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CALCULATION OF TRANSONIC FLOWS

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Abstract

A brief survey is given of methods for calculation of plane transonic flow around airfoils. Two hodograph based methods for shock-free flows and two physical space methods for flow with shock waves are discussed. The last method which is a relaxation procedure for equations of mixed type is discussed in more detail. Comparison of the results of the different methods for shock-free cases is made. Some calculations are also presented for flows with shock waves.

1. Introduction

There has been a recent revival of interest in transonic flow calculations for various reasons. Engineers feel that efficient transonic design of high subsonic speed aircraft can be achieved with the resulting improvement in performance. Upon further investigation it may turn out that aircraft which operate very close to sonic may have some advantages. Also, current aircraft turbines typically operate with at least part of the blade in the transonic region. Last, but not least, is the availability of large scale computing facilities which enable a different approach to these problems to be taken, in comparison to that of twenty years ago.

In this paper which we will review briefly four different methods which have been developed recently for calculation of steady, plane, transonic flow about airfoils. All these methods have in common the feature that the essential results could not have been achieved without the practical possibility of high speed calculations. Furthermore, each method represents a basically different mathematical approach to the problem. The comparison of these different methods will enable us to see how they complement one another and which will serve best for particular future developments. All of the methods discussed here aim at an exact solution of a flow problem, although one method is applied to the simplified transonic equations. Other approaches, such as integral methods which are based on approximations difficult to assess, are not considered here.

Two of the methods are based on the hodograph representation of the flow problem; one uses the classical method of superposition of elementary solutions, the other, essentially computational, uses a method of complex characteristics. The two methods discussed next rely on direct finite difference calculations of the equations of motion in physical coordinates; one approaches the steady flow as a limit of unsteady flow ($t \to \infty$) computations proceeding as t increases, the second uses a method which is essentially relaxation, adapted to a mixed flow (elliptic-hyperbolic). Thus, in the nature of the methods at their present stage the first two methods find the airfoil shapes which produce shock-free flow; shock waves can not be handled. The second two methods on the other hand calculate the flow about given profiles and one of the main points of these methods is the possibility of including shock waves.

The problem of plane inviscid flow past a profile is a logical starting point for the development of these computational methods. The essential difficulty of transonic flow is contained in this problem and even the results for this restricted class of problems has some engineering interest. This is true even though there is no transonic analogue to lifting line theory. Further developments can proceed in various ways. The hodograph methods can be extended to generate wider classes of shock-free shapes. The direct methods can be extended to account for viscous boundary layers and eventually to three-dimensional flows.

In the next four sections of the paper each of these methods will be described briefly and results of calculations based on each method will be presented and compared when possible with the results of the other methods and experiment. Finally, some remarks will be made comparing the methods and suggesting future developments.

2. Hodograph — Direct Method

The method developed by Nieuwland[1] and applied by the other Dutch workers (see also the paper by Boerstoel in this meeting) is based essentially on Lighthill's earlier work.

Analytic solutions are obtained for the hodograph equation (perfect inviscid gas) for the stream function $\psi(\tau, \theta)$

$$\tau(1-\tau)\psi_{\tau\tau}+\left(1+\frac{2-\gamma}{\gamma+1}\tau\right)\psi_{\tau}+\frac{1-\frac{\gamma+1}{\gamma-1}\tau}{4\tau}\psi_{\theta\theta}=0 \qquad (2.1)$$

where $\tau = (q/q_{max})^2$; q = local flow speed

θ = flow inclination

The particular solutions $\psi_\nu(\tau)$ of product form $\psi_\nu(\tau)e^{\pm i\nu\theta}$ are hypergeometric functions. An operator is constructed which maps a given incompressible flow into the space of the product solutions. The flow may have circulation but the calculations are simpler in the nonlifting case. Taking advantage of the linearity further particular solutions of (2.1) are added to produce desired detailed changes in the airfoil shape. However, the airfoil of course can not be prescribed in advance and comes out in the calculation; variations in shape and Mach number are obtained by varying the parameters. From the point of view of obtaining practical results the outstanding difficulty in this method is computational. The high order hypergeometric functions ψ_n are tricky to calculate, especially in the supersonic region and further slowly convergent infinite series of these functions must be calculated. Even with a modern computer (Telefunken TR4) summability methods (Wynn's ϵ-algorithm) must be used to accelerate convergence so that sufficient accuracy can be obtained. 18 hours was required to compute a table of ψ_n up to $n = 100$ ($0 < \tau < .32$) and about 15 minutes to calculate an airfoil defined by coordinates and two derivatives at 40 boundary points.

In this work of Nieuwland many difficulties have been overcome. A systematic set of results is reported in Ref. 2. In this survey systematic variations of parameters produced systematic variation of pressure distributions, many of which contain substantial supersonic zones with smooth transonic flow. See for example Figs. 1, 2, 3. The extreme sensitivity of the flow to the details of the shape is evident in these calculations. The existence of smooth flow depends on a delicate balance of successive reflections of Mach waves from the body and sonic line. The free stream Mach number can not be increased too far or the smooth flow breaks down; the limit line which is inside the body for smooth flow penetrates the surface and causes infinite acceleration.

3. Hodograph — Method of Complex Characteristics

Essentially the same problem as studied by Nieuwland is taken up by Korn[3] and the same type of results, smooth flow past a family of airfoils is achieved. The method is based on some earlier work of P. Garabedian in which all the variables are complexified.

The system of equations to be solved is basically

$$u_\xi = \lambda_+ v_\xi \quad , \quad u_\eta = \lambda_- v_\eta \qquad (3\text{-}1)$$

$$y_\xi = -\lambda_- x_\xi \quad , \quad y_\eta = \lambda_+ x \qquad (3\text{-}2)$$

where (u, v) = velocity components

(x, y) = physical space coordinate

(ξ, η) = characteristic coordinate

$$\lambda\pm = \frac{uv \pm \sqrt{u^2+v^2-c^2}}{c^2 - u^2} \qquad c\text{=local sound speed}$$

(Later a more convenient choice of characteristic parameters to replace (ξ, η) is made.) The characteristics are of course complex in the subsonic region and all the variables are considered complex. The characteristics are two-dimensional surfaces in a four-dimensional space. [The method is again based on the incompressible flow past an ellipse which fixes the topology of the hodograph singularity corresponding to the free-stream.] The corrections to the basic singular solutions are found by solving a characteristic initial value problem in the complex space. A version of Massau's method with complex arithmetic is used. The solution can be continued around the complex sonic line into the supersonic region.

A typical calculation in the subsonic region uses 200 grid points on each characteristic and takes about 20 sec. on the CDC 6600 machine (about 40000 sixty bit words of central memory are needed). Near the sonic line the solution is accurate to four digits. To compute the whole supersonic zone a typical initial grid uses 400 points on each characteristic and takes nearly two minutes for four place accuracy. The complete solution is found and plotted in six minutes.

The shape of the airfoil depends on the initial conditions chosen and the location of the initial characteristics. In these calculations the initial condition is taken as the (complex) Riemann function plus log terms which give singularities inside the nose and tail. There are seven parameters including M and E, the eccentricity of the corresponding ellipse. Figs. 4, 5 show some typical results, with the Mach lines drawn to scale.

4. Numerical Method: Marching Forward in Time

A numerical procedure for obtaining the steady transonic flow past a given profile with a shock wave, if necessary, has been carried out by Magnus and Yoshihara.[4] (Convair, San Diego).

The unsteady equations of continuity and momentum in conservation form are "solved" numerically.

$$\frac{\partial}{\partial t}\begin{Bmatrix}\rho \\ \rho u \\ \rho v\end{Bmatrix} = -\frac{\partial}{\partial x}\begin{Bmatrix}\rho u \\ \rho u^2 + p \\ \rho uv\end{Bmatrix} - \frac{\partial}{\partial y}\begin{Bmatrix}\rho v \\ \rho uv \\ \rho v^2 + p\end{Bmatrix} \tag{4-1}$$

The pressure is eliminated by assuming isentropic flow, a good approximation since transonic shock waves are weak, and a hyperbolic system in (u, v, ρ) is obtained. Initially and at infinity, the flow is taken as uniform, the airfoil surface is solidified* at $t = 0$ and the resulting flow computed as it tends toward a steady state.

The system (4-1) is replaced by a finite difference system in the main on a rectangular net $(\Delta x, \Delta y, \Delta t)$. A modification of the explicit Lax-Wendroff second-order accurate scheme is used. The truncation error of this scheme contributes both dispersion and damping. Near the boundary a special mesh and a special procedure is used. The conditions near infinity are handled by mapping the exterior of a circle in the (x, y) space into the interior of a circle in a transformed space. Extra fine meshes are introduced when rapid gradients are expected. The size of the time step is chosen as a suitable fraction of the allowable value as determined by a local linearized stability analysis.

From previous experience in computing one-dimensional unsteady shock waves it is to be expected that shock waves should appear in these calculationa as rapid transition zones spread over several mesh points centered around the correct location.

Two samples of calculations are presented. Fig. 6 shows one of Nieuwland's shock free profiles compared with experiment. The numerical results did not achieve the full suction peak (due to rapid acceleration on the profile) and consequently reflected waves arose downstream. Another example is the flow over an NASA 64A-410 profile at 4^{o} angle of attack (Fig. 7) and a comparison with wind tunnel experiments. (Fig. 8) shows the corresponding Mach number profiles.

In these calculations there were approximately 3700 mesh points, 53 points on the upper side of the airfoil. About 530 time steps were required to attain a reasonably steady result. The total computer time is approximately 2.2 hours on the CDC 6400 machine. The large amount of time required can be attributed to the large number of lattice points required to resolve the flow in the nose region adequately and to the very crude initial conditions.

*the Kutta condition is also imposed at a sharp trainling edge.

5. Direct Calculation — Relaxation Method

A relaxation method of calculating the mixed subsonic-supersonic steady flow field of an airfoil has been developed by Murman (Boeing Co.) and Cole.[5] This method also has the possibility of incorporating shock waves.

This method differs from those described previously in that the approximate transonic potential equation for $\phi(x,\tilde{y})$ is "solved"

$$\left(K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right)_x + \phi_{\tilde{y}\tilde{y}} = 0 \tag{5-1}$$

The exact potential is Φ

$$\Phi(x, y) = U\{x + \delta^{2/3}\phi(x, \tilde{y}; K) + \ldots\} \tag{5-2}$$

where U = free stream speed

(x, y) = dimensionless coordinates (airfoil chord equal one length unit.)

δ = airfoil thickness ratio;

$\tilde{y}$ = transonic coordinate = $\delta^{1/3} y$

K = transonic similarity parameter

$= \dfrac{1-M_\infty^2}{M_\infty \delta^{2/3}}$ (a version of K which is best according to our calculations)

The Equation (5-1) is of mixed type, hyperbolic for local supersonic flow $\phi_x > \dfrac{K}{\gamma+1}$ and elliptic for local subsonic flow $\phi_x < \dfrac{K}{\gamma+1}$. It can be shown (Ref. 5) that the correct shock-jump conditions are included in the conservation form (5-1)

The boundary value problem is illustrated in Fig. 9 for the case of a symmetric nonlifting flow. The condition at infinity is replaced by the first term of the far field solution. In the numerical work the finite difference calculations are carried out to that distance from the airfoil where the errors in neglecting the higher terms of the far field are of the same order as other errors. It can be shown by studying the integral equations obtained from (5-1) and the boundary conditions that

$$\phi(x,\tilde{y}) \cong \frac{1}{2\pi K^{1/2}} \mathscr{D} \frac{x}{x^2+K\tilde{y}^2} + \ldots \quad (5\text{-}3)$$

where

$$\mathscr{D} = \text{doublet strength} = 2\int_{-1}^{+1} F(\xi)d\xi + \frac{\gamma+1}{2}\int_{-\infty}^{\infty}\int u^2 d\xi d\eta$$

The doublet strength consists of the usual term proportional to the airfoil volume and a nonlinear contribution, unknown in advance. In the numerical procedure $\mathscr{D}$ has to be claculated as one of the unknowns in the problem. The result (5-3) is valid even if shock waves are present in the field.

The usual relaxation procedure fails when a local supersonic zone appears. The key to adapting a relaxation procedure to mixed flow is to use a local difference system which depends on the type of the equation, and has the correct domain of dependence. The following procedure is used: at each mesh point the velocity is computed and tested to determine if the flow is supersonic or subsonic. The appropriate implicit hyperbolic or elliptic difference scheme is then selected for that point. The implicit hyperbolic scheme has the advantage of being unconditionally stable with respect to mesh size and of fitting naturally into a line relaxation algorithm.

In the numerical procedure a mesh (i, j) is set-up in the finite part of the plane and values of ϕ are solved for along a vertical (x=const.) line. Each vertical line is successively relaxed proceeding in the + x (time-like) direction. The "latest" values of ϕ are used as they become available. After about 10 sweeps through the field the value of the doublet strength $\mathscr{D}$ is recalculated — it did not change rapidly. The sonic line and shock wave evolve naturally in the course of the iteration.

The following difference formulas have been used

$$\frac{1}{2}\left(\phi_x^2\right)_x = \phi_x \phi_{xx}$$

$$\rightarrow \left\{\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x}\right\}\left\{\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{(\Delta x)^2}\right\} \text{ elliptic} \quad (5\text{-}4)$$

$$\rightarrow \frac{\left(\phi_i-\phi_{i-1}\right)^2-\left(\phi_{i-1}-\phi_{i-2}\right)^2}{2(\Delta x)^2} \quad \begin{array}{l}\text{hyperbolic}\\ \text{1st order}\end{array} \quad (5\text{-}5)$$

$$\phi_{\tilde{y}\tilde{y}} \rightarrow \frac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{(\Delta\tilde{y})^2} \quad (5\text{-}6)$$

Sometimes a second-order accurate hyperbolic scheme was also used. The airfoil boundary condition was treated by using a mesh cell adjacent to the boundary. The test formula at each point which seemed to work best was:

$$(\zeta_x)_{test} = \frac{\zeta_{i+1}-\zeta_{i-1}}{2(\Delta x)} \quad (5\text{-}7)$$

A subcritical (subsonic) solution was first computed, by ordinary relaxation and then solutions for decreasing K were computed in sucession. The initial guess for $\phi_{ij}(K)$ was obtained from final values for a larger (or smaller) K. Convergence was established by computing answers at several K with initial guesses from above and below.

For the results reported here there are 74 mesh points along the x-axis (40 equally spaced along the airfoil) and 41 along the $\tilde{y}$-axis. The mesh was unequally spaced in the $\tilde{y}$-direction and in the x-direction ahead of and behind the airfoil. A typical calculation took about 400 iterations with a corresponding computing time of about 30 min. on an IBM 360/44. The same computation would be faster by more than an order of magnitude on a CDC 6600.

Some of the results achieved are shown for circular arc airfoils in Figs. 10, 11

$$C_p = \frac{\delta^{2/3}}{M_\infty^{2/3}} \bar{C}_p(x;K_s) \quad ; \quad K_s = \frac{1-M_\infty^2}{(M_\infty^2\delta)^{2/3}}$$

(Spreiter's similarity parameter). Fig. 12. For a Nieuwland airfoil symmetric fore and aft we have Fig. 13, 14 which show both on and off design calculations. A comparison with Korn's calculations is given in Figs. 15, 16, again for both on and off-design conditions. The good agreement for the design condition and the occurrence of shocks in the off-design condition is easily seen.

6. Remarks

I think that the previous sections have amply demonstrated the possibility of various methods for the calculation of transonic flow.

The good agreement between both hodograph methods and the finite difference calculation serves as a corroboration of both methods and enables the shock-free solutions to be extended by transonic similarity.

For future work the hodograph methods should be extended to obtain more general classes of shapes – finite difference calculations can also be done in the hodograph.

The unsteady procedures can be made much more efficient. Also an artificial time can be used so that a parabolic system is integrated instead of a hyperbolic one. This procedure is closely related to a relaxation procedure. Some experimentation on the unsteady small-disturbance equations seems advisable.

The modified relaxation procedure is sufficiently simple that thought can be given to doing lifting airfoils, axi-symmetric bodies, and even three-dimensional flows.

Numerical work for these types of problems is not cut and dried but by approaching the difficulties in an experimental way much progress can be made.

Acknowledgment

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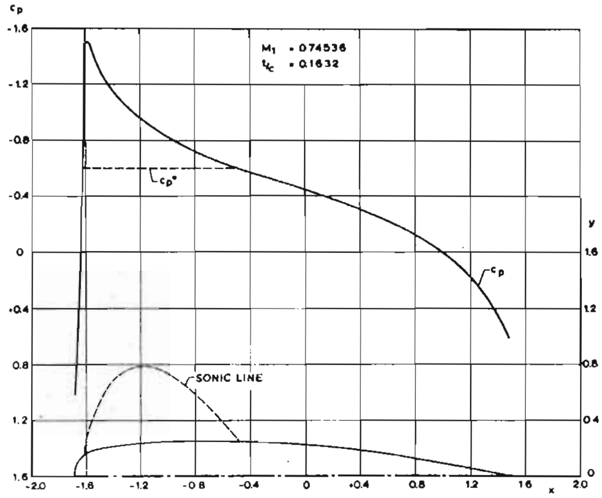

Fig. 1 Aerofoil section 0. 10-0. 675-1. 6.

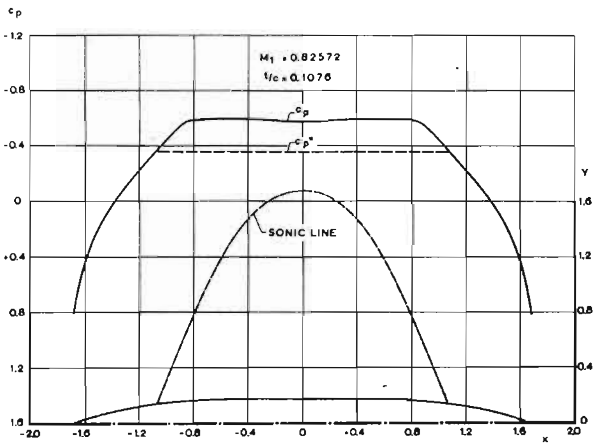

Fig. 2 Aerofoil section 0. 12-0. 7-0. 0.

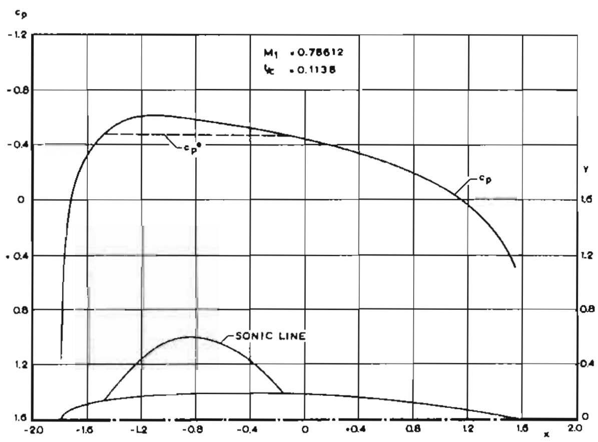

Fig. 3 Aerofoil section 0. 11-0. 75-0. 9·

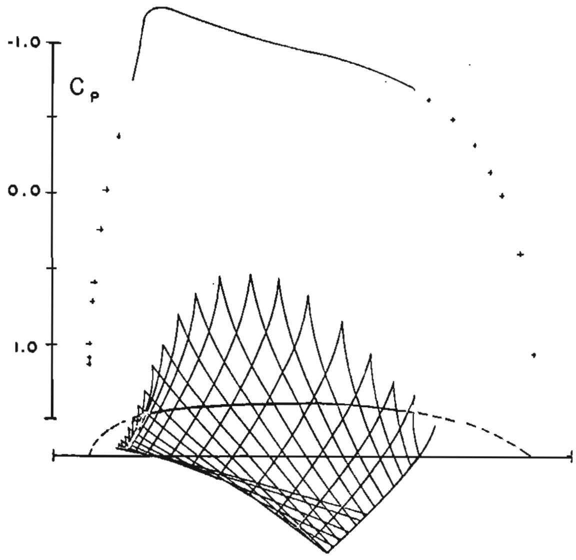

Fig. 4 Flow at $M_\infty = .7278$, $E = .381$, $\xi_o = -.40$ with $\epsilon_o = \epsilon_1 = 0$. $M_{max} = 1.2560$ and $T/C = .2418$.

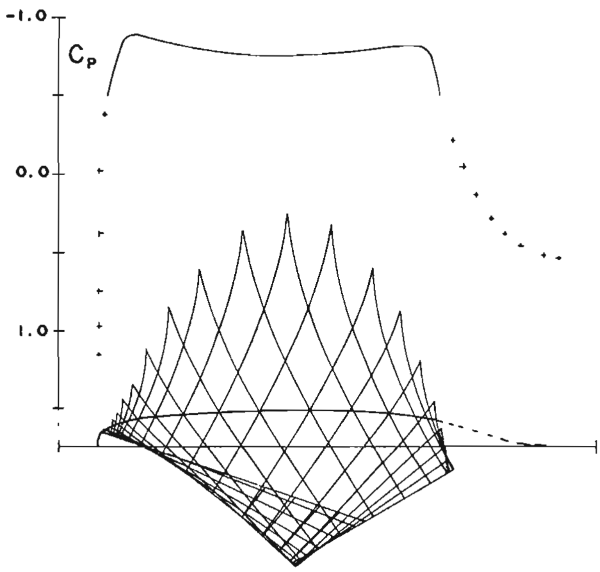

Fig. 5 Flow at $M_\infty = .7906$, $E = 0$, $\xi_o = .04$, $\epsilon_o = .4$, $\alpha_o = .59$, $\epsilon_1 = .15$, $\alpha_1 = -1.50$. $M_{max} = 1.2168$ and $T/C = .1582$.

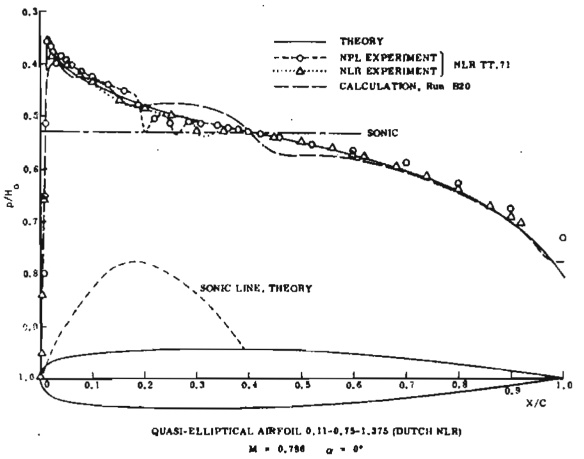

Fig. 6 Comparison of Calculations with Experiments for the Shockless Profile.

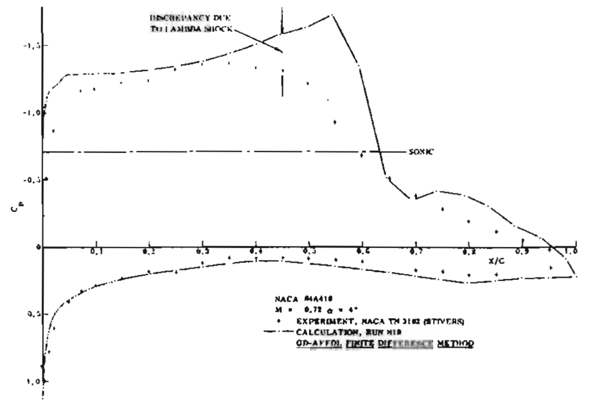

Fig. 7 Comparison of the Calculation with Experiments.

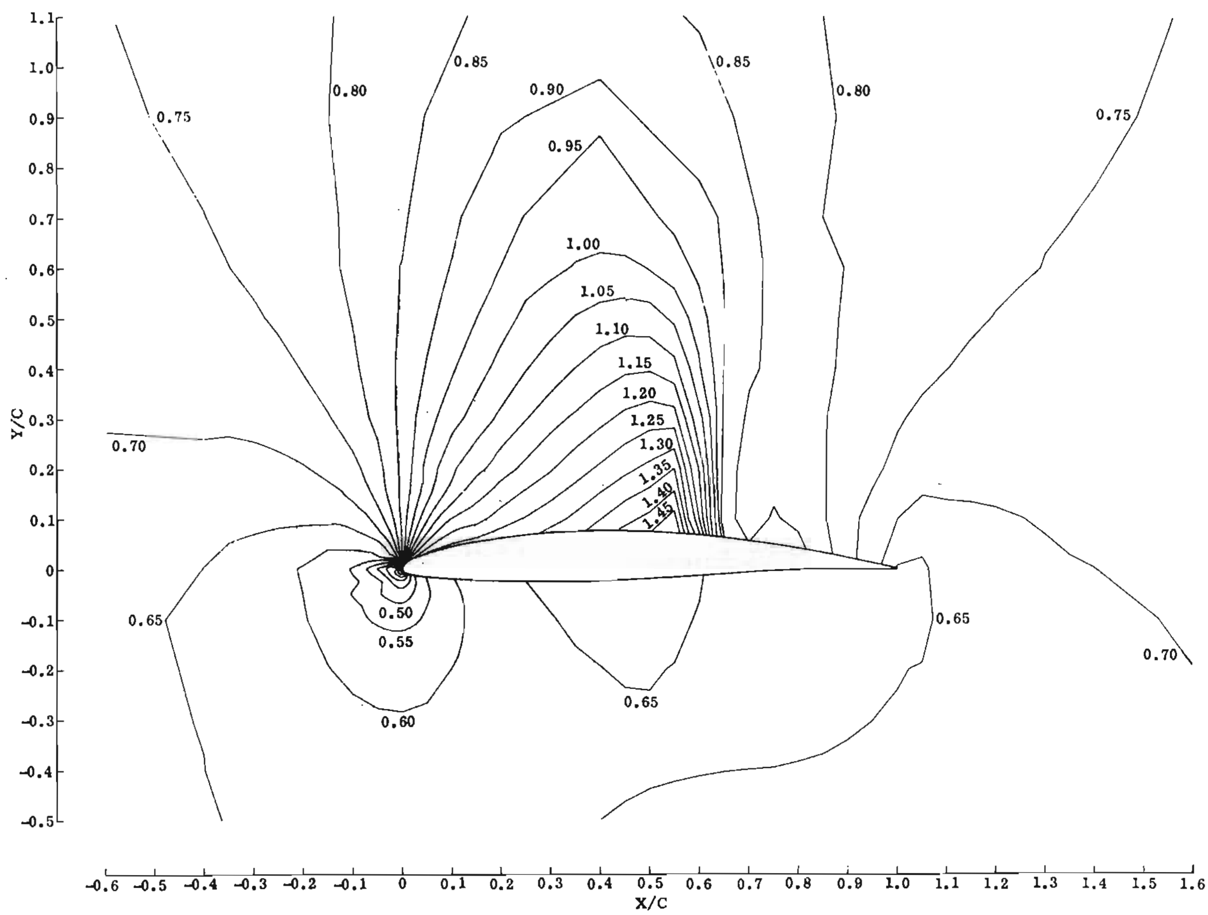

Fig. 8 Mach Number Contours for NACA 64A410 Airfoil at Mach 0.72 and α = 4 Degrees.

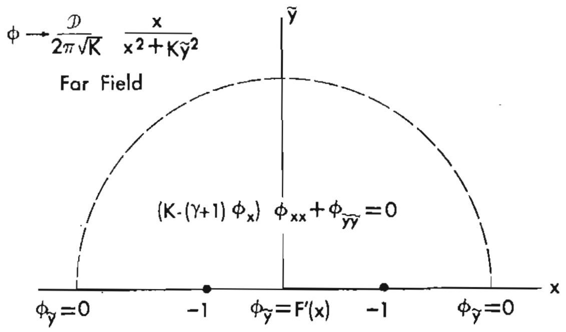

Fig. 9 Transonic boundary value problem.

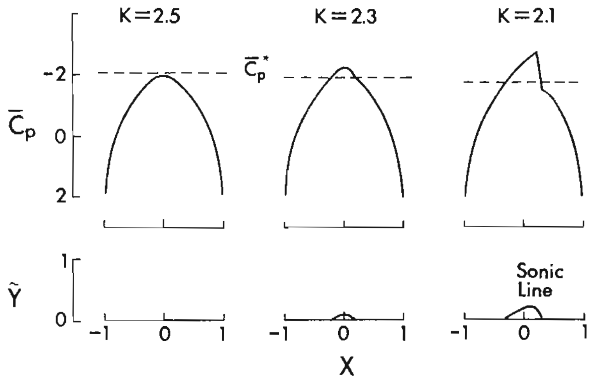

Fig. 10 Pressure distributions and sonic line locations for circular arc airfoil. Transition from subcritical to super-critical flow.

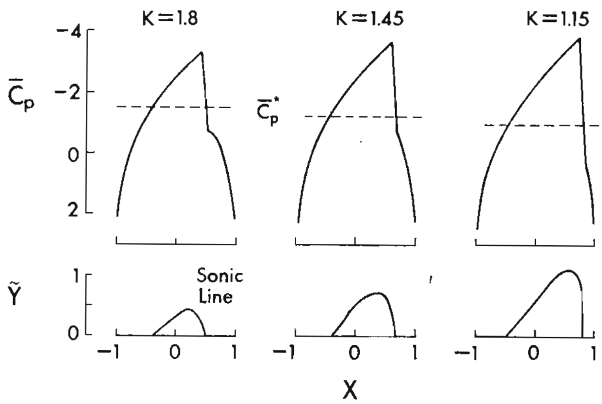

Fig. 11 Pressure distributions and sonic line locations for circular arc airfoil.

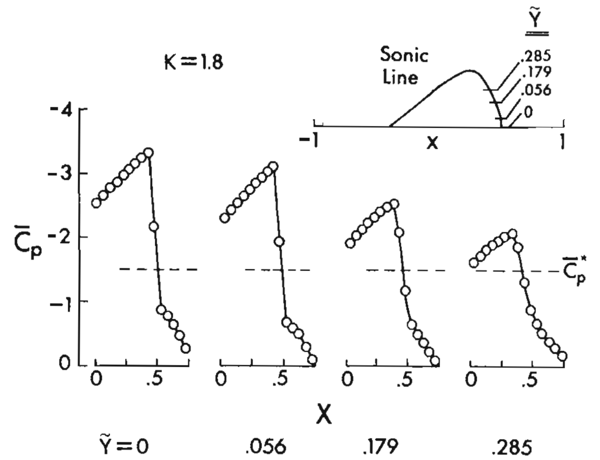

Fig. 12 Computed pressure distribution through the shock wave at various distances above a circular arc airfoil.

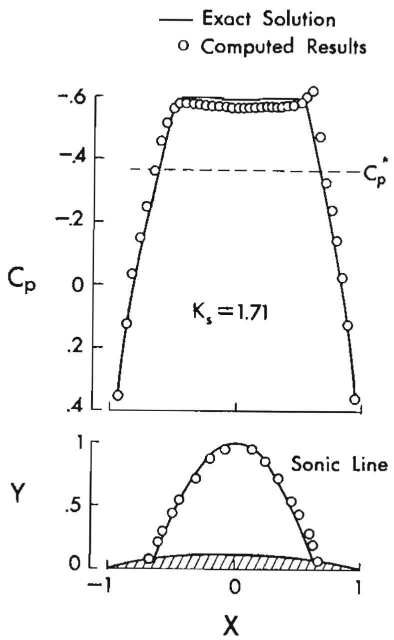

Fig. 13 Comparison of exact theory and computed results for a Nieuwland airfoil (NLR .12-.70-.00).

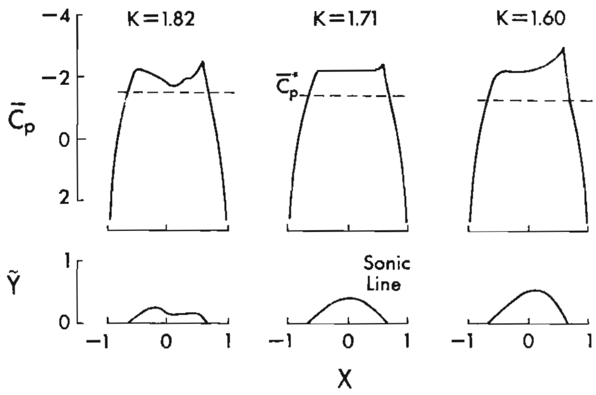

Fig. 14 Computed pressure distributions and sonic line locations for three values of K for a Nieuwland airfoil (NLR . 12- . 70- . 00).

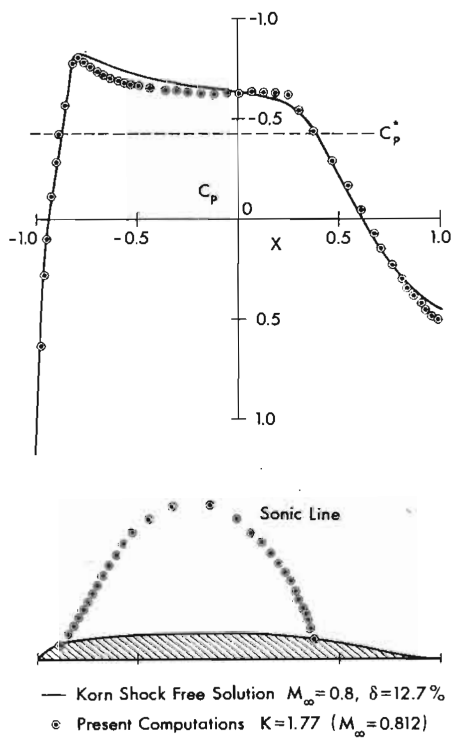

Fig. 15 Comparison of Relaxation of Complex Characteristic Method.

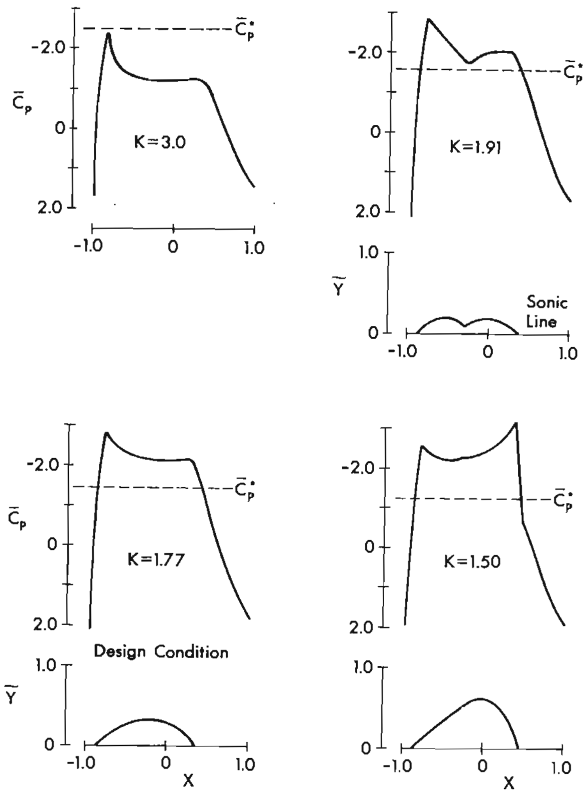

Fig. 16 Korn Airfoil—Relaxation Calculations of off-design conditions.