ICAS Paper No. 68-47

SPECTRAL GUST ALLEVIATION FACTOR

by

E. Huntley
Department of Applied Mathematics
and Computing Science
University of Sheffield
Sheffield, U.K.

The Sixth Congress of the International Council of the Aeronautical Sciences

DEUTSCHES MUSEUM, MÜNCHEN, GERMANY/SEPTEMBER 9-13, 1968

A STUDY OF THE SPECTRAL GUST ALLEVIATION FACTOR *)

E. Huntley
University of Sheffield, England

Abstract

Results are presented of a theoretical study of the spectral gust alleviation factor K, assuming a rigid aircraft and using the Dryden model of turbulence. For the aircraft responding only in heave an explicit expression for K in terms of the single parameter $\mu_g c/L$ is derived. The effects of incompressible unsteady aerodynamics for various aspect ratios are systematically studied and are shown to be less significant than previous studies suggest.

The effect of the pitching degree of freedom is investigated for aircraft both with and without tailplanes.

The importance of turbulence scale length L is clearly demonstrated.

The relation between K and the discrete factor F is considered and the relevance of the ratio F/K to the consistent estimation of loads on a new aircraft via the discrete and spectral techniques is discussed.

The RMS response ratio σ_n/σ_{w_g} is shown to depend only upon the two parameters $\mu_g c/L$ and V/L and from observations of the operating conditions of a wide range of transport aircraft it appears that they all have a similar sensitivity to gusts with $\sigma_n/\sigma_{w_g} = 0.016$ g units/ft/sec using L = 1000 ft.

1. Introduction

A great deal of research effort is going into the measurement of turbulent air velocities in relation to aircraft flight. There is a continuing need to review our knowledge of aircraft responses to air turbulence in order to appreciate the significance of the new information as it becomes available and in order to demonstrate which parameters in the problem are of most importance and therefore deserving of the closest study.

At the present stage of aircraft development we can see dramatic changes in aircraft speed, size and shape. Increased aircraft size, speed and height of operation imply larger values of the gust mass parameter μ_g than hitherto; they also mean that structural flexibility is becoming increasingly significant. Change of shape towards the slender configuration involves us not only in new appraisals of aircraft aerodynamics, steady and unsteady, but also in different modes of structural distortion from those typical of current subsonic turbojet transport aircraft. Because of the large number of parameters involved the task of sorting out which parameters are really significant (as opposed to those which may safely be neglected) is likely to be a lengthy one.

In 1953, Zbrozek [1] did a systematic study of the discrete gust alleviation factor taking into account the effects of unsteady lift functions for various aspect ratios and subsonic Mach numbers. Although power spectral techniques have been established for several years for the study of aircraft responses to random turbulence there does not appear to be a corresponding definitive study of the spectral gust alleviation factor although various aspects have been discussed in many studies, some of which are listed in the References [2-14].

One of the troubles in the past has been the magnitude of the computations involved which although by no means insurmountable, has nevertheless made parametric studies tedious to perform. In recent publications [15-17] the author has described new techniques which simplify response analyses of linear systems for deterministic or random inputs. Since aircraft dynamics can, in many circumstances, be adequately described by a system of linear time-invariant ordinary differential equations, the response of an aircraft to discrete gusts or random turbulence can be easily studied by the application of these techniques.

This paper should be regarded as a first stage of a study of aircraft responses to random turbulence based on stationary random process theory. It serves partly as an illustration of the techniques mentioned above but also gives some fundamental results for the acceleration response of the rigid aircraft.

The greater part of it is based upon Ref 20 in which only the heaving degree of freedom was considered and all sections with the exception of Section 4 are based upon that assumption. Subsequent work on the pitching degree of freedom is included in Section 4.

2. Background theory and summary of the serial/matrix technique

We shall make the following basic assumptions:

(i) that the vertical component of the random velocity field may be regarded as a stationary random process with zero mean and a power spectral density function described by the Dryden model (i.e. with an Ω^{-2} high frequency power law),

(ii) that it is a Gaussian process,

(iii) that the aircraft can be idealised as a linear system,

* This research was begun when the author was a member of Aerodynamics Department, R.A.E.

(iv) that spanwise variations in gust velocity may be neglected.

Assumptions (ii) and (iii) together imply that each response variable must also be a Gaussian process. The statistical analysis is then considerably simplified for in order to define meaningful statistical quantities such as the number of times per second a given response variable will cross a specified level it is sufficient to know its mean square value together with that of its derivative. [This defines N_0 when the variable concerned is c.g. acceleration]. Consequently we are interested primarily in the mean square responses of a linear system subjected to a stationary random process with zero mean and known power spectral density function.

Most of the theoretical work done so far has been by spectral analyses using the standard result

$$S_{oo}(\omega) = |H(\omega)|^2 S_{ii}(\omega) \quad (1)$$

where $|H(\omega)|^2$ represents the system frequency response function modulus squared; $S_{ii}(\omega)$ and $S_{oo}(\omega)$ are the input and output spectra respectively. For the complete aircraft system taking into account flexibility and unsteady aerodynamics the frequency response function can be a most complicated function. To obtain the response mean square value the equation

$$\sigma_o^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{oo}(\omega)d\omega$$

is used. Although it is possible to evaluate this integral analytically by contour integration, generally speaking this is impractical for all but the very simple cases. Determinantal expressions for σ^2 have been given by several authors* but frequently the integration is done numerically which then brings in questions of step length, cut-off frequency, and so on.

Another alternative is to replace equation (1) by the equivalent convolution relation in the time domain in terms of autocorrelation functions:

$$\phi_{oo}(t) = \int_{-\infty}^{\infty} \phi_{hh}(\tau)\phi_{ii}(t-\tau)d\tau \quad (2)$$

$\phi_{ii}(\tau)$ and $\phi_{oo}(\tau)$ are the input and output autocorrelation functions respectively. $\phi_{hh}(\tau)$ is the system autocorrelation function defined by

$$\phi_{hh}(\tau) = \int_{-\infty}^{\infty} h(t)h(t+\tau)d\tau$$

where $h(t)$ is the system unit impulse response. $\phi_{hh}(\tau)$ and $|H(\omega)|^2$ form a Fourier Transform pair so that

$$|H(\omega)|^2 = \int_{-\infty}^{\infty} \phi_{hh}(\tau)e^{-j\omega\tau}d\tau$$

$$\phi_{hh}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega\tau}d\omega$$

The mean square value is given by

$$\sigma_o^2 = \phi_{oo}(0) \quad (3)$$

The author has shown[16] how equation (2) can be interpreted as a transformation of the input autocorrelation function by the system to give the output autocorrelation function. The system transfer function is factorised into first order and second order factors, in both numerator and denominator, and the original system is replaced by a chain of filters each representing just one factor in the transfer function. The transformation of $\phi_{ii}(\tau)$ by the original system is then replaced by the successive transformations of $\phi_{ii}(\tau)$ as it passes through the various filters. It is demonstrated that for a very large class of input autocorrelation functions the transformation effected by an elementary filter is simply calculated by a single matrix operation - hence the description as the serial/matrix method. An explicit formulation of the output autocorrelation function is obtained and the mean square response given by equation (3).

As a simple demonstration of the method, Appendix A contains the derivation of mean square responses of a first order system when the input autocorrelation function is one representing random turbulence. More complicated cases are analysed by the use of comprehensive computer programmes written in ALGOL[17].

3. Vertical acceleration response and spectral gust alleviation factors for the heaving aircraft.

In this and all sections other than Section 4 we shall be concerned with the dynamics of an aircraft free to respond only in heave. The most significant measure of the response is that of c.g. acceleration, through its mean square value σ_n. Rather than work with the dimensional quantity σ_n g/ft/sec it is preferable to use the non-dimensional spectral gust alleviation factor K, defined by

$$\sigma_n = \frac{\rho Va}{2W/S} K \sigma_{w_g} = n_s K \sigma_{w_g} \quad (4)$$

where n_s is a statical standard of normal acceleration; σ_n is in g units.

We now establish the forms of the input autocorrelation function.

3.1 Autocorrelation functions for vertical gust velocity

A summary of most of the expressions for power spectra and autocorrelation functions of random air turbulence currently in use is provided by Taylor[11]. For the Dryden model, which gives a power spectral density function proportional to $1/\Omega^2$ at high frequencies, the autocorrelation function for vertical gust velocities is

$$\sigma_{w_g}^2 f_{w_g}(r) = \sigma_{w_g}^2 \left[1 - \frac{|r|}{2L}\right] \exp\left[\frac{-|r|}{L}\right]. \quad (5)$$

* The recent paper by Fuller[18] contains a survey of this development.

r is distance in feet, L is scale of turbulence in feet and subscript w_g denotes vertical gust velocity. $f_{w_g}(r)$ is the normalised autocorrelation function with $f_{w_g}(0) = 1.0$.

For an aircraft flying through the turbulence with velocity V, the random input to the aircraft has a power spectral density function which depends upon frequency ω in radians/sec where $\omega = \Omega V$, and an autocorrelation function depending upon time lag τ in seconds. Thus replacing r in equation (5) by $V\tau$ we obtain

$$\sigma^2_{w_g} f_{w_g}(\tau) = \sigma^2_{w_g}\left[1-\frac{V}{2L}|\tau|\right]\exp\left[-\frac{V}{L}|\tau|\right].$$

Now suppose that the time variable in the equation of motion for the aircraft is non-dimensionalised by the use of the time parameter $\hat{t}$, where $\hat{t} = W/g\rho SV$. Then, to be consistent, the time lag τ must be divided by the same parameter. If $\xi = \tau/\hat{t}$, we have

$$\sigma^2_{w_g} f_{w_g}(\xi) = \sigma^2_{w_g}\left[1-\frac{V\hat{t}}{2L}|\xi|\right]\exp\left[-\frac{V\hat{t}}{L}|\xi|\right]. \quad (6)$$

When we come to consider unsteady lift it is easier to work in terms of non-dimensional distance flown rather than time. Using the mean chord c as the unit of distance and writing $y = r/c$ the autocorrelation function becomes

$$\sigma^2_{w_g} f_{w_g}(y) = \sigma^2_{w_g}\left[1-\frac{c}{2L}|y|\right]\exp\left[-\frac{c}{L}|y|\right]. \quad (7)$$

3.2 Aircraft equation of motion

The following assumptions are made:

(i) the aircraft is rigid

(ii) the aircraft is free to heave but not to pitch

(iii) unsteady aerodynamics are neglected (in this section and in 3.3).

The equation of motion, in non-dimensional form, is then

$$(D + \frac{a}{2})\hat{w} = -\frac{a}{2}\hat{w}_g \quad (8)$$

where $\lambda = t/\hat{t}$ is non-dimensional time, $D \equiv d/d\lambda$, $\hat{w}$ is aircraft heaving velocity and $\hat{w}_g$ vertical gust velocity (each being non-dimensionalised by aircraft speed V).

Taking Laplace Transforms

$$H_{\hat{w}}(s) = \hat{w}(s)/\hat{w}_g(s) = (-a/2)/(s+a/2).$$

Heaving acceleration may be represented non-dimensionally by

$$\hat{n} = n\frac{\hat{t}}{V}$$

so that

$$-\hat{n} = Dw$$

and

$$H_{\hat{n}}(s) = \frac{\hat{n}(s)}{\hat{w}_g(s)} = \frac{a}{2}\cdot\frac{s}{s+a/2}. \quad (9)$$

3.3 Spectral gust alleviation factor, no unsteady lift

To obtain the mean square value of $\hat{n}$, we use the result derived in Appendix A with $A = a/2$, $k = a/2$, $b = V\hat{t}/L$, so that

$$\sigma^2_{\hat{n}} = \sigma^2_{w_g}\left(\frac{a}{2}\right)^2\left[\frac{V\hat{t}}{L}\left(2\frac{V\hat{t}}{L}+3\frac{a}{2}\right)\Big/ 2\left(\frac{a}{2}+\frac{V\hat{t}}{L}\right)^2\right]. \quad (10)$$

We now introduce the gust mass parameter μ_g defined by

$$\mu_g = \frac{2W/S}{\rho g c a} \quad (11)$$

Since $\hat{t} = W/g\rho SV$ it follows that

$$\mu_g c/L = 2V\hat{t}/aL. \quad (12)$$

Also

$$\sigma_{\hat{n}}/\sigma_{\hat{w}_g} = g\hat{t}(\sigma_n/\sigma_{w_g}),$$

so that

$$K = \frac{2W/S}{\rho Va}\cdot\frac{1}{g\hat{t}}\frac{\sigma_{\hat{n}}}{\sigma_{\hat{w}_g}} = \frac{2}{a}\cdot\frac{\sigma_{\hat{n}}}{\sigma_{\hat{w}_g}} \quad (13)$$

and, finally, from equations (10), (12) and (13)

$$K = \left[\frac{\mu_g c}{L}\left(2\frac{\mu_g c}{L}+3\right)\Big/ 2\left(\frac{\mu_g c}{L}+1\right)^2\right]^{\frac{1}{2}} \quad (14)$$

Within the stated assumptions, the spectral gust alleviation factor is seen to be a function of a single mass-scale parameter $\mu_g c/L$. This simultaneously characterises the ratio of inertia to aerodynamic forces, through μ_g, and also the aircraft scale to turbulence scale through c/L. K is plotted against $\mu_g c/L$ in a linear-log plot in Fig. 1. This agrees with the plot given by Taylor(11) who does not however quote any expression for it.

Taking typical current values of $\mu_g c$ as lying between say 250 and 1000 ft and the turbulence scale length as 1000 ft, K is seen to vary between 0.5 and 0.8. Since modern aircraft tend to have larger values of $\mu_g c/L$ than in the past the trend is towards larger values of K. The relevance of scale length L is fully discussed in Section 5.

3.4 The contribution from unsteady lift

The equation of motion for the heaving aircraft (equation 8) may be represented by the block diagram shown in Fig. 2(a). Since unsteady lift was neglected this meant that the force produced by $\hat{w}_g$ acted instantaneously. Thus a constant gain factor a/2 was sufficient to represent the transformation from gust velocity to force. The same was true for $\hat{w}$.

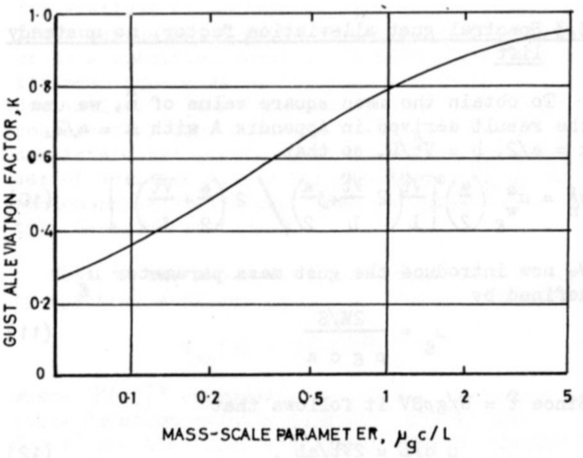
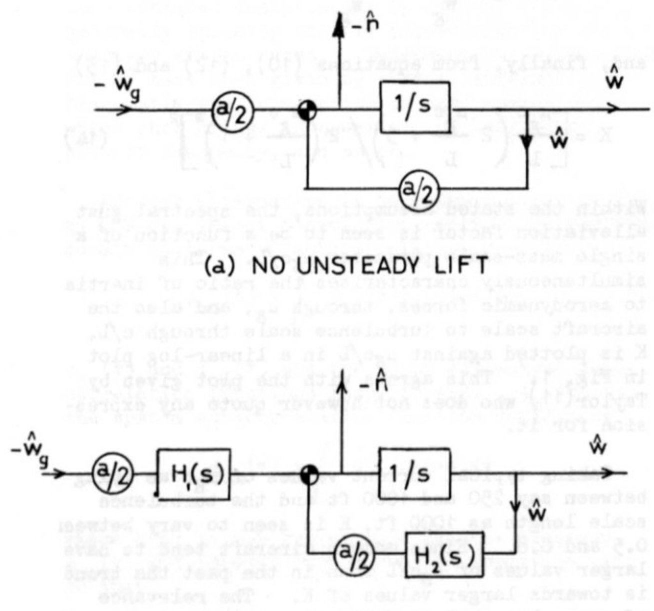


FIG.1 SPECTRAL GUST ALLEVIATION FACTOR, NO UNSTEADY LIFT



(a) NO UNSTEADY LIFT

(b) WITH UNSTEADY LIFT

FIG. 2 BLOCK DIAGRAMS FOR RESPONSES OF HEAVING AIRCRAFT

If we now wish to allow for unsteady lift, we have to include an additional transfer function $H_1(s)$ to provide the necessary lag in response to $\hat{w}_g$. Similarly a transfer function $H_2(s)$ is needed for the response to $\hat{w}$. The situation is then as shown in Fig. 2(b).

The transfer function for $\hat{n}$ becomes

$$H_{\hat{n}}(s) = \frac{\hat{n}(s)}{\hat{w}_g(s)} = \frac{a}{2}\cdot\frac{s\,H_1(s)}{s + \frac{a}{2}H_2(s)} .$$

Now the lag in production of force following a change in $\hat{w}$ or $\hat{w}_g$ is associated with the classical Wagner and Küssner functions respectively. These are most conveniently expressed in terms of non-dimensional distance travelled so we shall make a transformation from an equation based on non-dimensional time (λ or $t/\hat{t}$) to one based on non-dimensional distance (y or x/c or $V\hat{t}\lambda/c$.). Use p to denote the Laplace Transform variable associated with y so that

$$f(p) = \frac{V\hat{t}}{c}\,f(s) \quad \text{and} \quad p = s\Big/\frac{V\hat{t}}{c} .$$

The transfer function for $\hat{n}$ then becomes

$$H_{\hat{n}}(p) = \frac{\hat{n}(p)}{\hat{w}_g(p)} = \frac{a}{2}\cdot\frac{p\,H_1(p)}{p+\frac{ac}{2V\hat{t}}H_2(p)} = \frac{a}{2}\cdot\frac{p\,H_1(p)}{p+\frac{1}{\mu_g}H_2(p)} \quad (15)$$

The Küssner function, normally denoted by $\Psi(y)$, represents the transient response to a unit step in $\hat{w}_g$ whereas $H_1(p)$ denotes the Laplace Transform of the response to a unit impulse in $\hat{w}_g$. Consequently

$$H_1(p) = p\,\Psi(p)$$

where

$$\Psi(p) = \mathcal{L}\,[\Psi(y)] .$$

Similarly the Wagner function, $\Phi(y)$, represents the transient force following a unit step in $\hat{w}$. [There is in addition an air inertia term which we shall neglect since it is only of the order of 2% of the aircraft inertia term]. Hence

$$H_2(p) = p\,\Phi(p) .$$

3.4.1. Results for infinite aspect ratio

In the classical case of infinite aspect ratio, M = 0, these functions can be approximated by the following expressions:(19)

$$\Phi(y) = 1 - 0.165\exp(-0.09y) - 0.335\exp(-0.60y)$$

$$\Psi(y) = 1 - 0.50\exp(-0.26y) - 0.50\exp(-2.0y). \quad (16)$$

Hence,

$$p\,\Phi(p) = \frac{0.50p^2 + 0.561p + 0.054}{(p+0.09)(p+0.60)}$$

and

$$p\,\Psi(p) = \frac{1.13p + 0.52}{(p+0.26)(p+2.0)}$$

Finally,

$$H_{\hat{n}}(p) = \frac{a}{2} \cdot \frac{1.13(p+0.4601)(p+0.09)(p+0.60)}{(p+0.26)(p+2.0)\left\{p(p+0.09)(p+0.6) + \frac{1}{\mu_g}(0.500p^2+0.561p+0.054)\right\}} \tag{17}$$

For a particular value of $1/\mu_g$ the expression in braces has to be factorised into second order and/or first order factors. The computer programme then deals with the problem from this stage onwards and performs the analysis along the same lines illustrated by the simpler problem in Appendix A. The appropriate input autocorrelation function is that given in equation (7) but with normalised gust velocity. Thus,

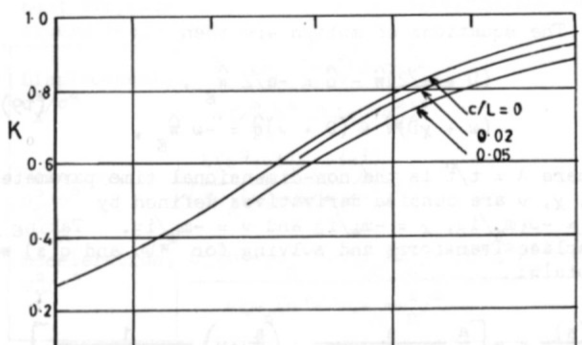
$$\phi_{ii}(y) = \sigma^2_{w_g}\left[1-\tfrac{1}{2}\tfrac{c}{L}|y|\right]\exp\left[-\tfrac{c}{L}|y|\right] . \tag{18}$$

Even without any further analysis it is apparent from the two equations (17) and (18) that K can depend only upon μ_g and c/L.

The results of the calculations are tabulated in Ref. 20 and are shown plotted here in Fig. 3(a). A set of curves for different values of c/L are produced running below and more or less parallel to the basic (no unsteady lift) curve. For the most extreme case treated with c/L = 0.05, the unsteady lift contribution is a reduction in K of around 0.08, or a 12% reduction at $\mu_g c/L = 0.5$.

3.4.2. Results for other aspect ratios

For these calculations the gust velocity is assumed constant across the span at any instant. Consequently variations in aspect ratio affect only the unsteady lift functions. The expressions used are those given by Zbrozek in his discrete gust alleviation factor study[1].



(a) INFINITE ASPECT RATIO

FIG.3 THE EFFECT OF UNSTEADY LIFT

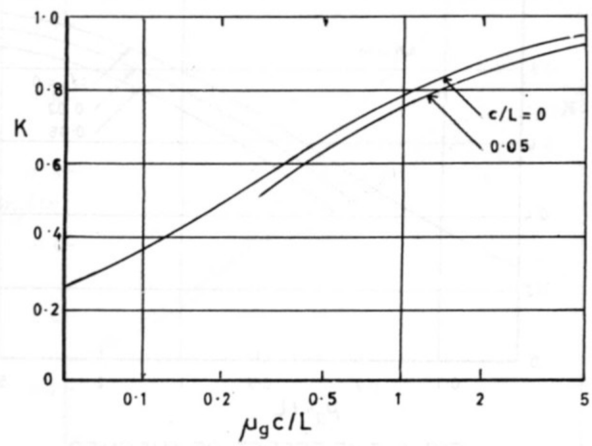


FIG. 3(b) ASPECT RATIO 6

Results for AR = 6 are shown in Fig. 3(b). The unsteady lift effect may be seen to be reduced in comparison with the effect for infinite aspect ratio. Thus a typical value for ΔK is 0.05 for c/L = 0.05, corresponding to an 8% change in K when $\mu_g c/L = 0.5$. Results for AR = 3[20], not shown here, indicate even less effect of unsteady lift and in most circumstances it could probably be ignored.

3.4.3. The separate effects of Küssner and Wagner functions

It appears that several previous studies (in particular those by Fung[2], Hall[10] and Taylor[11]) are based upon the assumption that the Wagner function may be ignored whilst retaining the Küssner function. The results of calculations made to test this hypothesis are shown in Fig. 4.

It may be seen that the two functions tend to work in opposite senses. Thus keeping the Küssner function whilst ignoring the Wagner function results in an overestimate of the effect of unsteady lift. This is seen more clearly in Fig. 5 which shows K cross-plotted against c/L for one value of $\mu_g c/L$. For the infinite aspect ratio case the effect of the Wagner function is quite important, relatively speaking, since it is around 33% of the Küssner effect. However for finite aspect ratios its effect is less important being, for example, only 16% of the Küssner effect for the AR = 6 case.

The results shown in Fig. 10.1 of Ref. 11 for infinite aspect ratio and neglecting the Wagner function appear to be identical with those presented here.

3.4.4. Comparisons with other calculations

The earliest systematic calculations appear to be those done by Fung[2], though a more comprehensive set of results based on Fung's approach are given by Press et al.[3]. These have been re-analysed and are shown in Fig. 6. Pratt and Bennett[6] give a set of design charts covering the effects of pitch as well as unsteady aerodynamics.

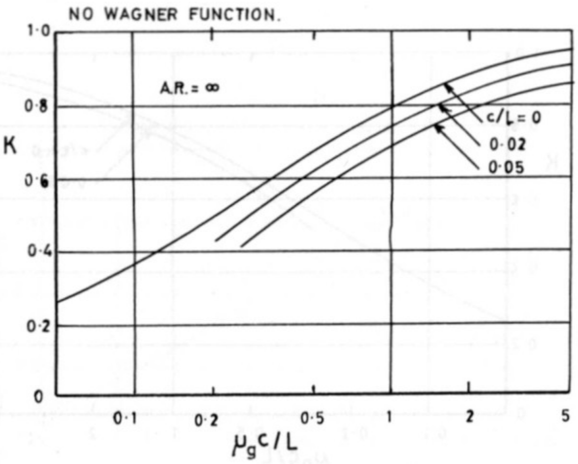


FIG. 4 THE EFFECT OF KÜSSNER FUNCTION ALONE

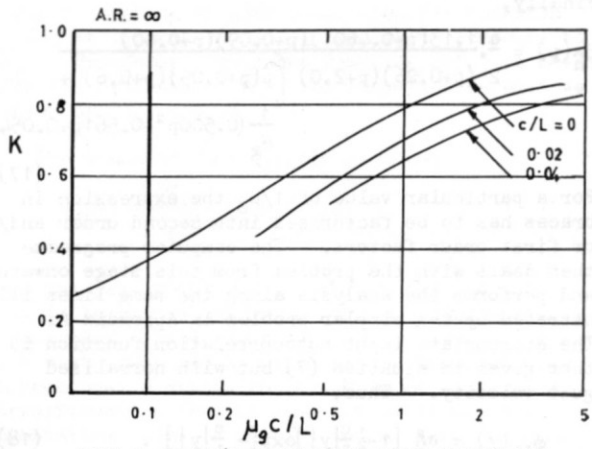


FIG. 6 FUNG'S RESULTS FOR SPECTRAL GUST ALLEVIATION FACTOR

Their data for the heaving-alone case agree substantially with those by Fung and are based on the same assumptions so we can deal with both sets of data simultaneously.

It is apparent from Fig. 6 that these earlier results follow the established trend with increasing $\mu_g c/L$ but that the unsteady aerodynamic forces have a much greater effect. (Fig. 7). Fung assumed that Wagner effect could be neglected so we might have expected closer agreement between Fung's and Hall's results. It can be shown that the remaining discrepancy is associated with Fung's inaccurate approximation to the Küssner function.

The overall conclusion from this work is that the effect of unsteady aerodynamic lift functions is appreciably less than had been suggested by earlier studies. In the earlier work two assumptions had been made regarding the form of these functions and both assumptions had worked in the same sense so as to considerably overestimate the effect upon the spectral gust alleviation factor. Further, we should remember that the comparison just made was based on unsteady lift functions for infinite aspect ratio and that for a finite aspect ratio the unsteady aerodynamic functions assume even less importance.

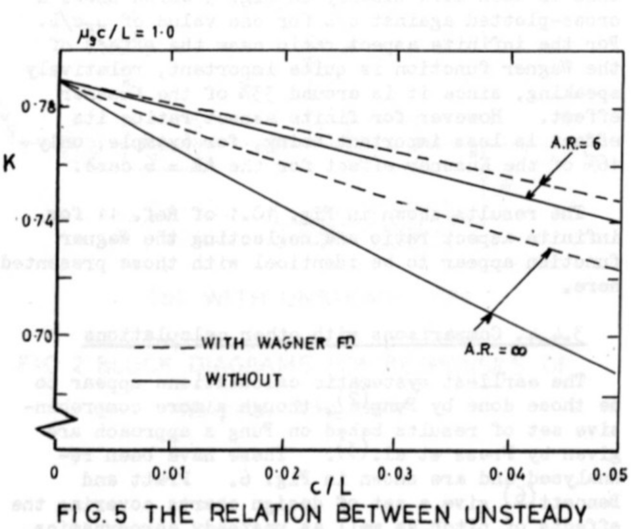


FIG. 5 THE RELATION BETWEEN UNSTEADY LIFT EFFECT AND AIRCRAFT SCALE

4. Spectral gust alleviation factor for the pitching aircraft

4.1 Tail-less aircraft

We make the assumptions:

(i) that the aircraft is rigid [the effects of static aeroelasticity could be included by using quasi-static derivatives]

(ii) that the phugoid is neglected

(iii) unsteady aerodynamic effects are neglected.

The equations of motion are then

$$(D + a/2)\hat{w} - \hat{q} = -a/2\ \hat{w}_g\ ,$$
$$(\omega + \chi D)\hat{w} + (D + \nu)\hat{q} = -\omega\ \hat{w}_g\ , \qquad (19)$$

where $\lambda = t/\hat{t}$ is the non-dimensional time parameter. ω, χ, ν are concise derivatives defined by $\omega = -\mu_1 m_w/i_B$, $\chi = -m_{\dot{w}}/i_B$ and $\nu = -m_q/i_B$. Taking Laplace Transforms and solving for $\hat{w}(s)$ and $\hat{q}(s)$ we obtain:

$$\frac{\hat{w}(s)}{\hat{w}_g(s)} = -\left[\frac{a}{2}\cdot\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2} + \left(\frac{a}{2}\nu+\omega\right)\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

and

$$\frac{\hat{q}(s)}{\hat{w}_g(s)} = \left(\frac{a}{2}\chi - \omega\right)\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

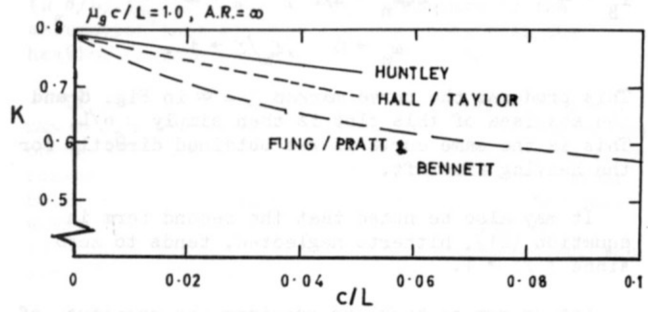


FIG. 7 COMPARISONS WITH PREVIOUS CALCULATIONS

where

$$2\zeta\omega_n = a/2 + \nu + \chi$$

$$\omega_n^2 = \frac{a}{2}\nu + \omega.$$

The transfer function for normal acceleration follows from

$$\hat{n} = n\hat{t}/V = -(D\hat{w} - \hat{q})$$

so that

$$\hat{n}(s) = -[s\hat{w}(s) - \hat{q}(s)].$$

Consequently,

$$\frac{\hat{n}(s)}{\hat{w}_g(s)} = \frac{a}{2}\,\frac{s^2}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{a}{2}(\nu+\chi)\,\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

In these expressions we can identify the various terms such as $s^2/(s^2+2\zeta\omega_n s+\omega_n^2)$ as the transfer functions of a second order system. Mean square responses of this system have been determined for input random processes having autocorrelation functions of the form of equation (6). The resulting closed form expressions are presented in Table 1 using the notation σ_x^2, $\sigma_{\dot{x}}^2$ and $\sigma_{\ddot{x}}^2$ to represent displacement, velocity and acceleration mean square responses.

Displacement, σ_x^2/σ^2	$\dfrac{b(b+2\zeta\omega_n)^2+\omega_n^2(3b+4\zeta\omega_n)}{4\zeta\omega_n^3(b^2+2b\zeta\omega_n+\omega_n^2)^2}$
Velocity, $\sigma_{\dot{x}}^2/\sigma^2$	$\dfrac{b(b^2+4b\zeta\omega_n+3\omega_n^2)}{4\zeta\omega_n(b^2+2b\zeta\omega_n+\omega_n^2)^2}$
Acceleration, $\sigma_{\ddot{x}}^2/\sigma^2$	$\dfrac{b[(b^2+3\omega_n^2)(4b\zeta\omega_n+\omega_n^2)+12b^2\zeta^2\omega_n^2]}{4\zeta\omega_n(b^2+2b\zeta\omega_n+\omega_n^2)^2}$
System:	$H_x(s) = \dfrac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$
Input:	$\phi_{ii}(\xi) = \sigma^2(1-\frac{b}{2}\lvert\xi\rvert)\exp(-b\lvert\xi\rvert).$

TABLE 1. Mean square responses for the second order system

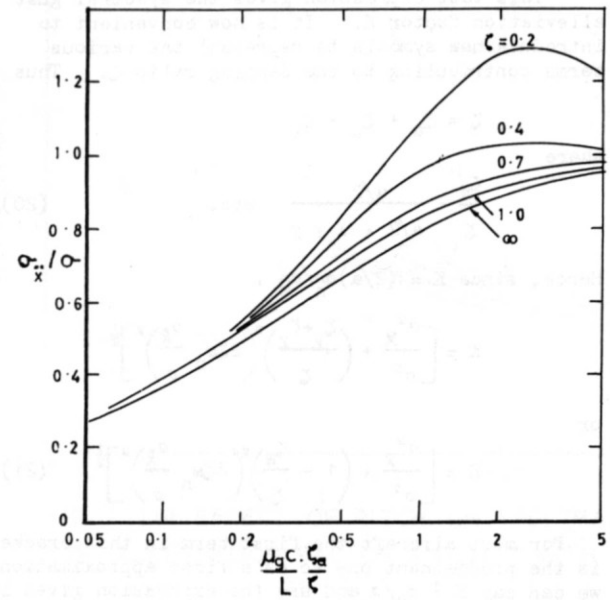


FIG.8 ACCELERATION ROOT MEAN SQUARE RESPONSE OF SECOND ORDER SYSTEM

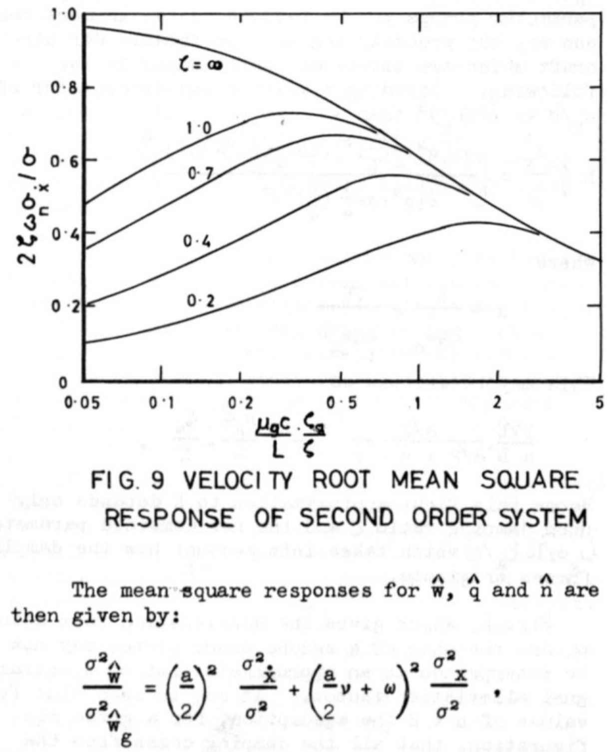


FIG. 9 VELOCITY ROOT MEAN SQUARE RESPONSE OF SECOND ORDER SYSTEM

The mean square responses for $\hat{w}$, $\hat{q}$ and $\hat{n}$ are then given by:

$$\frac{\sigma_{\hat{w}}^2}{\sigma_{\hat{w}_g}^2} = \left(\frac{a}{2}\right)^2\frac{\sigma_{\dot{x}}^2}{\sigma^2} + \left(\frac{a}{2}\nu + \omega\right)^2\frac{\sigma_x^2}{\sigma^2},$$

$$\frac{\sigma_{\hat{q}}^2}{\sigma_{\hat{w}_g}^2} = \left(\frac{a}{2}\chi - \omega\right)^2\frac{\sigma_{\dot{x}}^2}{\sigma^2},$$

and

$$\frac{\sigma_{\hat{n}}^2}{\sigma_{\hat{w}_g}^2} = \left(\frac{a}{2}\right)^2\left[\frac{\sigma_{\ddot{x}}^2}{\sigma^2} + (\nu+\chi)^2\frac{\sigma_{\dot{x}}^2}{\sigma^2}\right].$$

This last expression gives the spectral gust alleviation factor K. It is now convenient to introduce new symbols to represent the various terms contributing to the damping ratio ζ. Thus

$$\zeta = \zeta_a + \zeta_\nu + \zeta_\chi$$

where

$$\frac{\zeta_a}{\zeta} = \frac{a/2}{a/2 + \nu + \chi} \quad \text{etc.} \tag{20}$$

Hence, since $K = (2/a)\sigma_{\hat{n}}/\sigma_{\hat{w}_g}$,

$$K = \left[\frac{\sigma^2_{\chi}}{\sigma^2} + \left(\frac{\zeta_\nu + \zeta_\chi}{\zeta}\right)^2 \left(2\zeta\omega_n \frac{\sigma_{\dot{\chi}}}{\sigma}\right)^2\right]^{\frac{1}{2}}$$

or

$$K = \left[\frac{\sigma^2_{\chi}}{\sigma^2} + \left(1 - \frac{\zeta_a}{\zeta}\right)^2 \left(2\zeta\omega_n \frac{\sigma_{\dot{\chi}}}{\sigma}\right)^2\right]^{\frac{1}{2}} \tag{21}$$

For most aircraft the first term in this bracket is the predominant one so as a first approximation we can say $K \doteq \sigma_\chi/\sigma$ and use the expression given in Table 1 with $b \doteq V\hat{t}/L$.

K is then dependent upon three main parameters, namely, b defined above, undamped natural frequency ω_n and damping ratio ζ. The number of independent parameter groups may be reduced to two in more than one way but probably the most profitable for aircraft which are moderately well damped is the following. Dividing numerator and denominator of σ_χ/σ by $(2\zeta\omega_n)^5$ then

$$K \doteq \frac{\sigma_\chi}{\sigma} = \left\{\frac{\alpha[(\alpha^2+\frac{3}{4}(\frac{1}{\zeta})^2)(2\alpha+\frac{1}{4}(\frac{1}{\zeta})^2)+3\alpha^2]}{2[\alpha^2+\alpha+\frac{1}{4}(\frac{1}{\zeta})^2]^2}\right\}^{\frac{1}{2}}$$

where

$$\alpha = \frac{b}{2\zeta\omega_n} = \frac{V\hat{t}}{2\zeta\omega_n L}$$

This may be written as

$$\frac{2V\hat{t}}{a\,L}\cdot\frac{a/2}{a/2 + \nu + \chi} \quad \text{or} \quad \frac{\mu_g c}{L}\cdot\frac{\zeta_a}{\zeta}\ .$$

Hence this first approximation to K depends only upon damping ratio ζ and the new combined parameter $(\mu_g c/L).(\zeta_a/\zeta)$ which takes into account how the damping forces originate.

Fig. 8, which gives the acceleration root mean square response of a second order system may now be interpreted as an approximate plot of spectral gust alleviation factor. It may be seen that for values of $\alpha < 2$ the assumption, for a given configuration, that all the damping comes from the heaving motion, or that $\zeta_a/\zeta = 1$, results in an overestimate of the gust alleviation factor.

It is apparent from Fig. 8 how the results for the aircraft with two degrees of freedom go over into the results for the aircraft responding only in heave. Assume that aircraft pitch inertia increases without any change in aircraft mass i.e. the radius of gyration increases via the inertia parameter i_B. In the limit as $i_B \to \infty$ any concise aerodynamic derivative term such as ν or χ, having i_B in the denominator, tends to zero. Hence as

$$i_B \to \infty \qquad 2\zeta\omega_n \to a/2\ , \quad \zeta \to \infty$$
$$\omega_n \to 0 \quad , \zeta_a/\zeta \to 1\ .$$

This produces the curve marked $\zeta = \infty$ in Fig. 8 and the abscissa of this plot is then simply $\mu_g c/L$. This is the same curve as was obtained directly for the heaving aircraft.

It may also be noted that the second term in equation (21), hitherto neglected, tends to zero since $\zeta_a/\zeta \to 1$.

Let us now go back and consider the magnitude of this second term. We have

$$2\zeta\omega_n\sigma_{\dot{\chi}}/\sigma = \left\{\frac{\alpha[\alpha^2 + 2\alpha + \frac{3}{4}(\frac{1}{\zeta})^2]}{2[\alpha^2 + \alpha + \frac{1}{4}(\frac{1}{\zeta})^2]^2}\right\}^{\frac{1}{2}}$$

and this is shown plotted against α in Fig. 9, for several values of ζ. Unlike σ_χ/σ, this term increases in magnitude with increased damping ratio. However it should be borne in mind that large ζ is likely to be associated with values of ζ_a/ζ near unity and hence the term $(1 - \zeta_a/\zeta)^2(2\zeta\omega_n\sigma_{\dot{\chi}}/\sigma)^2$ may still be appreciably smaller than σ^2_χ/σ^2, except for the very small values of α.

Once this second term is taken into account the combined parameter $(\mu_g c/L)(\zeta_a/\zeta)$ is no longer so convenient to use. It is better to separate the mass scale parameter $\mu_g c/L$ from the damping parameters ζ and ζ_a. For a constant value of $\mu_g c/L$ (taken in this case to be 1.0) one can use the data in Figs. 8 and 9 to plot firstly K against ζ_a and then to derive the carpet plot (Fig. 10) of K for constant values of ζ_a and $\zeta_{\nu+\chi}$ (writing $\zeta_{\nu+\chi}$ for $\zeta_\nu + \zeta_\chi$).

In Fig. 10 the continuous lines are lines of constant $\zeta_{\nu+\chi}$; the dashed lines are lines of constant ζ_a. Since $\zeta_a + \zeta_{\nu+\chi} = \zeta$, vertical lines are lines of constant ζ. The continuous line denoted $\zeta_{\nu+\chi} = 0$ corresponds to a vertical section through the curves of Fig. 8 at a value of

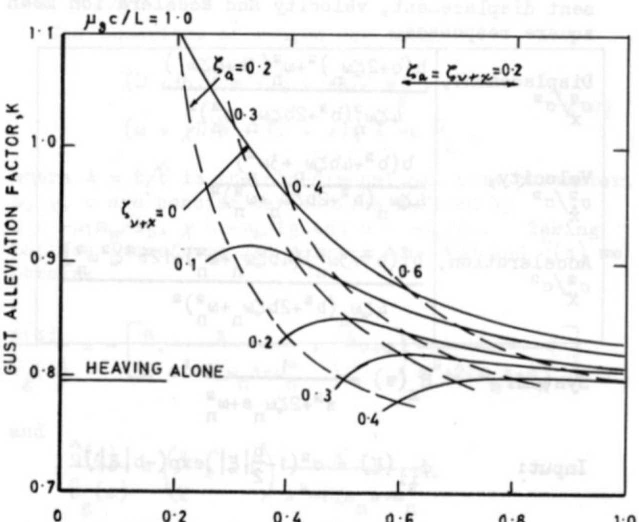


FIG.10 THE EFFECT OF PITCH ON SPECTRAL GUST ALLEVIATION FACTOR, TAIL-LESS AIRCRAFT

$(\mu_g c/L).(\zeta_a/\zeta) = 1.0$. It corresponds to the assumption that all the damping comes from the heaving freedom and none from pitching.

It may be noted that for a given value of the mass-scale parameter the alleviation factor decreases on the one hand, with increasing damping ratio ζ for constant damping due to heave ζ_a and, on the other hand, with increasing rotary damping $\zeta_{\nu+\chi}$ for constant ζ. In all cases as the total damping increases, K approaches the value indicated for the aircraft responding only in heave.

One may also judge from Fig. 10 that for many aircraft with moderate damping (say $\zeta \geq 0.5$) the pitching degree of freedom contributes an increment in K of no more than about 0.04, a percentage increase of 5%.

4.2 Aircraft with tail

In this section unsteady aerodynamic effects are not being considered. However the time delay between a gust arriving firstly at the wing and then the tail can be accounted for fairly easily using Zbrozek's quasisteady equations[4]:

$$(D+a/2)\hat{w} - \hat{q} = -a/2\ \hat{w}_g$$

$$(\omega+\chi D)\hat{w} + (D+\nu)\hat{q} = -[\omega - \overline{\nu_T - \chi D}]\hat{w}_g$$

where ν_T is the rotary damping coefficient due to the tail alone. By analogy with equation (20) we write

$$\zeta_{\nu_T} = \frac{\nu_T}{a/2 + \nu + \chi} \ .$$

It can be shown that for the tailed aircraft

$$K = \left[\frac{-\sigma_{\dot{x}}^2}{\sigma^2} + \left(\frac{\zeta_\nu + \zeta_{\nu_T}}{\zeta}\right)^2 (2\zeta\omega_n\sigma_{\dot{x}}/\sigma)^2\right]^{\frac{1}{2}} \qquad (22)$$

With a reasonable assumption for the magnitude of the downwash at the tail one can show that the contribution of the second term in equation (22) is approximately twice as large as in equation (21). The effect of this on the carpet plot of K is shown in Fig. 11, derived on the assumption $\zeta_{\nu_T} = \zeta_\nu$.

It may be seen that the general level of values of K is raised in comparison with those for the tailless aircraft (Fig. 10). As an estimate one might say that the pitching degree of freedom now gives rise to an increment in the alleviation factor of around 0.07, a percentage increase of 9%.

5. The importance of scale length L

We have seen that K is primarily dependent upon the combined mass-scale parameter $\mu_g c/L$, and only slightly upon aircraft scale directly, through the parameter c/L. Now although it has been the practice over the last decade or so to take L = 1000 ft as the standard scale length for turbulence at heights above 1000 ft, it has been suggested[13,14] that as spectra are measured to lower and lower frequencies due to improved instrumentation, at the same time estimates of L should be revised upwards. In fact Houbolt et al[13] quote values of L near 5000 ft for spectra

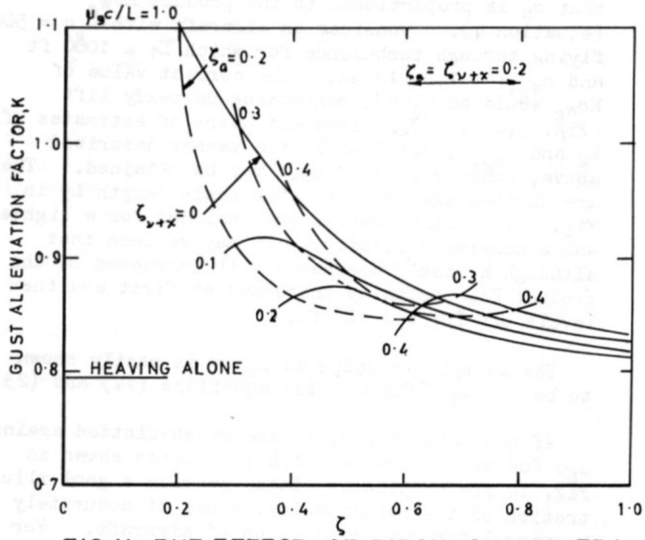


FIG.11 THE EFFECT OF PITCH ON SPECTRAL GUST ALLEVIATION FACTOR, AIRCRAFT WITH TAILS

of storm turbulence.

On increasing L, both parameters $\mu_g c/L$ and c/L are decreased. It may be seen from Fig. 3(a) that the gust alleviation factor is therefore reduced and the effect of unsteady lift upon K is likewise smaller. Assuming, for example, that pitch and the unsteady lift effect are neglected then for an aircraft with $\mu_g c = 500$, we see that K = 0.667 for L = 1000 ft but only 0.364 for L = 5000 ft. Thus at first sight it appears that the effect of greater scale length is entirely beneficial since it reduces K.

However, this point must not be taken in isolation since any estimate of L from experimental data is tied up with the simultaneous estimation of σ_{w_g}. If L is over-estimated, so, also, is σ_{w_g} and in a calculation of aircraft loads these two effects tend to compensate. Consider Fig. 12 in which are shown two spectral plots of $S_{w_g}(\Omega)$ with the same high frequency content, one for L = 1000 ft, $\sigma_{w_g} = 1.0$ ft/sec and the other for L = 5000 ft, $\sigma_{w_g} = \sqrt{5}$ ft/sec. Now suppose that in an experiment a sample record is taken from a process in which $L_1 = 1000$ ft and the root mean square gust velocity is specified by $\sigma_{w_g,1}$ but that, because of instrumentation difficulties, one succeeds only in obtaining the spectrum beyond, say, $\Omega = 2 \times 10^{-3}$ rads/ft. It is then apparent that since the 'knee' of the curve is poorly defined and could easily be lost in experimental scatter, several different pairs of values of L and σ_{w_g} could be found which would give a spectrum adequately representing the data. In general, with $L = L_2$, say, the corresponding root mean square velocity would be given by

$$\sigma_{w_g,2} = \sigma_{w_g,1}\left(\frac{L_2}{L_1}\right)^{\frac{1}{2}} \ . \qquad (23)$$

In particular with $L_2 = 5000$ ft, the corresponding estimate of $\sigma_{w_g,2}$ would be $\sqrt{5}\ \sigma_{w_g,1}$, as indicated in Fig. 12.

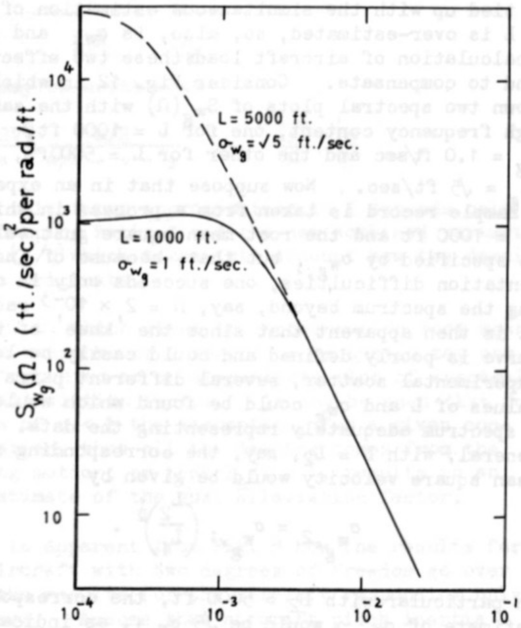
Now consider the effect that this uncertainty would have upon gust load estimation. We know that σ_n is proportional to the product $K\sigma_{w_g}$ (equation 4). Consider an aircraft with $\mu_g c = 500$ flying through turbulence for which $L_1 = 1000$ ft and $\sigma_{w_g,1} = 1.0$ ft/sec. The correct value of $K\sigma_{w_g}$ would be 0.667, neglecting unsteady lift (Fig. 3(a)). For different pairs of estimates of L_2 and $\sigma_{w_g,2}$, obtained in the manner described above, other values of $K\sigma_{w_g}$ may be obtained. These are plotted against estimated scale length L_2 in Fig. 13. [Other curves are included for a lighter and a heavier aircraft]. It may be seen that although K itself decreases with increased L, the product $K\sigma_{w_g}$ actually increases at first and then levels off for $L_2 > 10^4$ ft.

The asymptotic value as $L_2 \to \infty$ is easily shown to be $(3\mu_g c/2000)^{\frac{1}{2}}$ using equations (14) and (23).

If the data in Fig. 13 are cross-plotted against $\mu_g c$ for various values of L the curves shown in Fig. 14 are obtained. These provide a good illustration of the relative importance of accurately knowing L for different types of aircraft. For the lighter aircraft with $\mu_g c$ around 250 ft it matters very little, as far as rigid body loads are concerned, what value of L is assumed. However, at the other end of the range for, say $\mu_g c = 1000$ ft, errors of up to 40% in estimates of σ_n could arise from uncertainties in the value of L.

The exercise just described was repeated using $L_1 = 2500$ ft as the basis and the results of this calculation are presented in Fig. 15. The errors in $K\sigma_{w_g}$ that would result from incorrect estimates of L_2 from sample records of the process are seen to be much smaller than those mentioned in the previous paragraph. Again taking the values for $\mu_g c = 1000$ ft, the errors can be seen to lie roughly in the range ±15%.

There is therefore a strong argument for increasing the standard length L used in the model of



FREQUENCY, Ω rad./ft.

FIG.12 TWO GUST SPECTRA WITH THE SAME HIGH FREQUENCY CONTENT

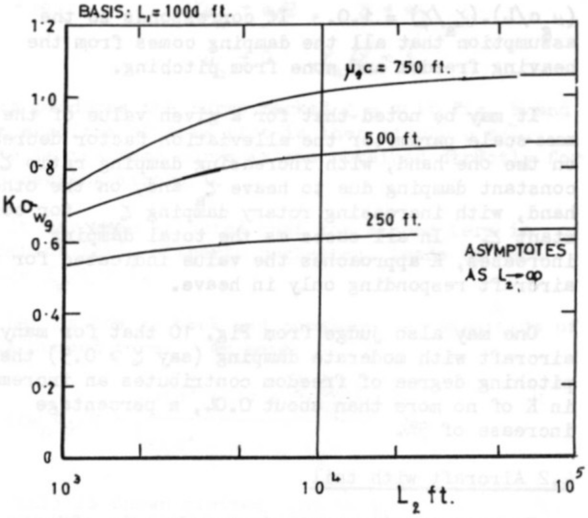


FIG.13 VARIATION OF PARAMETER $K\sigma_{w_g}$ WITH ESTIMATED SCALE LENGTH

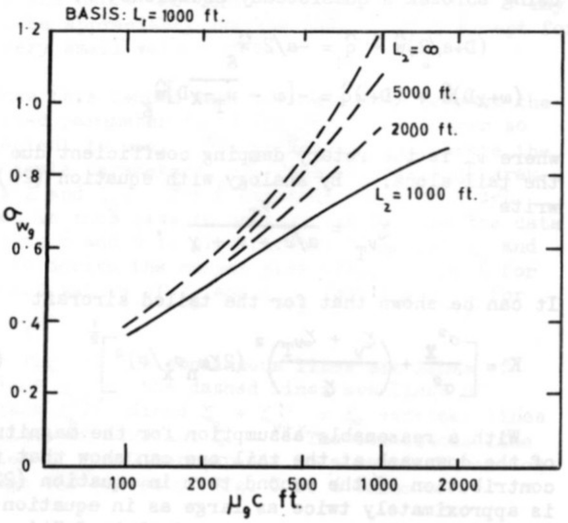


FIG.14 VARIATION OF $K\sigma_{w_g}$ WITH $\mu_g c$ FOR VARIOUS ESTIMATED SCALE LENGTHS

turbulence from the present value of 1000 ft to one in the range 2000-3000 ft (as is suggested in proposed F.A.A. criteria). L is known to be at least 1000 ft (for heights above 1000 ft) and is probably less than 6000 ft so that a standard scale length in the range 2000 to 3000 ft would not only accord better with the experimental evidence but would also reduce the possible errors arising from assuming an incorrect value.

6. Discussion on gust alleviation factors, discrete and spectral

6.1 Discrete gust alleviation factor

For the analysis of gust - c.g. acceleration data (Bullen[8]) the discrete gust concept is used following a procedure laid down in AP 970 based on the paper by Zbrozek[1]. This uses a discrete gust alleviation factor F determined by computing

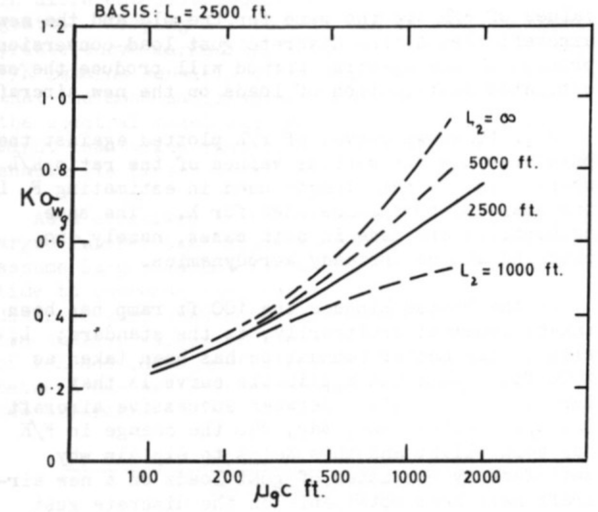


FIG.15 VARIATION OF $K\sigma_{w_g}$ WITH $\mu_g c$ FOR VARIOUS ESTIMATED SCALE LENGTHS

the peak acceleration of the aircraft when responding in heave to a 100 ft ramp gust. Then

$$n = \frac{\rho V a}{2W/S} F w_g \qquad (24)$$

where n is the peak acceleration in g units. [For stressing calculations aircraft firms will, of course, do more extensive calculations of discrete gust responses involving more modes and a variety of gust shapes and lengths.]

Zbrozek produced a series of curves for various ramp lengths. However, for the case when unsteady lift is neglected, it is apparent either by analogy with the analysis presented here for the spectral gust alleviation factor, or by direct consideration of the equation of motion that his set of curves of F against μ_g for various values of R/c (ramp length to chord ratio) could be replaced by a single curve of F against $\mu_g c/R$. It is easily shown that in this case

$$F = \frac{\mu_g c}{R}\left[1 - \exp\left(-\frac{1}{\mu_g c/R}\right)\right]; \qquad (25)$$

this function is shown as the continuous line in Fig. 16. It may be seen that F shows a similar increase in magnitude with increased $\mu_g c/R$ as that previously noted for K.

When unsteady lift is taken into account it is not possible to get a simple closed expression for F but nevertheless it is beneficial to plot the results obtained numerically against $\mu_g c/R$ for various values of c/R. The results for infinite aspect ratio shown in Fig. 16 were computed from the transfer function $H_{\wedge}(p)$ derived for the spectral calculations but using the deterministic input ALGOL programme[15,17]. The curves for various values of c/R in the main lie below those for the no unsteady lift case, particularly at the higher values of $\mu_g c/R$. For finite aspect ratios the curves are all found to bunch closer to the basic curve as was observed in the curves for the spectral factor and as one might expect from a knowledge of the unsteady lift functions.

6.2 Comparison between discrete and spectral gust alleviation factors

Neglecting unsteady lift, we have in Fig. 17 compared F as a function of $\mu_g c/R$ with K plotted as a function of $\mu_g c/L$. To indicate the significant parts of these curves, typical values of $\mu_g c$ are inserted most of which have been taken from Hall's paper. These range from 169 foraDC3, through 300 for a DC4, to 660 for a Comet I and 1500 for a Boeing 720; they are presented in Fig. 17 as values of $\mu_g c/L$ assuming L = 1000 ft and $\mu_g c/R$ assuming R = 100 ft.

This figure shows a somewhat striking similarity in shape between the two curves. If the curve for F is shifted to the left by some constant amount then it is possible to nearly superimpose one curve upon the other; if the best fit is made within the range of current values of $\mu_g c/L$ this means in effect

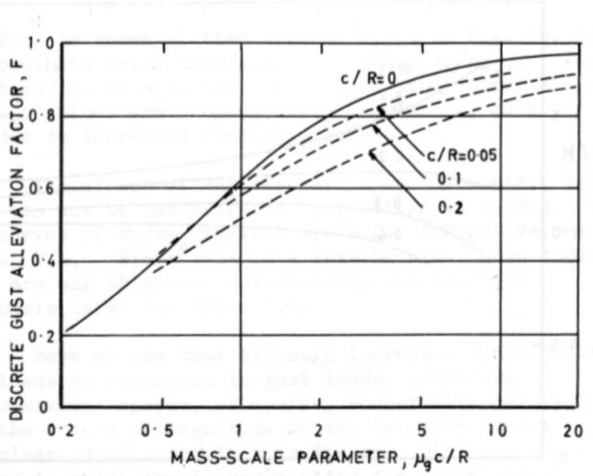


FIG.16 DISCRETE GUST ALLEVIATION FACTOR

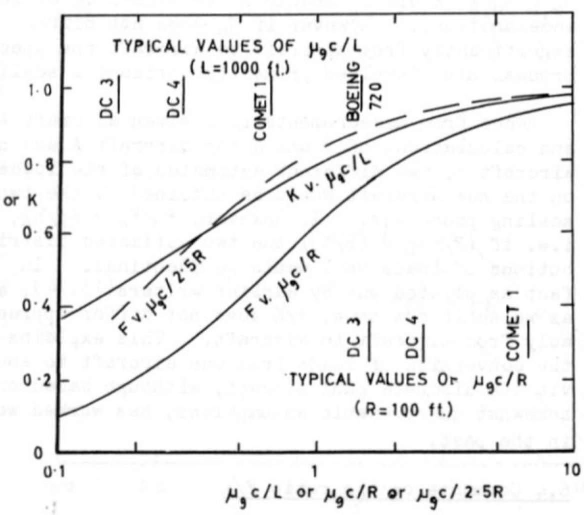


FIG.17 COMPARISON BETWEEN SPECTRAL AND DISCRETE GUST ALLEVIATION FACTORS

plotting F not against $\mu_g c/R$ but against $\mu_g c/2.5R$. Having done this, the differences between the two curves for $\mu_g c/L > 0.30$ are, on the whole, less than 3%.

Thus at this stage we can say that F and K are both heavily dependent on $\mu_g c$ and also on the shape or structure of the turbulence. In both the deterministic ramp type gust model and in the spectral model there is a length parameter (R or L) which is not yet completely identified. We have demonstrated a sort of equivalence between R and L in that for the assumed models of turbulence, for a given aircraft (i.e. prescribed $\mu_g c$) the same values of K and F are obtained when $L = 2.5R$. Note that we are not saying anything about the actual structure of the turbulence; we are merely indicating possible computational advantages of being able to infer values of K knowing F and vice versa. In the next section plot we shall explore further the relation between F and K to see whether these similarities do throw light on present methods of analysis of gust loads.

6.3 Estimation of loads on a new aircraft

We take as the starting point a measured distribution of crossings of acceleration levels from an existing aircraft or, rather, a multiplicity of them from several aircraft. These distributions can either be converted directly to give estimated distributions of loads on the new aircraft or used to give an intermediate gust model. In either case two different classes of conversion process are used; that based on the discrete gust concept and the other on the random process concept(13,14). Let us consider the direct conversion of accelerations.

In the discrete gust load process the only parameter embodying the aircraft dynamics is F. The actual conversion process of loads from aircraft A to aircraft B then simply involves horizontal scaling using the factor $(n_s F)_B/(n_s F)_A$. When using the spectral gust load process two parameters are required, namely a horizontal scaling factor $(n_s K)_B/(n_s K)_A$ and a vertical scaling factor $N_{o,B}/N_{o,A}$ (where N_o measures the crossings of zero acceleration). However if N_o does not differ significantly from aircraft to aircraft the spectral process also involves primarily horizontal scaling.

Hence from measurements in a given aircraft A, and calculations of F and K for aircraft A and new aircraft B, two different estimates of the loads on the new aircraft would be obtained by the two scaling processes. If however, $F_B/F_A = K_B/K_A$, i.e. if $(F/K)_B = (F/K)_A$ the two estimated distributions of loads on B would be identical. In fact as pointed out by earlier writers(13,14), and as we shall now show, F/K does not differ appreciably from aircraft to aircraft. This explains why the conversion of loads from one aircraft to another via the discrete gust concept, although based on somewhat questionable assumptions, has worked well in the past.

6.4 Comments on the ratio F/K

In the previous sections we have established that one important parameter concerned in the obtaining of consistent conversion of loads from aircraft to aircraft is the parameter F/K. If the values of F/K are the same for the old and the new aircraft then either discrete gust load conversion process or the spectral method will produce the same estimated distribution of loads on the new aircraft.

Fig. 18 shows curves of F/K plotted against the parameter $\mu_g c$ for various values of the ratio R/L where R is the ramp length used in estimating F, L the scale of turbulence used for K. The same assumptions are made in both cases, namely, no pitching and no unsteady aerodynamics.

In the United Kingdom the 100 ft ramp has been taken, somewhat arbitrarily, as the standard; L, with little better foundation has been taken as 1000 ft. Thus the applicable curve is that labelled $R/L = 0.1$. Between successive aircraft for $\mu_g c$ greater than, say, 250 the change in F/K has been slight and this helps to explain why satisfactory estimates of gust loads on a new aircraft have been obtainable on the discrete gust hypothesis. Even if spectral techniques had been sufficiently developed only small improvements in estimates of loads on new aircraft would have been achieved.

Now although the spectral approach to the estimation of gust loads is well-established, in aircraft firms it has not supplanted the discrete gust approach and both methods tend to be used side by side. Hence it may still be thought worthwhile if some comments were made regarding the other curves in Fig. 18 and, in particular, if consideration be given to the question what, with hindsight, would have been the most satisfactory value assumed for R. Most of the counting accelerometer data in the U.K. has been converted to discrete derived gusts, $N(w_{gd})$, by the use of discrete gust alleviation factors based on the 100 ft ramp. However no one claims that the distribution $N(w_{gd})$ actually represents the distribution of true gust velocities which a meteorologist would recognise. $N(w_{gd})$ provides a convenient half way stage to unify and collate all the information available from a variety of test aircraft. Supposing now that the 300 ft ramp length had arbitrarily been chosen as the standard so that all the discrete gust data would have been processed using different values of F, and resulted

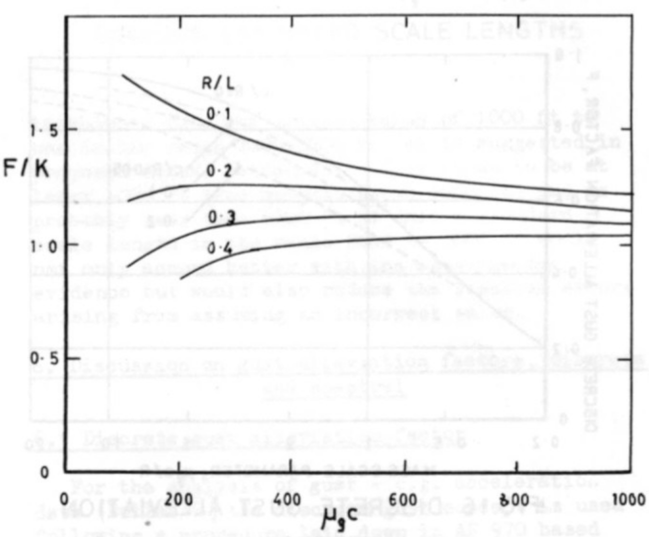


FIG.18 VARIATION OF GUST ALLEVIATION FACTOR RATIO F/K WITH $\mu_g c$

in different distributions of discrete derived gusts. Keeping L = 1000 ft and taking the appropriate curve marked R/L = 0.3, it may be seen that F/K remains sensibly constant. If one assumes that the most satisfactory model of turbulence is the spectral model and uses this as a basis one might then argue that all the distributions of w_{Ed} should lie closer together.

As a corollary to this discussion one might argue further that since there is a proposal to assume larger values for L one ought at the same time to increase the standard length R used in the discrete gust load conversion process. Otherwise the smaller values of R/L would imply larger values of F/K and consequently accentuate differences between the two methods of load estimation for a new aircraft.

7. The dependency of gust loads on the main parameters

In the main this paper is concerned with the calculation of the spectral gust alleviation factor K. With the assumptions of a rigid non-pitching aircraft it has been shown how K depends on just the two non-dimensional parameters $\mu_g c/L$ and c/L. We now go a stage further to see how σ_n/σ_{w_g} depends upon these and any other parameters through the use of the equation $\sigma_n = n_s K \sigma_{w_g}$.

σ_n/σ_{w_g} depends not only upon K but also upon the constants occurring in the statical standard $n_s = \rho V a/2W/S$. However, this can be re-arranged to give $n_s = V/g\mu_g c$. Thus

$$\frac{\sigma_n}{\sigma_{w_g}} = \frac{1}{g}\cdot\frac{V}{L}\cdot\frac{1}{\mu_g c/L}\cdot K\left(\frac{\mu_g c}{L}\cdot\frac{c}{L}\right) \qquad (26)$$

At this stage we can afford to leave unsteady lift out of the discussion knowing that in any practical case and a particular value of $\mu_g c/L$, K can be obtained by reducing the K for no unsteady lift by a suitable percentage, depending upon c/L.

In this case we can absorb the $(\mu_g c/L)^{-1}$ term in with K and consider the new combination

$$\frac{1}{\mu_g c/L}\cdot K\left(\frac{\mu_g c}{L}\right).$$

This is shown plotted against $\mu_g c/L$ in Fig. 19, the ordinate being displayed as $(Lg/V).(\sigma_n/\sigma_{w_g})$. The trend is twoards the lower right hand corner since $\mu_g c$ values are progressively increasing, mainly due to increased operating height.

The effect of the aircraft speed parameter, V/L, can now be demonstrated by plotting as in Fig. 20, curves of σ_n/σ_{w_g} against $\mu_g c/L$ for various values of V/L. Since this is a log-log plot these curves are all obtained by vertically shifting the basic curve for V/L = 1.0.

Here we see that although increased $\mu_g c/L$ values leads to reduction in gust loads, increased operating speeds, of course, accentuates them and the orders of magnitude of the two effects are clearly indicated in this figure. To give a guide as to which trend predominates the operating points of various transport aircraft are inserted on the graph. Most of the data are taken from Hall's paper but to bring the list up to date additional data are inserted for the Boeing 720, and two

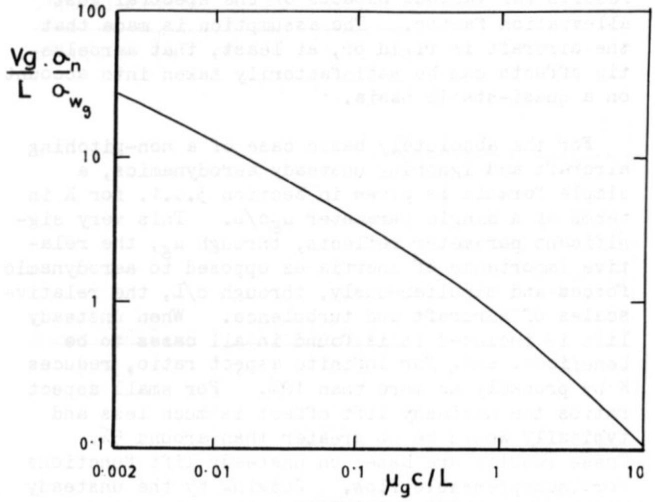


FIG.19 VARIATION OF NON-DIMENSIONAL C.G. ACCELERATIONS WITH $\mu_g c/L$

supersonic transport configurations.

This figure is interesting as an illustration of the fact that despite the vast range of types covered (a twenty fold increase in $\mu_g c/L$, a ten fold increase in speed between DC3 and S.S.T.) nevertheless values of σ_n/σ_{w_g} all lie close to a mean of, about, 0.016. This is of course just another way of saying that they have all been designed to meet a similar gust requirement.

8. Discussion

As the first stage of a study to be made of aircraft responses to random turbulence based on stationary random process theory, this paper presents

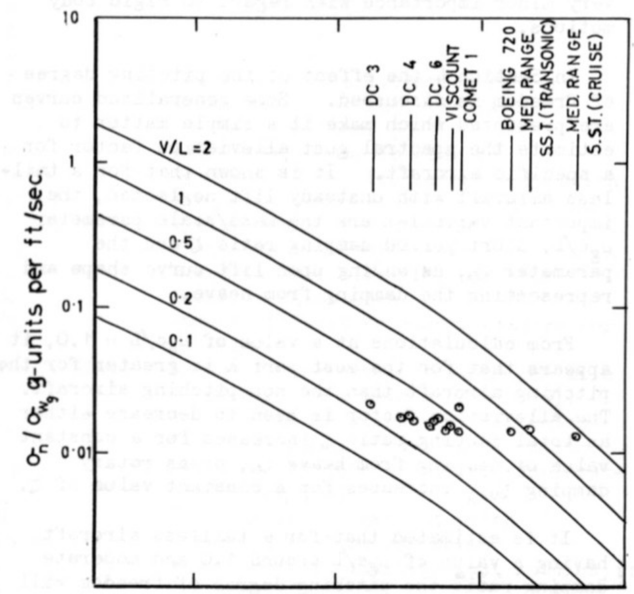


FIG. 20 GUST SENSITIVITY— ACTUAL C.G. ACCELERATIONS

results for various aspects of the spectral gust alleviation factor. The assumption is made that the aircraft is rigid or, at least, that aeroelastic effects can be satisfactorily taken into account on a quasi-static basis.

For the absolutely basic case of a non-pitching aircraft and ignoring unsteady aerodynamics, a simple formula is given in Section 3.4.1. for K in terms of a single parameter $\mu_g c/L$. This very significant parameter reflects, through μ_g, the relative importance of inertia as opposed to aerodynamic forces and simultaneously, through c/L, the relative scales of aircraft and turbulence. When unsteady lift is included it is found in all cases to be beneficial and, for infinite aspect ratio, reduces K by probably no more than 10%. For small aspect ratios the unsteady lift effect is much less and typically would be no greater than around 5%. These results are based on unsteady lift functions for incompressible flow. Judging by the unsteady lift functions shown in Ref. 1 for subsonic compressible flow we might expect greater reductions in the alleviation factor for flight at high subsonic speeds.

In Section 3.4.3., unsteady aerodynamic effects are broken down further and Küssner function and Wagner function separately discussed. It is shown that taking the Küssner function alone and ignoring the Wagner function results in overestimates of the beneficial reduction in K arising from the unsteady aerodynamics. The contrary effect of the Wagner function is around 33% of that of the Küssner function for infinite aspect ratio but much less than this for finite aspect ratios.

It is shown that the two simplifying assumptions regarding the form of the unsteady lift functions, made in previous calculations(2,6), led to considerable overestimates of the effect of unsteady aerodynamics. This would now appear to be of very minor importance with regard to rigid body motions.

In Section 4 the effect of the pitching degree of freedom is discussed. Some generalised curves are presented which make it a simple matter to estimate the spectral gust alleviation factor for a specific aircraft. It is shown that for a tailless aircraft with unsteady lift neglected, the important variables are the mass/scale parameter $\mu_g c/L$, short period damping ratio ζ and the parameter ζ_a, depending upon lift curve shape and representing the damping from heave.

From calculations at a value of $\mu_g c/L = 1.0$, it appears that for the most part K is greater for the pitching aircraft than the non-pitching aircraft. The alleviation factor is seen to decrease either as total damping ratio ζ increases for a constant value of damping from heave ζ_a, or as rotary damping $\zeta_{\nu+\chi}$ increases for a constant value of ζ.

It is estimated that for a tailless aircraft having a value of $\mu_g c/L$ around 1.0 and moderate damping ratio the pitching degree of freedom will contribute a 5% increase in the alleviation factor. For an aircraft with tail this figure may be nearer 10%. In this latter case general conclusions regarding trends are more difficult to state.

In the discussion in Section 5 on the relevance of turbulence scale length L the question of measurement accuracy of L is brought in. Since the measurement of L depends critically on the long wavelength part of the spectrum it is difficult to put a precise value on L, but there is evidence to support the contention that the present standard of 1000 ft is too low and should be replaced by a value in the range 2000 to 3000 ft. It is therefore important that we clearly realise how significant such a change would be with regard to aircraft response. It is shown firstly that an increase in assumed scale length L means a decrease in the value of the gust alleviation factor. However, uncertainty in the value of L from experiment is associated with uncertainty in σ_{w_g} and the two have to be taken together when estimating gust loads. It is demonstrated that in the case of a light aircraft (say $\mu_g c = 250$) these uncertainties may mean no more than a 10% variation in estimates of gust loads. However, for a heavier aircraft (say $\mu_g c = 1000$) there may be anything up to 40% variation. Since L is known to be at least 1000 ft (for heights above 1000 ft) and probably less than 6000 ft, replacing the standard length of 1000 ft by a value in the range 2000 to 3000 ft would not only accord better with the experimental evidence but would also considerably reduce the possible errors arising from the use of an incorrect value.

The root cause of the difficulty in estimating loads on a very large aircraft such as the Boeing 720 or the S.S.T. is, of course, the fact that so much power comes from the very long gust wavelengths where the spectrum is ill-defined. There is also a further difficulty to contend with, namely the greater contribution from the pilot. In this study we have been assuming stick-fixed aerodynamics but in the next stage the pilots contribution or that of an autopilot may be studied by the addition of some suitable transfer function in the block diagram.

The relation between the discrete and spectral gust alleviation factors is discussed in Section 6. It is shown firstly that the discrete factor F may be very conveniently plotted against $\mu_g c/R$, where R is ramp length in ft, and, for the case when unsteady lift is neglected, a simple expression is derived for F. Again when unsteady lift is included it leads to a set of curves for various values of c/R lying in the main below the basic curve.

One might have expected the general trend of K and F with increasing $\mu_g c$ to be similar to each other but it is slightly surprising to find the graphs of the two factors to be quite so similar in shape. It is demonstrated that in the conversion of loads from one aircraft to another an important factor leading to consistent estimations by the spectral and discrete gust techniques is that both old and new aircraft should have the same values of the parameter F/K. In Fig. 18 it is shown that F/K depends only upon $\mu_g c$ and the ratio of the two scale lengths R and L used in the two gust models. For the normal standard lengths of R = 100 ft and L = 1000 ft, F/K has varied little from aircraft to aircraft in the past which at least partly explains why loads have been satisfactorily predicted by the discrete gust conversion process. The spectral technique would have provided only slight improvement.

Agreement between spectral and discrete gust methods would have been even closer had a longer standard ramp length been used, say R = 300 ft, since this would have resulted in almost negligible variation in F/K with $\mu_g c$. Perhaps it should be stressed that whilst this somewhat arbitrary definition of a standard ramp length may be satisfactory in order to produce consistent c.g. acceleration estimates it will almost certainly not be adequate when discussing structural responses. For example structural responses of slender aircraft at low speed are known to be very sensitive to ramp length in the range 100 to 200 ft.

Lastly, in Section 7, the estimation of normal accelerations for a new aircraft is discussed. It is shown that to estimate σ_n/σ_{w_g}, using the relation

$$\sigma_n/\sigma_{w_g} = n_s K$$

there are basically only two important parameters, namely $\mu_g c/L$ and V/L. Curves are presented showing σ_n/σ_{w_g} as a function of $\mu_g c/L$ for various values of V/L. It appears from observations of the trend of $\mu_g c$ and V for a very wide range of transport aircraft that they all have approximately the same sensitivity to gusts, namely

$$\sigma_n/\sigma_{w_g} = 0.016 \text{ g units/ft/sec}$$

(assuming L = 1000 ft). This constant value probably simply reflects the fact that the aircraft considered have all been designed to satisfy somewhat similar gust strength requirements.

Acknowledgment

Crown Copyright, presented with the permission of the Controller, Her Majesty's Stationary Office.

Appendix A

Mean square response of the first order system (Method of Ref. 16, Part II).

Consider the system described by the differential equation

$$(D+k)x(\lambda) = A\,y(\lambda)$$

where λ is a non-dimensional time parameter, $D \equiv d/d\lambda$, A is a constant, $y(\lambda)$ and $x(\lambda)$ are random processes.

Taking Laplace Transforms,

$$H_x(s) = \frac{x(s)}{y(s)} = \frac{A}{s+k}$$

and

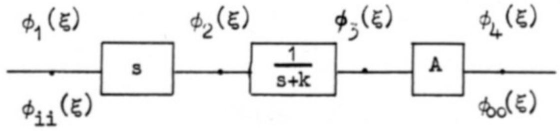
$$H_{\dot{x}}(s) = s\frac{x(s)}{y(s)} = \frac{sA}{s+k} .$$

We shall concentrate on the second of these two transfer functions.

Now assume that the stationary input process $y(\lambda)$ has autocorrelation function given by

$$\phi_{ii}(\xi) = \sigma_y^2[1-\tfrac{b}{2}|\xi|]\exp[-b|\xi|]$$

where ξ is a non-dimensional time lag based on the same time parameter as λ; σ_y^2 is the mean square value of y. The mean square value of the response, $\sigma_{\dot{x}}^2$, is obtained by calculating the transformations of $\phi_{ii}(\xi)$ as it passes through the following system.



Input function, $\phi_1(\xi)$ — s-matrix

$$\sigma^2\left[1, -\frac{b}{2}\right]\begin{bmatrix} e^{-b|\xi|} \\ |\xi|e^{-b|\xi|} \end{bmatrix} \qquad \begin{bmatrix} 2b & -b^2 & 0 \\ -2 & 2b & -b^2 \end{bmatrix}$$

Output function, $\phi_2(\xi)$

$$\sigma_y^2\left[3b, -2b^2, \frac{b^3}{2}\right]\begin{bmatrix} u_1(\xi) \\ e^{-b|\xi|} \\ |\xi|e^{-b|\xi|} \end{bmatrix}$$

Input function $\phi_2(\xi)$ — 1/(s+k)-matrix

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2k} \\ 0 & \frac{1}{k^2-b^2} & 0 & \frac{-b}{k(k^2-b^2)} \\ 0 & \frac{-2b}{(k^2-b^2)^2} & \frac{1}{k^2-b^2} & \frac{k^2+b^2}{k(k^2-b^2)^2} \end{bmatrix}$$

The function $\phi_3(\xi)$ is then given by

$$\sigma_y^2\left[0, \frac{-b^2(2k^2-b^2)}{(k^2-b^2)^2}, \frac{b^3}{2(k^2-b^2)}, \frac{bk(3k^2-b^2)}{2(k^2-b^2)^2}\right]\begin{bmatrix} u_1(\xi) \\ e^{-b|\xi|} \\ |\xi|e^{-b|\xi|} \\ e^{-k|\xi|} \end{bmatrix}$$

The scale constant filter merely multiplies $\phi_3(\xi)$ by A^2. Putting $\xi = 0$ in $\phi_4(\xi)$ we then obtain the expression for the mean square value for $\dot{x}$, which after simplification becomes

$$\sigma_{\dot{x}}^2 = \sigma_y^2\,\frac{A^2 b(2b+3k)}{2(k+b)^2}$$

Symbols

a	Lift curve slope
c	Wing mean chord
D	Operator $d/d\lambda$
F	Discrete gust alleviation factor
f(r) etc.	Normalised gust autocorrelation function (Eqn. 5)
g	Acceleration due to gravity
h(t)	System unit impulse response function
$\lvert H(\omega)\rvert$	Frequency response function modulus
H(s), H(p)	System transfer functions
i_B	Pitch inertia coefficient
K	Spectral gust alleviation factor
L	Scale of turbulence
N_o	Number of positive zero crossings per mile
N(n), N(w_g)	Number of positive crossings per mile of arbitrary levels of n or w_g.
n	c.g. normal acceleration in g units
$\hat{n}$	Non-dimensional normal acceleration
n_s	Statical standard of normal acceleration (Eqn. 4)
$\hat{q}$	Non-dimensional rate of pitch
p, s	Laplace Transform variables
R	Ramp length in discrete gust studies, ft.
r	Distance, ft.
S	Wing area
$S_{ii}(\omega)$, $S_{oo}(\omega)$	Input and output spectra respectively
t	Time, secs.
$\hat{t}$	W/gρSV, unit of time, aerosecs.
V	Aircraft forward speed
W	Aircraft weight
w	Aircraft vertical velocity

w_g	Gust vertical velocity
$\hat{w}$, $\hat{w}_g$	Non-dimensional velocities
w_{gd}	Gust velocities used in gust models
x	Distance travelled in ft.
y	x/c, non-dimensional distance travelled
λ	Non-dimensional time (Appendix A)
μ_g	$2W/S/\rho g c a$, gust mass parameter
ξ	$\tau/\hat{t}$, non-dimensional time lag in autocorrelation function
ρ	Air density, slugs/ft³
σ	R.m.s value
τ	Time lag, secs.
$\phi_{hh}(\tau)$	System autocorrelation function
$\phi_{ii}(\tau)$, $\phi_{oo}(\tau)$	Input and output autocorrelation functions
$\Phi(y)$	Wagner function
$\Phi(p)$	Laplace transform of $\Phi(y)$
$\Psi(y)$	Küssner function
$\Psi(p)$	Laplace transform of $\Psi(y)$
Ω	Gust spatial frequency (Radians/ft)
ω	ΩV, frequency in radians/sec.
ω, ν, χ	Concise aerodynamic derivatives (equation 19)
ω_n	Short period mode undamped natural frequency
ζ	Short period mode damping ratio
$\zeta_a, \zeta_\nu, \zeta_\chi$	Terms contributing to ζ

References

1. Zbrozek, J. K. Gust Alleviation Factor. A.R.C. R&M 2970, 1953
2. Fung, Y. C. Statistical aspects of dynamic loads. Jour. Aero. Sci. Vol. 20, No. 5, 1953
3. Press, H. et. al. A re-evaluation of data on atmospheric turbulence and airplane gust loads for application in spectral calculations. NACA Rep. 1272, 1956
4. Zbrozek, J. K. Longitudinal response of aircraft to oscillatory vertical gusts. R.A.E. Rep. Aero. 2559, 1955
5. Zbrozek, J. K. A study of the longitudinal response of aircraft to turbulent air. R.A.E. Rep. Aero. 2530, 1955
6. Pratt, K. G. and Bennett, F. V. Charts for estimating the effects of short period stability characteristics on airplane vertical acceleration and pitch angle response in continuous atmospheric turbulence. NACA TN 3992, 1957
7. Diederich, F. W. The response of an aircraft to random atmospheric disturbances. NACA Rep. 1345, 1958
8. Etkin, B. A theory of the response of airplanes to random atmospheric turbulence. Jour. Aero/Space Sciences, Vol. 26, No. 7, 1959
9. Atmospheric turbulence and its relation to aircraft. (Proceedings of a symposium at the R.A.E., Farnborough 1961). H.M.S.O.
10. Hall, J. C. A re-analysis of existing gust data using the power spectrum technique. (Symposium, Hatfield College of Technology, 1962).
11. Taylor, J. Manual on aircraft loads. AGARDograph 83, 1964
12. Huntley, E. The longitudinal response of a flexible slender aircraft to random turbulence. A.R.C. R&M 3454, 1964
13. Houbolt, J. C. et al. Dynamic response of airplanes to atmospheric turbulence including flight data on input and response. NASA TR R-199, 1964
14. Zbrozek, J. K. Atmospheric gusts. Jour. Royal Aero. Soc., Vol. 69, 1965
15. Huntley, E. A matrix formulation for the time response of time-invariant linear systems to discrete inputs. Int. J. Control, 1966, Vol. 4, No. 1, 49-72
16. Huntley, E. Matrix methods for the analytic determination of the output autocorrelation functions of linear systems for stationary random inputs. Parts I and II. R.A.E. TR 66387, 1966 (To be pub. Int. J. Control).
17. Hazelwood, L. and Huntley, E. ALGOL programmes for the response analysis of linear systems with deterministic or random inputs (R.A.E. Report to be published)
18. Fuller, A. T. The replacement of saturation constraints by energy constraints in control optimization theory. Int. J. Control, 1967, Vol. 6, No. 3.
19. Bisplinghoff, R. L. et al. Aeroelasticity. Addison-Wesley, 1955
20. Huntley, E. Spectral Gust Alleviation Factor R.A.E. Technical Report 68182, July 1968