5

The set of planes prescribed by Whitham's procedure for shock determination will be related to those parts of the wave front corresponding to AB and A'B' in (x_o, y_o, z_o), but not to the parts corresponding to BB', which embraces an angle of $2\psi = 2\sin^{-1}(1/\bar{\sigma})$. Mathematically, the shaded regions in Fig. 5 are conical-hyperbolic, while the unshaded region in influence zone of the wing is conical-elliptic. For the determination of shock front bounding the conical-hyperbolic regions, a set of planes passing through the apex and perpendicular to AB or A'B' in the plane x_o = constant, as laid down by the procedure of Whitham, should be taken for study of the flow. For the determination of the shock front bounding the conical-elliptic region, as is in the present case, the flow should however be studied in meridian planes passing through the x_o-axis (or x-axis). The problem resembles then the conical-elliptic problem for a wing with subsonic leading edges.

IX. Results and discussion

For a triangular plate with constant lift distribution, $I(x_o', z_o')$ which denotes the function proportional to the vortex intensity in (11) is given by: $I(x_o', z_o') = 2\Delta u_1 = M_\infty \Delta p / \frac{1}{2}\rho_\infty U_\infty^2$ = constant, where Δu_1 and Δp denote the absolute values of differences in u_1 and p respectively between the upper and lower sides of the wing and $\Delta p = A\cot\alpha_\infty/\bar{\sigma} L_W^2$, A and L_W being again the total lift and the maximum chord length of the wing respectively. With reference to a flat triangular plate with an angle of incidence ε, one finds the relation $\varepsilon = \frac{1}{\pi} I$ for the same wing area and the same total lift.

It becomes obvious from the present investigation of front shock attached to the wing that the results for the regions of flow a and b (Fig. 2) are of major interest.

Region a (Fig. 2) - conical-elliptical region of flow:

The wave front for this region corresponds to BB' in Fig. 5. According to (4) and the appropriate boundary conditions, the problem in the coordinate system of (x_o, y_o, z_o) is also a conical one. The full velocity components u_1 and v_1 (for (5)) can be found in closed form as follows:

$$u_1 = -\frac{I}{2\pi}\tan^{-1}\frac{2\bar{\sigma}\sqrt{\xi}}{\sqrt{\eta-\xi}}$$

$$v_1 = -\frac{I}{2\pi\bar{\sigma}}\ln\left[\frac{\eta-\xi}{\eta+\xi+2\sqrt{\xi\eta}}\right] + \frac{I\sqrt{\bar{\sigma}^2-1}}{2\pi\bar{\sigma}}\left[\tan^{-1}\frac{2\sqrt{(\bar{\sigma}^2-1)\xi\eta}}{\eta+\xi}\right], \qquad (19)$$

ξ and η being related to x_o and y_o according to (3).

On the shock or in its vicinity, because of the condition $\xi/\eta \ll 1$ (corresponding to $\xi/2y \ll 1$), it follows that:

$$u_1 = - \frac{I\bar{\sigma}}{\pi} \left(\frac{\xi}{\eta}\right)^{1/2} - \frac{4\bar{\sigma}^2}{3} \left(\frac{\xi}{\eta}\right)^{3/2} + \ldots.$$

$$v_1 = \frac{I\bar{\sigma}}{\pi} \left(\frac{\xi}{\eta}\right)^{1/2} - \frac{4\bar{\sigma}^2}{3} \left(\frac{\xi}{\eta}\right)^{3/2} + \ldots. \qquad (20)$$

With due attention paid to the choice of the plane for flow analysis (cf. VIII), one would obtain the first terms of the expressions in (20) as results for u_1 and v_1 by a direct application of the usual procedure of Whitham's theory, as ξ/η here may be replaced by $\xi/2y$ in this approximation.

Because of the conical nature of flow, the solution of (9) is found under the assumption of $\bar{\sigma}^2\xi/\eta << 1$ to be:

$$\left(\frac{\xi}{\eta}\right)^{\frac{1}{2}} = \frac{3M_\infty^3(\gamma+1)\bar{\sigma}I}{8\pi\cot^2\alpha_\infty}, \qquad (21)$$

and the shock is then simply expressed by:

$$\frac{Y}{X} = \tan\alpha_\infty \left[1 + \frac{3M^6\bar{\sigma}I^2(\gamma+1)^2}{32\pi^2\cot^4\alpha_\infty}\right] \qquad (22)$$

The shock strength s, which is constant here, is given by:

$$s = \frac{3M_\infty^4\gamma(\gamma+1)(\hat{\sigma}I)^2}{8\pi^2} \qquad (23)$$

It may be interesting to compare the shock strength for region a in (23) with the corresponding value for a flat incident triangular wing with the same planform and the same lift coefficient. The shock strength for the latter is[7], because of $\varepsilon = \frac{I}{\pi}$:

$$s = \frac{3M^6\gamma(\gamma+1)(\hat{\sigma}I)^2}{8\pi^2\cot^2\alpha_\infty} \qquad (24)$$

$\hat{\sigma}$ being $\tan\omega$ in Fig. 4.

Thus the ratio between the values for a flat wing and a wing of constant loading is seen to be $M^2/M_\infty^2 - 1$, which approaches one for large M_∞. This is quite plausible, as, with a fixed planform, the zone of non-constant lift distribution for a flat wing will diminish accordingly with increasing M_∞.

The point A in Fig. 2 marking the outermost point of region a on the shock is indicated by:

$$Y_a = \left[\frac{64\pi^2}{9M_\infty^6(\gamma+1)^2C_L\sigma^2} - 1\right] L_W \tan\alpha_\infty \qquad (25)$$

with again $\hat{\sigma} = \tan\omega = \bar{\sigma}\tan\alpha_\infty$ (Fig. 4).

The corresponding value for a flat wing with the same $\hat{\sigma}$ and C_L is from (18):

$$Y_a = \left[\frac{64\pi^2(M_\infty^2 - 1)}{9M_\infty^8(\gamma+1)^2C_L^2\sigma^2} - 1\right] L_W \tan\alpha_\infty \qquad (26)$$

For large values of Y_a, the ratio between the latter and the former is roughly $(M_\infty^2 - 1)/M_\infty^2$. Thus region a for an incident triangular wing with constant loading extends farther away from the wing.

From the above results it becomes evident that the parameter $\bar{\sigma}\sqrt{\xi/\eta}$ or $\bar{\sigma}\sqrt{\xi/2y}$ at the shock is of great significance in the sonic boom analysis for a wing. As now in region a $\xi/\eta = O(\bar{\sigma}^2I^2) = O(\bar{\sigma}^2C_L^2)$ (cf. (21)) the parameter $\bar{\sigma}\sqrt{\xi/\eta}$ or $\bar{\sigma}\sqrt{\xi/2y}$ may be replaced by $\bar{\sigma}^2C_L$ or for constant M_∞ by $\hat{\sigma}^2C_L$. For cases where $\bar{\sigma}^2C_L << 1$ applies, a direct application of the procedure of representing the flow field by its far-field values would give good approximation to the shock front and shock strength for the conical-elliptic region of flow here. For cases where $\bar{\sigma}^2C_L << 1$ no longer applies (despite ξ/η or $\xi/2y << 1$), correction terms or even a higher-order approximation might then be necessary.

Owing to the assumption of constant lift distribution on the wing, logarithmic singularity of v_1 in the plane $z = 0$ ($z_o = 0$) arises at the wing surface, which, however should not upset the present analysis of shock front and boom intensity.

Region b (Fig. 2) - region with nature of a plane flow:

This is the region in which the flow is characterized by the expansion from the sharp trailing edge of the wing. The jumps in the velocity components u_1 and v_1 in this region are found to be:

$$\Delta u_1 = - \Delta v_1 = \frac{I}{4} \tag{27}$$

In the plane $y_o = 0$ (also $y = 0$), this can be easily understood, since behind the trailing edge the velocity component u_1 vanishes in the projection plane of the wing (cf. (19)). However, the result is in so far of significance as these jumps in velocity components remain unaltered away from the wing in the system of (x_o, y_o, z_o) and do not tend to zero in the far field. This reminds one of the acoustic solution for a plane flow, for example, by Ackeret. Certainly, in the actual flow, these velocity jumps will be curtailed downstream by the flow in region d (Fig. 2) and by the compression waves coming from the straight leading edges (corresponding to AB and A'B' in Fig. 5), but this would not affect the study of front shock at present.

Such features of plane flow in region b, which may be regarded as an example of very typical near-field influence on the far field flow, cannot be adequately revealed by the theory of Whitham in its prevalent form. In its present form, Whitham's theory is capable of accurately approximating a trailing-edge expansion or an expansion of Prandtl-Meyer type only at the wing surface, and perhaps at very large distance from the wing, but cannot adequately reproduce the behaviour of flow in the intermediate regions which form, as a matter of fact, the main range of interest for a sonic boom study. This is because the otherwise very powerful theory approximates in its prevalent form a wing with no exception by an equivalent axisymmetrical body, for which any velocity disturbance will inevitably die away like $y^{-\frac{1}{2}}$ in the flow field. In this connexion, it should also be pointed out that a determination of the shock waves coming from genuinely supersonic leading edges (corresponding for a triangular wing to the plane waves AB and A'B' in Fig. 5) would also be inadequate by adoption of the prevalent procedure in Whitham's theory. Again this is because, in the range of interest for sonic boom study, such plane waves cannot be replaced by disturbances caused by an axisymmetrical body. Consequently, just as the relative proximity of aircraft to ground has necessitated in many cases the replacement of the usual asymptotic approximation in Whitham's theory by non-asymptotic approximation, so will this relative proximity also necessitate a revision of the conception that in each plane normal to the wave front arising from a supersonic leading edge (e.g. AB and A'B' in Fig. 5) the wing may be replaced by an equivalent body of revolution.

Owing to the possibility that now Δu_1 and Δv_1 in region b, (26), can outweigh u_1 and v_1 inherited from region a particulary at the shock front (cf. (19) (20)), only a part of the characteristics in region b will converge into the front shock in such cases. The rest of them will very probably converge, due to the over-expansion in this region, into a rear shock which remains then to be determined.

In region a, the front shock is of constant strength. Away from region a, the strength of the front shock will decrease continuously until the shock degenerates into Mach line at infinity. Therefore, in region b some neutral Mach line ξ_N, which merges into the front shock only at infinity, should be expected (Fig. 6).

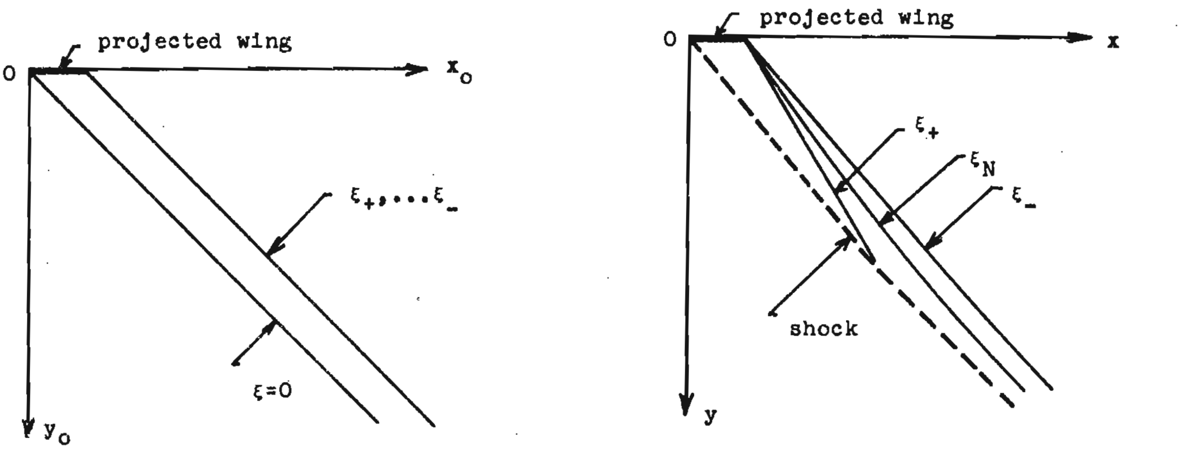


Fig. 6

To analyze the flow in region b, one may proceed in the following manner. Taking ξ_+ and ξ_- to be the two characteristics bounding region b in the x,y-plane (Fig.6) and noting that ξ_+ and ξ_-, and all the characteristics between them (in fact, infinite in number), fall together in the x_o,y_o-plane because of (3), one has:

$$\begin{aligned} \xi_+ &= x_o - y_o = (x-y)_+ - (x_1-y_1)_+ = \\ &= (x-y)_- + \frac{M_\infty}{2}\int_{\bar\eta=\xi}^{\bar\eta=\eta}\left[v_1 - k\,u_1\right]_+ d\bar\eta = \\ &= \xi_- = x_o - y_o = (x-y)_- + \frac{M_\infty}{2}\int_{\bar\eta=\xi_-}^{\bar\eta=\eta}\left[v_1 - k u_1\right]_- d\bar\eta \end{aligned} \tag{28}$$

with $K_1(\xi)$ in (5) set equal to zero and with subcripts + and - denoting values on ξ_+ and ξ_- respectively. This gives:

$$\begin{aligned} &(x-y)_+ - (x-y)_- = \\ &= \frac{M_\infty}{2}\int_{\bar\eta=\xi}^{\bar\eta=\eta}\left[(v_{1_-} - v_{1_+}) - k(u_{1_-} - u_{1_+})\right] d\bar\eta \end{aligned} \tag{29}$$

In principle, one could determine region b in all details by adequately subdividing the region between ξ_+ and ξ_- through intermediate characteristics $\xi_1, \ldots.\xi_n$ with a corresponding subdivision of Δu_1 and Δv_1 in (22). To determine any intermediate characteristic ξ_m, one has then just to substitute ξ_- in (28) and (29) by ξ_m.

From (5), the characteristic ξ_m is expressed by:

$$\begin{aligned} \xi_m &= 1 = x - y + \frac{M_\infty}{2}\int_{\bar\eta=\xi}^{\bar\eta=\eta}(v_1 - k\,u_1)\,d\bar\eta \\ &\approx x - \left[1+\mu_m\frac{M_\infty(1+k)I}{4}\right]y + \frac{M_\infty(1+k)I\bar\sigma}{\pi}\sqrt{2y} - \frac{M_\infty(1+k)I\bar\sigma}{\pi} \end{aligned} \tag{30}$$

with μ_m denoting the fractional increment of u_1 and v_1 assigned to ξ_m ($\mu_m = 0$ for ξ_+ and $\mu_m = 1$ for ξ_-) and with the trailing edge represented by $x_o = 1$. The shock front for this region can be determined either graphically or from (8) by noting $\mu = \mu(\xi,\eta)$ in this region. The shock strength is again given by (10).

X. Conclusions

Owing to the assumption of constant lift distribution over the triangular wing, analytical results relevant for the analysis of front shock of the wing have been obtained in closed form. The singular behaviour of v_1 in the plane $z = 0$ ($z_o = 0$) should not affect qualitatively the following conclusions:

(1) Validity of equivalence of a wing to a body of revolution

In the plane of symmetry vertical to the wing, the equivalence is seen to exist in region a (Fig. 2) under the assumption of $\bar{\sigma}(\xi/2y)^{1/2} << 1$ or $\bar{\sigma}^2 C_L << 1$. This is because this near-field region of conical flow is conical-elliptic in the analytical sense (cf. VIII, BB' in Fig. 5). However, in planes normal to wave fronts arising from the leading edges and corresponding to AB and A'B' in Fig. 5, flows with the nature of plane waves will prevail in the near field, and no equivalence to an axisymmetrical body will apply there. In region b, again in the symmetry plane z = 0, where the trailing edge begins to exert its influence, flow with the nature of a plane flow also makes its appearance. Flow of such a type will then imprint its character on a wide intermediate field of flow until the strong disturbances of plane-wave nature are finally attenuated in the far field. In this intermediate field of flow, the wing can again not be substituted by an equivalent body of revolution determined from conditions in the far field together with conditions at the wing surface. As the intermediate flow field seems to be of major interest to a study of sonic boom (cf. VII), the conception of equivalence of a wing to a body of revolution should therefore be subject to revision.

(2) Accuracy of the usual approximation procedure

The usually employed approximation procedure of replacing the velocity perturbations in (6) by its far-field values is allowable on the condition that $\bar{\sigma}(\xi/\eta)^{1/2}$ or $\sigma(\xi/2y)^{1/2} << 1$ applies on the shock. For those portions of the shock front where the above condition no longer applies, correction terms for the results obtained by the approximation procedure or even a higher-order approximation would then be necessary.

(3) Possibilities to influence the boom intensity

Because of the far-reaching effects of the above-mentioned intermediate field embodying flows of plane-wave character, interesting possibilities seem to offer themselves to influence the boom intensity. In this respect, however, a study of the whole flow field involving very likely a rear shock should be undertaken first. It appears that, among other factors such as the angle of incidence, the aspect ratio of the wing, etc., the shape of trailing edge of the wing, for example, could also influence both the front and the rear shocks to a considerable extent.

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