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ORTHOGONALLY STIFFENED CYLINDRICAL SHELLS SUBJECTED TO INTERNAL PRESSURE

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ORTHOGONALLY STIFFENED CYLINDRICAL SHELLS SUBJECTED TO INTERNAL PRESSURE

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Abstract

Comprehensive investigation of two configurations of stiffened cylindrical shells subjected to internal pressure are made and illustrated by C-5A fuselage-type examples. One configuration has equally spaced longitudinal stringers and circumferential rings, while the other has uniform circumferential reinforcing bands (straps) added between every two rings. The skin, stringers, and rings are treated as distinct individual elements. The interactions among the elements are coupled by requiring compatible deformations along all element intersections. Numerical results for stiffened shells with, and without, reinforcing circumferential straps based on C-5A fuselage material parameters and dimensions are shown as plots of displacements and of inner and outer fiber stresses at various locations. The plotted results show that the straps, which are located midway between every two rings, used on C-5A fuselage are highly effective in equalizing stress levels throughout the stiffened shell, with associated reduction of maximum stress levels. This reduction will produce significant benefits in fatigue strength performance where fuselage pressurization and de-pressurization are predominant, and in resistance to damage propagation.

Nomenclature

a	Shell radius
A_R	Ring cross-sectional area
b	Stringer spacing
D_i	Plate rigidities of skin
e_{ij}	Strain tensor
E	Modulus of elasticity
F	Airy stress function
I	Moment of inertia of the stringer cross-sectional area
I_R	Moment of inertia of the ring cross-sectional area
K_{ij}	Curvature tensor
ℓ	Ring spacing
M_s	Bending moment in stringer
M_{ij}	Stress couples in skin
M_r	Bending moment in ring
N_{ij}	Stress resultants in skin
p	Internal pressure
P	Skin normal loading function (along outward normal is positive)
$\bar{P} = \pi a^2 p$	Axial load
q	Load between skin and stringer
$2Q$	Load between skin and ring
R_o	Ring mid-surface radius
T_r	Normal stress resultant in ring
t_i	Effective shell thickness in skin material
V_s	Transverse shear in stringer
V_r	Transverse shear in ring
v	Circumferential ring displacement
w	Radial skin displacement (along inward normal is positive)
W_s	Stringer displacement
W_r	Radial ring displacement (along inward normal is positive)
x,y	Longitudinal and circumferential coordinates
γ	Poisson's ratio
X_{ij}	Change of curvature tensor
λ,μ	Lamé constants

Introduction

Cylindrical shells stiffened by longitudinal stringers and rings have been used widely for various structural pusposes. Due to the complexity of the configuration the structure has been analyzed in the past by considering an equivalent homogeneous orthotropic shell with effective extensional and flexural stiffnesses as may be seen in almost all references(1-9). The discussion on the determination of the rigidity properties may be referred to(10,11). Such idealization yields results which are quite acceptable for general stability and free vibration analyses when only the buckling load and the natural frequencies are needed respectively. If one wishes to investigate the actual behavior of the structure or to design the structure in order to provide adequate strength or fatigue endurance, it becomes necessary to know the actual deformation of the shell and the actual stress distribution in the structure, and the analysis based on the idealization of the structure to that of an equivalent homogeneous orthotropic shell becomes undesirable. Bartolozzi(12) has discussed the general solution for the free vibrations of longitudinally stiffened cylindrical shells by treating the shell and stringers as individual components. No numerical example is given. Egle and Sewall(13) have studied the free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements. Beam modal functions are used to represent the deformation of the stringer as well as the shell in longitudinal direction, and Rayleigh-Ritz procedure is utilized. The present study is concerned with the deformation and stress analysis of orthogonally stiffened cylindrical shell subjected to static internal pressure. The shell, stringers and rings are treated as separate structural components. The interactions among the elements are coupled by requiring compatible deformations of the shell, the stringers and the rings. Series solutions satisfying the governing differential equations of the three basic components are used to solve the problem. The convergence of the resulting series for

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calculating the deformation of the structure is excellent; however, the convergence of the series solution for calculating higher derivatives of the deformation is sometimes not rapid due to the truncation of terms in the series in actual numberical computations. The quantities which involve higher derivatives of displacements, such as stress couples, are therefore calculated by using a finite-difference technique. The displacements used in these finite-difference expressions are computed first according to the series solution and hence, any desirable mesh size may be used. Small deformation and linear theories are used in the analysis, and the closed end effects are evaluated separately and then superimposed upon the solution corresponding to an open-ended cylinder subjected to normal loading.

Analysis

Cylindrical shells which are orthogonally stiffened by uniform and equally-spaced stringers and rings are considered for analysis. In a modern large vehicle design such as the giant C-5A airplane, intermediate circumferential bands are provided between every two adjacent rings of the fuselage for more effective structural performance and fail safe considerations. Non-uniform skin thickness in the longitudinal direction is therefore accounted for. Furthermore, the shell is considered to be long and the effects of the supporting conditions at the remote ends are negligible. As a result, the deformation patterns between every two adjacent stringers and every two adjacent rings are considered to be identical throughout the shell structure, and therefore, only one typical portion needs to be analyzed. The basic structural components involved in the analysis as shown in Figure 1 are the shell, a stringer and a ring. The boundary conditions at the edges of the components are the vanishing of slope and transverse shear.

The general solutions according to linear theories for each component will first be obtained in terms of the interacting forces. The interaction among the components will then be coupled by requiring compatible deformations along the intersections. The analysis and equations governing the behavior of these components without consideration of closed-end effects are first presented, and the effects of closed ends on the deformation and stresses will be discussed separately. More detailed derivations and discussions may be referred to(14).

Open Ended Cylindrical Shells

1. Stringer: The following system of equations are taken directly according to the elementary beam theory

$$EI\left(\frac{d^4W_s}{dx^4},\ -\frac{d^3W_s}{dx^3},\ -\frac{d^2W_s}{dx^2}\right) = \left(q, V_s, M_s\right) \qquad (1)$$

where E is the modulus of elasticity, I is the moment of inertia of the cross-sectional area, W_s is the transverse displacement, V_s is the transverse shear, and M_s is the bending moment; the subscript s corresponds to the stringer, and the sign convention is shown in Figure 1a. The solution

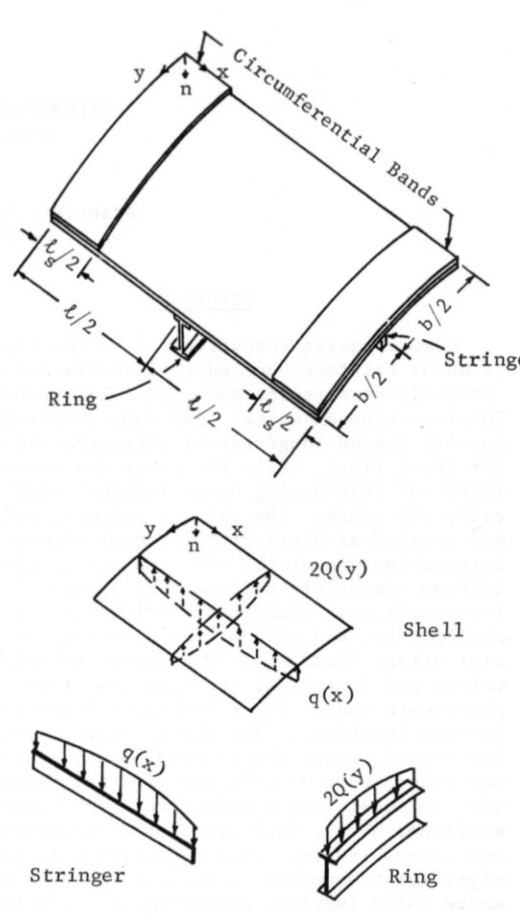

Fig. 1 Typical Section of Stiffened Shell Structure

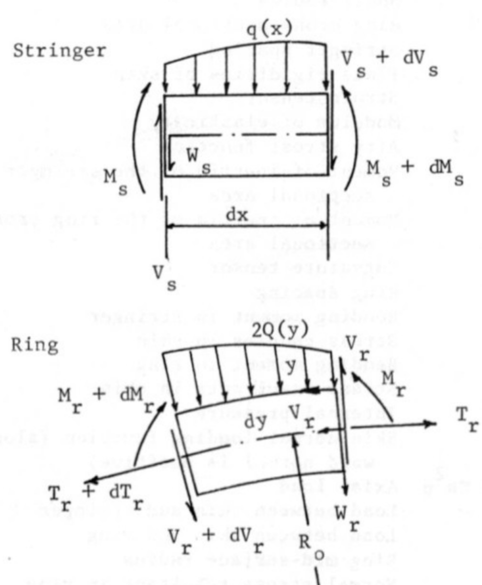

Fig. 1a. Elemental Sections of Stringer, and Ring

$$W_s = W_{so} + \sum_{m=2,4}^{\infty} \frac{q_m}{\alpha_m^4 EI} \cos \alpha_m x \tag{2}$$

Corresponding to the loading

$$q(x) = \frac{q_o}{2} + \sum_{m=2,4}^{\infty} q_m \cos \alpha_m x \tag{3}$$

is seen to satisfy the differential equation given in equation (1) as well as the boundary conditions $W_s'(0) = W_s'(\ell) = V_s(0) = V_s(\ell) = 0$. q_m represents the Fourier coefficient of the interacting load, q, $\alpha_m = \frac{m\pi}{\ell}$; W_{so} corresponds to the rigid body displacement of the stringer. Since the transverse shear forces are zero at x=0 and x=ℓ, one may conclude the $q_o/2$, the mean value of loading on the stringer, must be zero.

2. Ring: The equilibrium equations are

$$\frac{dT_r}{dy} - \frac{V_r}{R_o} = 0 , \tag{4}$$

$$\frac{dV_r}{dy} + \frac{T_r}{R_o} = - 2Q , \tag{5}$$

$$- \frac{dM_r}{dy} + V_r = 0 , \tag{6}$$

By using stress-strain and strain-displacement relationships, the following expressions are obtained:

$$T_r = EA_r \left(\frac{dV_r}{dy} - \frac{W_r}{R_o}\right) , \tag{7}$$

$$M_r = - EI_r \left(\frac{dV_r}{R_o dy} + \frac{d^2W_r}{dy^2}\right) , \tag{8}$$

Some symbols and sign convention are shown in Figure 1a. A_R is the cross-sectional area of the ring, and I_R is the moment of inertia of the cross-sectional area.

By eliminating V_r between equations (4) and (5) and then substituting equations (7) and (8) into these results and into equation (6) yields the following relationships:

$$I_R \frac{d^3W_r}{dy^3} - A_R \frac{dW_r}{dy} + R_o A_R \frac{d^2V_r}{dy^2} = 0 , \tag{9}$$

$$- I_R \frac{d^4W_r}{dy^4} - \frac{A_R}{a^2} W_r + \frac{A_R}{a} \frac{dV_r}{dy} = - \frac{2Q}{E} . \tag{10}$$

By eliminating V_r between equations (9) and (10) one obtains

$$\beta_1^* \frac{d^5W_r}{dy^5} + \beta_2^* \frac{d^3W_r}{dy^3} = 2 \frac{dQ}{dy} \tag{11}$$

where

$$\beta_1^* = EI_R - \frac{EI_R^2}{R_o^2 A_R} , \tag{12a}$$

$$\beta_2^* = 2 \frac{EI_R}{R_o^2} . \tag{12b}$$

The loading function, 2Q(y), is considered in the series form

$$2Q = Q_o + \sum_{n=2}^{\infty} 2 Q_n \cos \frac{n\pi y}{b} . \tag{13}$$

and the radial displacement W_r is represented by the following cosine series which satisfies the boundary conditions $dW_r/dy = V_r = 0$ at y = 0 and y = b:

$$W_r = W_{ro} + \sum_{n=2}^{\infty} W_{rn} \cos \frac{n\pi y}{b} . \tag{14}$$

Since Q_o represents the average uniform loading intensity on the ring, hence, the average ring deformation becomes

$$W_{ro} = \frac{Q_o a R_o}{EA_R} . \tag{15a}$$

Substituting equation (14) into equation (11) and compare like terms of the resulting series, the Fourier coefficients W_{rn} are found to be

$$W_{rn} = \frac{2Q_n}{\left(\beta_1^*\beta_n^2 - \beta_2^*\right)\beta_n^2} , \tag{15b}$$

and hence the general solution becomes

$$W_r = \frac{Q_o a R_o}{EA_R} + \sum_{n=2,4}^{\infty} \frac{2Q_n}{\left(\beta_1^*\beta_n^2 - \beta_2^*\right)\beta_n^2} \cos \beta_n y . \tag{16}$$

Other quantities V_r, T_r, and M_r may be obtained respectively from equations (6), (7), and (8) in conjunction with equation (16).

3. Shell: A typical portion of the shell under investigation is shown in Figure 1. The origin is chosen at midway between two rings and midway between two stringers. Linear bending theory for a cylindrical shell with non-uniform skin thickness along x-direction is considered. For convenience, tensor notation is used and the system of basic equations corresponding to normal loading are presented below. Similar derivation for uniform shell may be seen in [15,16], and the elasticity equations in tensor notation may be referred to any standard text book such as [17].

$$N_{ij,j} = 0 , \tag{17}$$

$$Q_{i,i} + K_{ij}N_{ij} - P = 0 , \tag{18}$$

$$Q_i = M_{ij,j} , \tag{19}$$

$$\frac{1}{t} N_{ij} = \frac{E\nu}{1-\nu^2} e_{kk}\delta_{ij} + 2\mu e_{ij} , \tag{20}$$

$$e_{ij} = \frac{1+\nu}{Et} N_{ij} - \frac{\nu}{Et} N_{kk}\delta_{ij}$$

$$= \frac{1}{2}\left(U_{i,j} + U_{j,i} - 2 K_{ij}w\right) , \tag{21}$$

$$M_{ij} = -\frac{1}{2} D \left[2\nu X_{kk}\delta_{ij} + (1-\nu)(X_{ij} + X_{ji})\right] \tag{22}$$

$$X_{ij} = W,_{ij} \text{ and } K_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 1/a \end{bmatrix} , \tag{23}$$

where N_{ij} are in-plane stress resultants, Q_i are transverse shear stress resultants, K_{ij} is the curvature tensor, P is the normal loading function, M_{ij} are stress couples, e_{ij} is the strain tensor, t is the shell thickness, δ_{ij} is the Kronecker delta, E is the modulus of elasticity, ν is the Poisson's ratio, μ is the Lamé constant, u_i are in-plane displacement components, w is the normal displacement, $D = Eh^3/12(1-\nu^2)$ is the plate rigidity, X_{ij} is the change of curvature tensor, and a is the radius of the cylinder. The relevant compatibility equation is

$$\epsilon_{ia}\epsilon_{jb}e_{ij,ab} = \epsilon_{ia}\epsilon_{jb}K_{ij}W,_{ab} \tag{24}$$

where ϵ_{ia} is the permutation symbol.

Substitution of equation (21) into equation (24) and introducing into the resulting equation the Airy stress function, F, such that

$$N_{ij} = \nabla^2 F\delta_{ij} - F,_{ij} , \tag{25}$$

one obtains

$$\bar{\nabla}^4 F = -\frac{1}{a} W,_{11} \tag{26}$$

where

$$\bar{\nabla}^4 = \frac{1}{Et}\left(\frac{\partial^4}{\partial y^4} - \nu\frac{\partial^4}{\partial x^2\partial y^2}\right) + \frac{\partial^2}{\partial x^2}\left[\frac{1}{Et}\left(\frac{\partial^2}{\partial x^2} - \nu\frac{\partial^2}{\partial y^2}\right)\right] + 2(1+\nu)\frac{\partial}{\partial x}\left(\frac{1}{Et}\frac{\partial^3}{\partial x\partial y^2}\right) \tag{27}$$

Elimination of Q_i between equations (18) and (19) and substitution of equations (22) and (25) into the resulting equation leads to the following equation:

$$\bar{\bar{\nabla}}^4 W = \frac{1}{a} F,_{11} - P \tag{28}$$

where

$$\bar{\bar{\nabla}}^4 = D\left(\frac{\partial^4}{\partial y^4} + \nu\frac{\partial^4}{\partial x^2\partial y^2}\right) + \frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\right] + 2(1-\nu)\frac{\partial}{\partial x}\left(D\frac{\partial^3}{\partial x\partial y^2}\right) , \tag{29}$$

and

$$P = p + q(x)\delta(y - \frac{b}{2}) + 2Q(y)\delta(x - \frac{\ell}{2}) \tag{30}$$

where $\delta(y-y_o)$ is the singularity function.

Equations (26) and (28) are the governing differential equation of the shell. The constant, 2, used in last term of equation (14) is merely for convenience.

For a uniform cylindrical shell stiffened by intermediate uniform circumferential bands, the extensional stiffness, 1/Et, and the flexural rigidity, D, of the shell are

$$\left(D,\frac{1}{Et}\right) = \sum_{i=1}^{N}\left(D_i,\frac{1}{Et_i}\right)\left[u(x - \ell_{i-1}) - u(x - \ell_i)\right] \tag{31}$$

where N represents the number of intervals of constant skin thickness between x=0 and x=ℓ/2, and $u(x - x_o)$ is the unit step-function. $\bar{\nabla}^4$ and $\bar{\bar{\nabla}}^4$ reduce to the well-known biharmonic operator ∇^4 for a shell having uniform skin thickness, i.e.

$$Et\bar{\nabla}^4 = \frac{1}{D}\bar{\bar{\nabla}}^4 = \nabla^4 . \tag{32}$$

The zero slope and vanishing of transverse shear as boundary conditions involve odd derivatives of normal displacement with respect to x and y along the edges x=0,ℓ and y=0,b, respectively. These boundary conditions suggest that the general solution of equations (26) and (28) may be taken in the following double cosine series form:

$$W = \sum_{m=0,2}^{\infty}\sum_{n=0,2}^{\infty} W_{mn} \cos\alpha_m x \cos\beta_n y , \tag{33}$$

and

$$F = \sum_{m=0,2}^{\infty}\sum_{n=0,2}^{\infty} F_{mn} \cos\alpha_m x \cos\beta_n y - \frac{1}{a}\int\int W_{oo}\, Et\, dx\, dx \tag{34}$$

where

$$\alpha_m = \frac{m\pi}{\ell} , \text{ and } \beta_n = \frac{n\pi}{b} . \tag{35}$$

m and n take on even integers because of symmetry about x=ℓ/2 and y=b/2. Therefore, the region bounded by x=0, ℓ/2 and y=0, b/2 only need be considered.

Since the series solution will be used in the analysis, the loading function shown in equations (28) and (30) will be expanded into appropriate series as follows:

$$P = \sum_{m=2,4}^{\infty}\sum_{n=2,4}^{\infty} P_{mn} \cos\alpha_m x \cos\beta_n y + \sum_{n=2,4}^{\infty} P_{on} \cos\beta_n y + \sum_{m=2,4}^{\infty} P_{mo} \cos\alpha_m x + P_{oo} \tag{36}$$

where

$$P_{mn} = \frac{4Q_n}{\ell}\cos\frac{m\pi}{2} + \frac{2q_m}{b}\cos\frac{n\pi}{2} ,$$

$$P_{on} = \frac{q_o}{b} \cos\frac{n\pi}{2} + \frac{2Q_n}{\ell} ,$$

$$P_{mo} = \frac{q_m}{b} + \frac{2Q_o}{\ell} \cos\frac{m\pi}{2} ,$$

$$P_{oo} = \frac{q_o}{2b} + \frac{Q_o}{\ell} + p . \qquad (37)$$

Note that the following relationship

$$\int^{\ell} f(\xi)\, \delta(\xi-\xi_o)\, d\xi = f(\xi_o) \qquad (38)$$

has been used.

The solution corresponding to a shell structure with uniform skin may be obtained directly from the general solution for shells with intermediate circumferential straps. However, unnecessary errors will be inherited in such limiting case because of series expansions of skin stiffness used in the analysis. Therefore, a separate solution for a shell with uniform skin thickness will be presented first and a general solution corresponding to shells with straps will be presented subsequently.

4. <u>Shell with Uniform Skin</u>: For this case, the linear operator shown in equation (32) will be used in the governing differential equations (26) and (28). The last term in equation (34) will simply be $Et/2a\, W_{oo}$. By substituting equations (33), (34) and (36) in conjunction with equations (32), (36) and (37) into equations (26) and (28), and by collecting like terms in the results, one may express the coefficients of w in terms of loading coefficients as follows:

$$W_{mn} = - \frac{\alpha_{mn}^4}{D\alpha_{mn}^8 + \frac{Et}{a^2}\alpha_m^4} \left(\frac{4Q_n}{\ell} \cos\frac{m\pi}{2} + \frac{2q_m}{b} \cos\frac{n\pi}{b} \right) ,$$

$$W_{on} = - \frac{1}{D\beta_n^4 + \frac{Et}{a^2}} \left(\frac{q_m}{b} + \frac{2Q_o}{\ell} \cos\frac{m\pi}{2} \right) ,$$

$$W_{oo} = \frac{a^2}{Et} \left(p + \frac{Q_o}{\ell} \right) , \qquad (39)$$

where

$$\alpha_{mn}^2 = \alpha_m^2 + \beta_n^2 . \qquad (40)$$

Expressions similar to equations (39) may also be obtained for the coefficient of the stress function, F.†

By matching the displacements between the shell and stringer along the intersecting line according to equations (2) and (33) in conjunction with equation (39), one obtains the following relationships:

$$\sum_n^{\infty} \frac{2 \cos\frac{n\pi}{2}}{D\beta_n^4 \ell} Q_n = W_{oo} = \frac{a^2}{Et} \left(p + \frac{Q_o}{\ell} \right) , \qquad (41)$$

and

$$\sum_n^{\infty} \frac{4\alpha_{mn}^4 \cos\frac{n\pi}{2} \cos\frac{m\pi}{2}}{\left(D\alpha_{mn}^8 + \frac{Et}{a^2}\alpha_m^4 \right)\ell} Q_n + \frac{2\cos\frac{m\pi}{2}}{\left(D\alpha_m^4 + \frac{Et}{a^2} \right)\ell} Q_o$$

$$+ \frac{q_m}{b} \left\{ \frac{b}{EI\alpha_m^4} + \frac{1}{D\alpha_m^4 + \frac{Et}{a^2}} + \sum_n^{\infty} \frac{2\alpha_{mn}^4 \cos^2(\frac{n\pi}{2})}{D\alpha_{mn}^8 + \frac{Et}{a^2}\alpha_m^4} \right\} = 0. \ (42)$$

In a similar manner, the following relationships are obtained according to equations (16) and (33) in conjunction with equation (39) in order to satisfy the condition for compatible deformeation between the skin and the ring:

$$-\sum_{m=2}^{\infty} \gamma_{mn} \cos\frac{m\pi}{2} \left(\frac{4Q_n}{\ell} \cos\frac{m\pi}{2} + \frac{2}{b} q_m \cos\frac{n\pi}{2} \right) - \frac{2}{D\beta_n^4 \ell} Q_n =$$

$$= \left[\beta_n^2 \left(\beta_1^* \beta_n^2 - \beta_2^* \right) \right]^{-1} (2Q_n) \qquad (43)$$

and

$$W_{oo} = \frac{a^2}{Et} \left(p + \frac{Q_o}{\ell} \right) = \left\{ \frac{a^2}{EA_R} \right.$$

$$\left. + \sum_{m=2}^{\infty} \frac{2}{\left(D\alpha_m^4 + \frac{Et}{a^2} \right)\ell} \right\} Q_o + \sum_{m=2}^{\infty} \eta_m q_m , \qquad (44)$$

where

$$\gamma_{mn} = \frac{\alpha_{mn}^4}{D\alpha_{mn}^8 + \frac{Et}{a^2}\alpha_m^4} , \qquad (45)$$

$$\eta_m = \frac{\cos\frac{m\pi}{2}}{\left(D\alpha_m^4 + \frac{Et}{a^2} \right) b} . \qquad (46)$$

Q_o, Q_n, and W_{oo} are eliminated in equations (41) through (44) and a system of an infinite number of simultaneous equations results:

$$[A_{mj}]\, [q_j] = [C_m] \qquad (47)$$

where

$$A_{mj} = \sum_{n=2}^{\infty} \left(\frac{4}{\ell} \cos\frac{m\pi}{2} \cos\frac{j\pi}{2} \gamma_{mn} \gamma_{jn} \zeta_n \right)$$

$$+ \epsilon_j\, \delta_{jm} + g_m\, \eta_j ,$$

$$C_m = - g_m \frac{a^2}{Et} ,$$

$$g_m = - \frac{2\cos\frac{m\pi}{2}}{\left(D\alpha_m^4 + \frac{Et}{a^2} \right)\ell} \left[\left(\frac{1}{Et\ell} + \frac{1}{EA_R} \right) a^2 \right.$$

$$\left. + \sum_j^{\infty} \frac{2}{\left(D\alpha_j^4 + \frac{Et}{a^2} \right)\ell} \right]^{-1} ,$$

† Detailed expressions may be referred to (14).

$$\zeta_n = -\frac{1}{b}\left[\frac{1}{D\beta_n^4 \ell} + \frac{1}{\beta_m^2\left(\beta_1^{*}\beta_n^2 - \beta_2^{*}\right)}\right.$$

$$\left. + \sum_{j=2}^{\infty} \gamma_{jn}\left(\frac{2}{\ell}\right)\right]^{-1} ,$$

$$\epsilon_j = \frac{1}{\left(D\alpha_j^4 + \frac{Et}{a^2}\right)b} + \frac{1}{EI\alpha_j^4} + \sum_k^{\infty} \gamma_{jk}\left(\frac{2}{b}\right), \quad (48)$$

δ_{mj} is the Kronecker delta.

In practice, only a finite number of equations (47) will be considered. After having solved for $q_1, q_2, \ldots q_m$, the following quantities can be calculated:

$$Q_n = \zeta_n (-1)^{n/2} \sum_{j=2}^{M} (-1)^{j/2} \gamma_{jn} q_j$$

$$= \zeta_n \sum_{j=2,4}^{M} (-1)^{\frac{1}{2}(n+j)} \gamma_{jn} q_j ,$$

$$p_{mn} = (-1)^{m/2} \frac{4Q_n}{\ell} + (-1)^{n/2} \frac{2}{b} q_j \delta_{jm} ,$$

$$W_{mn} = -P_{mn} \gamma_{mn} ,$$

$$W_{on} = \frac{2Q_n}{D\beta_n^4 \ell} ,$$

$$Q_o = \left[\frac{a^2}{E}\left(\frac{1}{t\ell} + \frac{1}{A_R}\right) + \sum_{i=2}^{M} \frac{2}{\left(\alpha_i^4 D + \frac{Et}{a^2}\right)\ell}\right]^{-1} \left[\frac{a^2}{Et} + \sum_{j=2}^{M} \frac{\eta_j q_j}{b}\right]$$

$$W_{mo} = -\left(\alpha_m^4 D + \frac{Et}{a^2}\right)^{-1} \left[\frac{2Q_o}{\ell}(-1)^{m/2} + \frac{q_m}{b}\right] ,$$

$$W_{oo} = -\frac{a^2}{Et}\left(1 + \frac{Q_o}{\ell}\right) ,$$

$$W(x,y) = \sum_{n=0,2}^{K} \sum_{m=0,2}^{M} W_{mn} \cos\alpha_m x \cos\beta_n y ,$$

$$N_{yy} = -Et\frac{W}{a} . \quad (49)$$

Since only a finite number of terms is taken in the series which accurately represents the displacement w, there is no assurance of the convergence of the higher derivatives of this truncated series. Therefore, the stress couple M_{ij}, bending moments M_s and M_r, transverse shears Q_x, Q_y, V_s and V_r, and the ring stress resultants T_r are calculated by using a finite-difference technique.

5. <u>Shell with Straps</u>: For this case, the skin is considered to have non-uniform thickness. The variation of the skin thickness takes the form shown in equation (31). Substitution of equations (33), (34), and (36) into equations (26) and (28) results in the following equations:

$$\frac{1}{Et}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} F_{mn}\left\{\beta_n^4 - \nu\alpha_m^2\beta_n^2\right\}\cos\alpha_m x \cos\beta_n y$$

$$+ \frac{1}{Et}\sum_{n=2}^{\infty}\beta_n^4 F_{on}\cos\beta_n y$$

$$-\frac{\partial^2}{\partial x^2}\left[\frac{1}{Et}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} F_{mn}\left\{\alpha_m^2 - \nu\beta_n^2\right\}\cos\alpha_m x\cos\beta_n y\right]$$

$$-\frac{\partial^2}{\partial x^2}\left[\frac{1}{Et}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} F_{mo}\,\alpha_m^2\cos\alpha_n x\right.$$

$$\left. - \frac{\nu}{Et}\sum_{n=2}^{\infty} F_{on}\,\beta_n^2\cos\beta_n y\right]$$

$$+ 2(1+\nu)\frac{\partial}{\partial x}\left[\frac{1}{Et}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\alpha_m\beta_n^2 F_{mn}\sin\alpha_m x\cos\beta_n y\right]$$

$$= \frac{1}{a}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} W_{mn}\,\alpha_m^2\cos\alpha_m x\cos\beta_n y$$

$$+ \frac{1}{a}\sum_{m=2}^{\infty} W_{mo}\,\alpha_m^2\cos\alpha_m x , \quad (50)$$

and

$$D\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} W_{mn}\left\{\beta_n^4 + \nu\alpha_m^2\beta_n^2\right\}\cos\alpha_m x\cos\beta_n y$$

$$-\frac{\partial^2}{\partial x^2}\left[D\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} W_{mn}\left\{\alpha_m^2 + \nu\beta_n^2\right\}\cos\alpha_m x\cos\beta_n y\right.$$

$$-\frac{\partial^2}{\partial x^2}\left[D\sum_{m=2}^{\infty}\alpha_m^2 W_{mo}\cos\alpha_m x + \nu D\sum_{n=2}^{\infty}\beta_n^2 W_{on}\cos\beta_n y\right]$$

$$+ 2(1-\nu)\frac{\partial}{\partial x}\left[D\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\alpha_m\beta_n^2 W_{mn}\sin\alpha_m x\cos\beta_n y\right]$$

$$+ D\sum_{n=2}^{\infty}\beta_n^4 W_{on}\cos\beta_n y$$

$$= -\frac{1}{a}\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\alpha_m^2 F_{mn}\cos\alpha_m x\cos\beta_n y$$

$$-\frac{1}{a}\sum_{m=2}^{\infty}\alpha_m^2 F_{mo}\cos\alpha_m x - P - \frac{W_{oo}}{a^2}Et . \quad (51)$$

The following quantities appearing in equations (50) and (51) are further expanded into Fourier series:

$$\left(Et, \frac{1}{Et}, \frac{1}{Et} \cos \alpha_m x\right) = \frac{1}{2}\left(\lambda_{ono}, \delta_{ono}, \delta_{mno}\right) + \sum_{s=2}^{\infty} \left(\lambda_{ons}, \delta_{ons}, \delta_{mns}\right) \cos \alpha_s x , \tag{52}$$

$$\left(\frac{1}{Et}, D\right) \sin \alpha_m x = \sum_{s=2}^{\infty} \left(\Delta_{mns}, E_{mns}\right) \sin \alpha_s x , \tag{53}$$

$$\left(D, D \cos \alpha_m x\right) = \frac{1}{2}\left(\epsilon_{ono}, \epsilon_{mno}\right) + \sum_{r=2}^{\infty} \left(\epsilon_{ons}, \epsilon_{mns}\right) \cos \alpha_s x . \tag{54}$$

In conjunction with equation (31), the Fourier coefficients, $\lambda_{ono} \ldots \epsilon_{mns}$, represented by the general set of symbols, A_{ono}, A_{ons}, A_{mno} and A_{mns} become

$$A_{ono} = \frac{4}{\ell} \sum_{i=1}^{N} A_i\left(\ell_i - \ell_{i-1}\right) ,$$

$$A_{ons} = \frac{4}{s\pi} \sum_{i=1}^{N} A_i\left(\sin \frac{s\pi\ell_i}{\ell} - \sin \frac{s\pi\ell_{i-1}}{\ell}\right) ,$$

$$A_{mno} = \frac{4}{m\pi} \sum_{i=1}^{N} A_i\left(\sin \frac{m\pi\ell_i}{\ell} - \frac{m\pi\ell_{i-1}}{\ell}\right) ,$$

$$A_{mns} = \frac{2}{\pi} \left[\frac{1}{m-s} \sum_{i=1}^{N} A_i\left\{\sin (m-s) \frac{\ell_i}{\ell} \pi - \sin (m-s) \frac{\ell_{i-1}}{\ell} \pi\right\} + \frac{1}{m+s} \sum_{i=1}^{N} A_i\left\{\sin (m+s) \frac{\ell_i}{\ell} \pi - \sin (m+s) \frac{\ell_{i-1}}{\ell} \pi\right\}\right]_{m \neq s} ,$$

$$A_{mns} = \frac{1}{\ell} \sum_{i=1}^{N} A_i\left(\ell_i - \ell_{i-1}\right) + \frac{2}{(m+s)\pi} \sum_{i=1}^{N} A_i\left\{\sin (m+s) \frac{\ell_i}{\ell} \pi - \sin (m+s) \frac{\ell_{i-1}}{\ell} \pi\right\} \quad m = s . \tag{55}$$

where N represents the number of intervals of constant skin thickness between $x=0$ and $x=\ell/2$. The corresponding distance for each region is denoted by ℓ_i.

Substitution of equations (20), (26), (27), (28) into equations (24) and (25) in conjunction with equations (21) and (29), and collection of like terms in the resulting equation lead to the following relationships:

$$\sum_{m}^{\infty} F_{mn}\left(\beta_n^4 - \nu\alpha_m^2\beta_n^2\right) \frac{\delta_{mno}}{2} + \beta_n^4 \frac{\delta_{ono}}{2} F_{on} = 0 , \tag{56}$$

$$\sum_{m}^{\infty} F_{mn}\left[\left\{\left(\beta_n^4 - \nu\alpha_m^2\beta_n^2\right) + \left(\alpha_n^2 - \nu\beta_n^2\right)\alpha_s^2\right\} \delta_{mns} + 2(1+\nu)\, \alpha_m \alpha_s \beta_n^2 \Delta_{mns}\right] + F_{on}\left(\beta_n^4 - \nu\beta_n^2\alpha_s^2\right) \delta_{ons} = \frac{1}{a} W_{sn}\, \alpha_s^2 , \tag{57}$$

$$\sum_{m}^{\infty} F_{mo}\, \alpha_m^2 \alpha_s^2 \delta_{mos} = \frac{1}{a} W_{so}\, \alpha_s^2 , \tag{58}$$

$$\sum_{m}^{\infty} W_{mn}\left(\beta_n^4 + \nu\alpha_m^2\beta_n^2\right) \frac{\epsilon_{mno}}{2} + \beta_n^4 \frac{\epsilon_{ono}}{2} W_{on} = -\frac{2Q_n}{\ell} , \tag{59}$$

$$\sum_{m}^{\infty} W_{mn}\left[\left\{\left(\beta_n^4 + \nu\alpha_m^2\beta_n^2\right) + \left(\alpha_m^2 + \nu\beta_n^2\right)\alpha_s^2\right\} \epsilon_{mns} \cdot 2\, (1-\nu)\, \alpha_m \beta_n^2 \alpha_s E_{mns}\right] + W_{on}\left(\beta_n^4 + \nu\beta_n^2\alpha_s^2\right) \epsilon_{ons} = -\frac{1}{a} \alpha_s^2 F_{sn} - \frac{4Q_n}{\ell} \cos \frac{s\pi}{2} - \frac{2q_r}{b} \cos \frac{n\pi}{2} , \tag{60}$$

$$\sum_{m}^{\infty} W_{mo}\, \alpha_m^2 \alpha_s^2 \epsilon_{mos} = -\frac{1}{a} \alpha_s^2 F_{so} - \frac{2Q_o}{\ell} \cos \frac{s\pi}{2} - \frac{q_r}{b} - \frac{W_{oo}}{a^2} \lambda_{ons} , \tag{61}$$

$$\frac{Q_o}{\ell} + p + \frac{W_{oo}}{2a^2} \lambda_{ono} = 0 . \tag{62}$$

By requiring compatible deformations of the skin, the stringer, and the ring, i.e.

$$w\left(x, \frac{b}{2}\right) = W_s , \tag{63}$$

and

$$w\left(\frac{\ell}{2}, y\right) = W_r , \tag{64}$$

Equations (2), (16) and (33) in conjunction with equations (56) through (62) are substituted into equations (63) and (64). After a lengthy mathematical manipulation (detailed derivation may be referred to [14]) the following system of relationships is obtained:

$$\sum_{j}^{\infty} \sum_{m}^{\infty} \left\{D_{mns} \frac{a}{\alpha_m^2} \left[e_{jnm} - \beta_n^2\left(\beta_n^2 + \nu\alpha_m^2\right) \epsilon_{onm} \cos \frac{j\pi}{2} - A_n \beta_n^2 \left\{\left(\beta_n^2 + \nu\alpha_j^2\right) \epsilon_{jno} - \beta_n^2 \epsilon_{ono} \cos \frac{j\pi}{2}\right\}\right.\right.$$

$$\cdot \left\{ \frac{\beta_n^2 + \nu\alpha_m^2}{\beta_1^*\beta_n^2 - \beta_2^*} \epsilon_{onm} + \frac{2}{\ell} \cos\frac{m\pi}{2} \right\} \Big] + \frac{1}{a} \alpha_s^2 \delta_{sj} \Big\} W_{jn}$$

$$+ \sum_k^\infty \sum_m^\infty D_{mns} \left(\frac{2a}{b}\right) \cos\frac{n\pi}{2} EI\, \alpha_m^2 \cos\frac{k\pi}{2} W_{mk}$$

$$+ \sum_m^\infty D_{mns} \left(\frac{2a}{b}\right) \cos\frac{n\pi}{2} EI\, \alpha_m^2 W_{mo} = 0 \ , \qquad (65)$$

$$\sum_j^\infty \sum_m^\infty \Big[\alpha_s^2\alpha_m^2\delta_{mos} \Big\{\alpha_j^2 \epsilon_{jom} + \frac{EI}{b} \alpha_m^2 \delta_{mj}$$

$$+ \frac{1}{a^2} \beta_m \cos\frac{j\pi}{2}\Big\} + \frac{\alpha_s^2}{a^2} \delta_{sj}\Big] W_{jo}$$

$$+ \sum_m^\infty \sum_k^\infty \alpha_s^2 \alpha_m^2 \delta_{mos} \frac{EI}{b} \cos\frac{k\pi}{2} W_{mk}$$

$$= p\alpha_s^2 \sum_m^\infty \alpha_m^2 \delta_{mos} \left[\frac{2}{\lambda_{ono}} \beta_n + \frac{\lambda_{onm}}{\alpha_m^2 \lambda_{ono}}\right] , \qquad (66)$$

where

$$A_n = \left[\frac{2}{\ell} + \epsilon_{ono}\, \beta^2 \left(\frac{1}{\beta_1^*\beta_n^2 - \beta_2^*}\right)\right]^{-1} ,$$

$$D_{mns} = \delta_{mns} \left[\beta_n^4 - \nu\alpha_m^2\beta_n^2 + \left(\alpha_m^2 - \nu\beta_n^2\right)\alpha_s^2\right]$$

$$+ 2\,(1+\nu)\, \alpha_m \alpha_s \beta_n^2 \Delta_{mns} - \left(\beta_n^2 - \nu\alpha_m^2\right)$$

$$\left(\beta_n^2 - \nu\alpha_s^2\right) \frac{\delta_{mno}\delta_{ons}}{\delta_{ono}} ,$$

$$e_{jnm} = \epsilon_{jnm} \left[\beta_n^4 + \nu\alpha_j^2\beta_n^2 + \left(\alpha_j^2 + \nu\beta_n^2\right)\alpha_m^2\right]$$

$$+ 2\,(1-\nu)\, \alpha_j \alpha_m \beta_n^2 E_{jnm} ,$$

$$B_m = \frac{2}{\alpha_m^2 \ell} \left(\cos\frac{m\pi}{2} - \frac{\lambda_{onm}}{\lambda_{ono}}\right)\left(\frac{2}{\lambda_{ono}\ell} + \frac{1}{EA_R}\right)^{-1} . \qquad (67)$$

Equations (65) and (66) represent a set of infinitely many simultaneous algebraic equations. In the actual numerical calculation, only a finite number of terms will be considered. After W_{mn} are determined, the displacements and internal loads will be calculated in the same manner as discussed in the previous case.

6. Closed End Effects: When the cylindrical shell subjected to internal pressure is closed at both ends, a total axial load of $\bar{P} = \pi a^2 p$ will be carried by the stringers and skins. The stresses due to $\bar{P}$ are small when compared to the effects due to lateral pressure alone. The following analysis without consideration of shear lag effect will be quite acceptable.

It is well known that the normal displacement of a thin cylinder, without consideration of constraints of rings and stringers, subjected to lateral pressure alone is

$$w_o = \frac{pa^2}{Et} . \qquad (68)$$

The corresponding final deformations, considering the effect of the constraints of the ring and the stringer, may be written as

$$w = W \frac{pa^2}{Et} . \qquad (69)$$

The reduction or addition of deformation, as well as stresses due to the constraints of the stringers and rings, may be indicated by the parameter, $\alpha\,(x,y)$, or

$$\alpha\,(x,y) = \frac{w - w_o}{|W_o|} = W - 1 \ . \qquad (70)$$

Substitution of equation (68) into (69) yields

$$w = (1-\alpha)\, w_o = (1-\alpha) \frac{pa^2}{Et} \qquad (71)$$

The state of stress and deformation, $\bar{w}$, without consideration of the constraints of the ring and the flexural constraint of the stringer due to axial load $\bar{P}$ or $pa/2$ intensity along the circumference, may be derived by satisfying the equilibrium conditions and the compatibility of linear strain of the skin and the stringer. The following results are obtained:

$$\bar{\sigma}_{xx} = \frac{pa/2}{\left(1 + \frac{A_s}{bt}\right)t} , \qquad (72)$$

and

$$N_{xs} = \bar{\sigma}_{xx} A_s = \frac{paA_s}{2t\left(1 + \frac{A_s}{bt}\right)} \qquad (73)$$

The corresponding circumferential stress is

$$\bar{\sigma}_{yy} = 0 \ .$$

As a result, the contraction of the skin in the radial direction corresponding to zero hoop stress becomes

$$\bar{w} = \frac{\nu pa^2}{2\,Et\left(1 + \frac{A_s}{bt}\right)} . \qquad (74)$$

The increase or decrease in the displacement due to constraints of the ring and the flexural rigidity of the stringer is $\alpha\bar{w}$. The parameter, α, has been defined in equation (70). The final displacement, w^*, of the skin due to the axial load $\bar{P}$ becomes

$$w^* = (1-\alpha)\,\bar{w} = \frac{\nu}{2\left(1 + \frac{A_s}{bt}\right)} (1-\alpha) \frac{pa^2}{Et} . \qquad (75)$$

The corresponding circumferential stress, σ_{yy}^*, becomes

$$\sigma_{yy}^* = \alpha E \frac{\bar{w}}{a} = \frac{\alpha\nu}{2t\left(1 + \frac{A_s}{bt}\right)} pa = \alpha\nu\sigma_{xx}^* \qquad (76)$$

which is considerably smaller than the hoop stress corresponding to lateral pressure alone.

From the above discussion, it is interesting to note that the deformation and stresses due to the effect of axial load may be determined as soon as the solution for an open-ended cylindrical shell is obtained.

Numerical Examples

Typical arrangements and dimensions of C-5A airplane fuselage are selected for example solutions. The material parameters and dimensions common to all cases listed in Table 1 are p=1 psi, ℓ=20", ν=0.3, E=10^7 psi, the mesh size Δx=Δy=0.25". The symbol, t_s, shown in the table represents the thickness of a titanium strap and ℓ_s is its width. The straps are placed symmetrically along the mid-line between two adjacent rings. The numbers shown in the last "Remark" column represent depths of the rings.

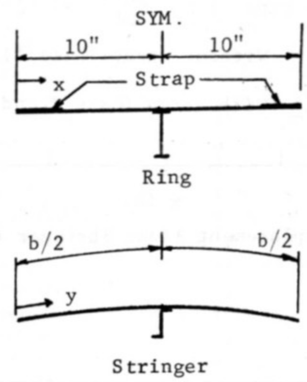

Fig. 2. Geometry and Dimensions for Example Problems

TABLE 1. Data for Numerical Examples

Case	a in.	b in.	R_o in.	I $in.^4$	A_s $in.^2$	A_R $in.^2$	I_R $in.^4$	t in.	t_s(Ti) in.	$\ell_s=2\ell_1$ in.	Remark
1	143	7.8	138.9	0.051	0.1881	0.721	6.56	0.07	0	0	8" Ring without strap
2	143	7.8	138.9	0.051	0.1881	0.721	6.56	0.07	0.02	4.5	8" Ring with strap
3	143	7.8	138.9	0.051	0.1881	0.600	3.92	0.07	0.02	4.5	6" Ring with strap

The results for the displacements and inner and outer fiber stresses along longitudinal and circumferential directions (see Fig. 2) corresponding to the combined effect of lateral pressure and axial load at the following locations are plotted in Fig. 3 through 12:

1. along mid-line between adjacent stringer (y=0),
2. along stringer (y=b/2),
3. along mid-line between two adjacent ring (x=0),
4. along quarter-line of ring spacing (x=ℓ/4),
5. along center-line of ring (x=ℓ/2).

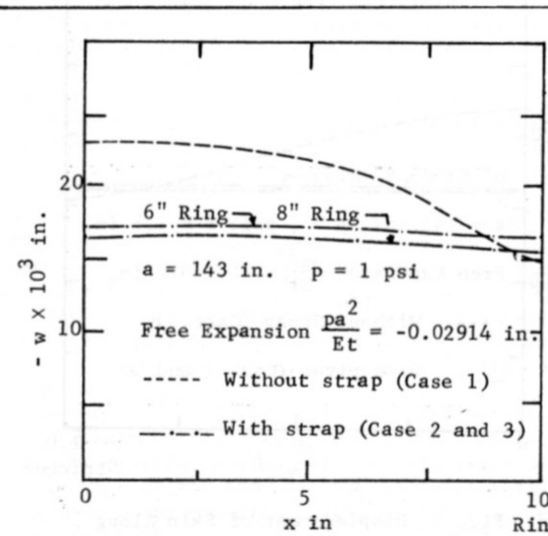

Fig. 3 Displacement of Skin Along Mid-line Between Two Adjacent Stringers (y=0)

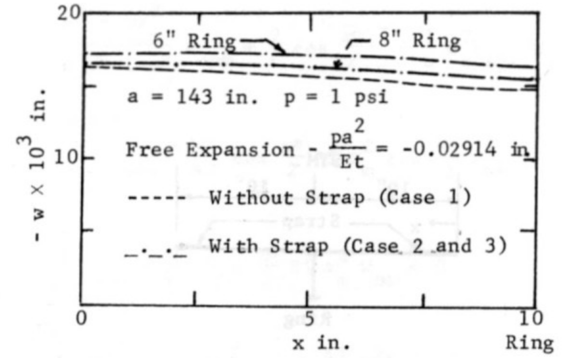

Fig. 4 Displacement Along Stringer (y = 3.9")

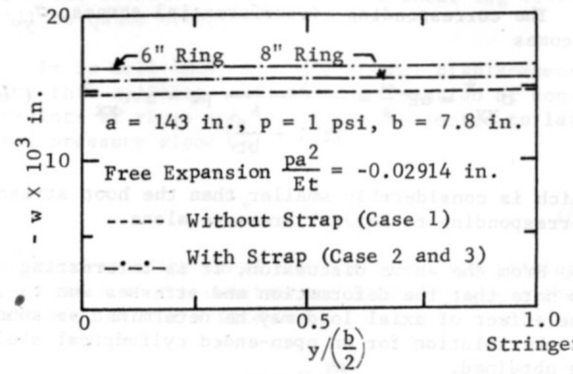

Fig. 7 Displacement Along Center Line of Ring (x = 10 in.)

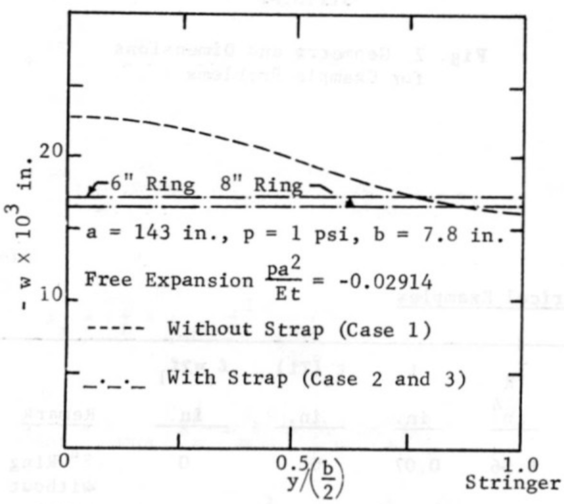

Fig. 5 Displacement of Skin (x = 0)

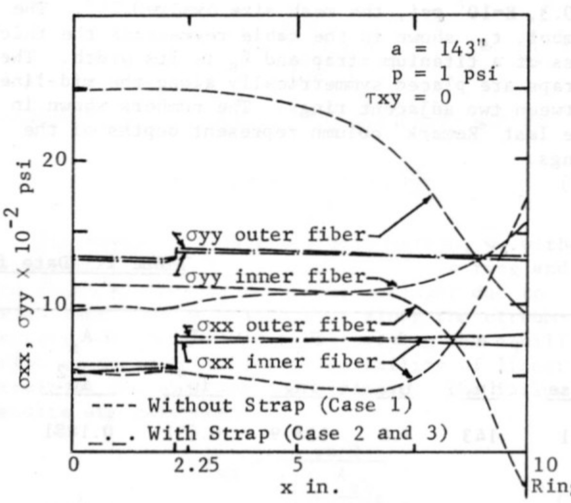

Fig. 8 Stresses Along Mid-line Between Two Adjacent Stringers (y=0)

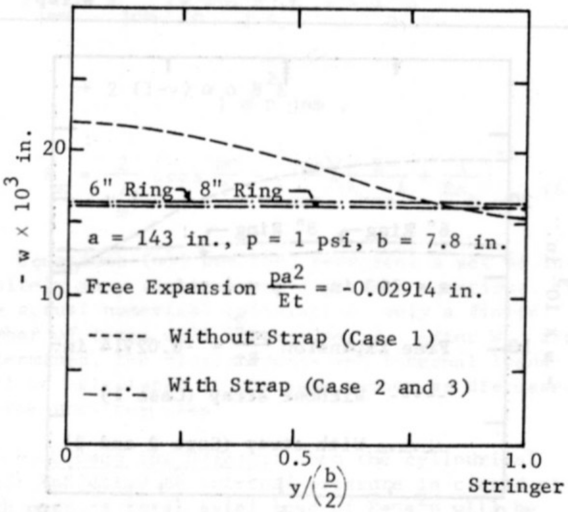

Fig. 6 Displacement of Skin Along Quarter-line of Ring Spacing (x = 5")

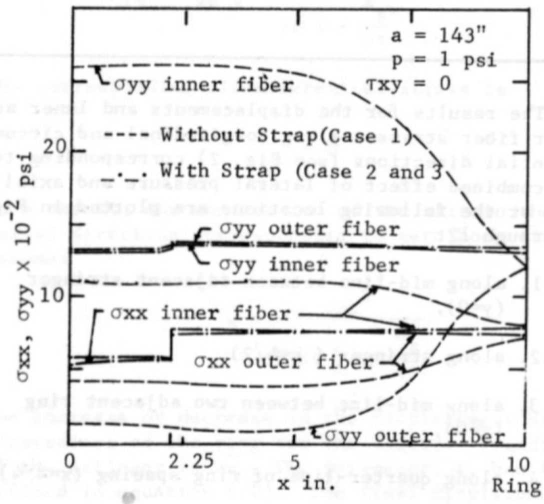

Fig. 9 Shell Stresses Along Stringer Location (y=3.9")

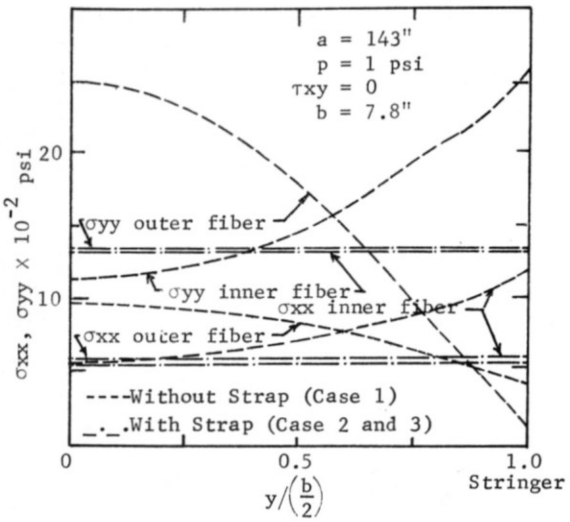

Fig. 10 Stresses Along Mid-line Between Two Adjacent Rings (x=0)

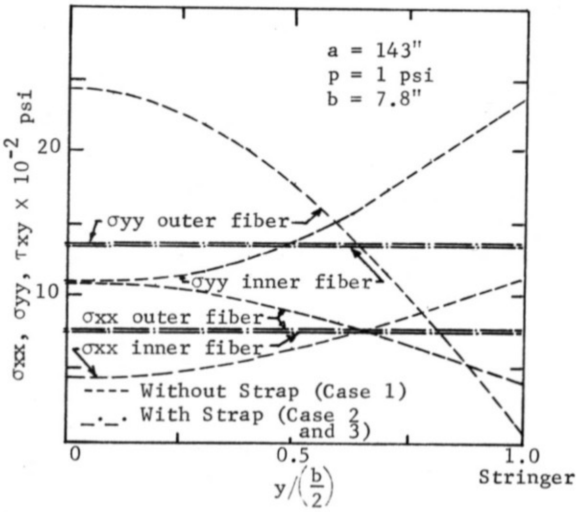

Fig. 11 Along Quarter-line Between Two Adjacent Rings (x=5")

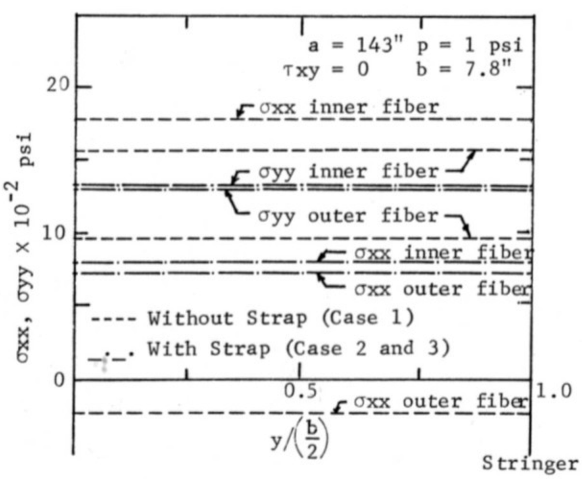

Fig. 12 Skin Stresses Along the Ring (x=10")

Discussion

The solution of cylindrical shells with uniform as well as non-uniform thickness orthogonally stiffened by uniform stiffeners subjected to internal pressure is obtained by treating the stiffeners as separate elements. The deformation and stress condition can be calculated for any point in the structure. Numerical examples based on C-5A fuselage material parameters and dimensions are presented. The results indicate that the variation of displacement of the shell along the longitudinal as well as the circumferential directions is significant for the cases where the circumferential bands are not included; as a result, large bending stresses are introduced. For a shell with intermediate bands, however, the variation of displacements and stresses along the circumferential direction is negligibly small, and the variations along longitudinal directions are small but detectable. As a result, stress levels are equalized and maximum stress levels are reduced. This reduction will increase the fatigue endurance where fuselage pressurization and de-pressurization are predominant, and in resistance to damage propagation. In short, it appears that the insertion of circumferential bands is structurally effective according to the cases considered. While the example problems consider only one strap located midway between every two rings, the analysis and computer program for IBM 7094 are prepared for multiple straps at variable locations.

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