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by

Václav Kovařík
Building Research Institute of the Czech Technical University in Prague
Prague, Czechoslovakia

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Doc.Ing.Václav Kovařík,PhD
Building Research Institute, Technical University
Prague, Czechoslovakia

Abstract

In this paper the behaviour of sandwich plates under a step heat input is investigated. With neglect of coupling, the problem considered separates into two distinct problems to be solved consecutively. The first is a problem of heat conduction, the second a problem of thermoelasticity which is the more direct concern of this paper. Solving this very last problem the use is made of the theory of sandwich plates given in previous papers of the author (3),(4),(5),(6). Similarly to R.D.Mindlin and L.E.Goodman (8) the solution is found as the sum of two partial solutions. The first solution is identical to that of thermal bending while the second one represents free forced vibrations of sandwich plates under the pulsating force consisting of the negative of inertia forces corresponding to the first solution. To illustrate the procedures employed the problem solved by B.A.Boley (1) for homogeneons plates is considered. Limiting process to the homogeneous plate enables the comparison of numerical results with those received by Boley.

I. Introduction

The sandwich plate under consideration has its middle plane in x_1 , x_2 -plane (Fig. 1). In the x_3- direction the thickness of the lower face extends from $x_3=-h$ to $x_3=-s$, the core from $x_3=-s$ to $x_3=s$ and the upper face layer from $x_3=s$ to $x_3=h$. The two face layers are of the same physical properties, and perfect bond between the adjacent layers is assumed. Individual layers in this analysis are assumed to be elastic orthotropic continua with main axes of orthotropy in the x_j -directions (j = 1,2,3).

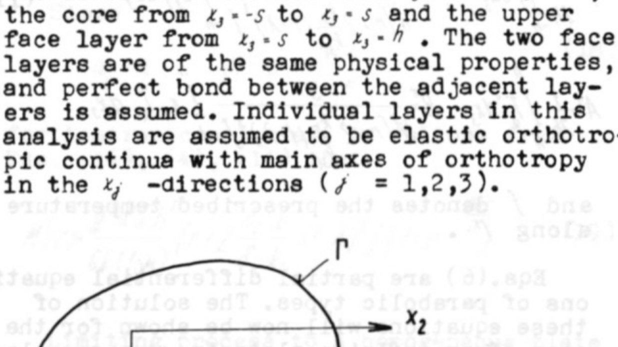

Figure 1. Planform and structure of sandwich plate considered

The following general assumptions are used in the derivation of the basic relations:

a) No heat is generated within the body
b) Thermal conductivity k of each layer is constant throughout the layer

for the heat conduction problem, and

1) Transverse normal strain ε_3 of the plate is negligible
2) Transverse normal stress σ_3 of the plate can be neglected
3) Supporting of the plate is arranged in such a way that it does not enable the motion of the plate as a whole
4) The Kirchhoff-Love s hypothesis on normals is not acceptable

for the thermoelastic problem, respect.

II. Heat conduction problem

The time - dependent heat conduction problem in an orthotropic body is governed by the Fourier's equation

$$\left(\sum_{j=1}^{3} k_j \frac{\partial^2}{\partial x_j^2} - c\rho^* \frac{\partial}{\partial t^*}\right) T^* = 0 \qquad (1)$$

provided the effect of the stresses and deformations upon the temperature distribution is quite small and can be neglected. In Eq.(1) the following notation is used: k_j denotes thermal conductivity in the x_j -direction, c the specific heat, ρ^* the density of the material, t^* time, and T^* the temperature.

In addition to Eq.(1) it is necessary to specify the appropriate boundary and initial conditions in order to describe fully the problem. There are five principal boundary conditions which are used in the mathematical theory of heat conduction. In what follows the use will be made of the conditions given by Boley - Weiner (2). Initial condition defines the temperature distribution in $t^*=t_0^*$

$$T^*(P,t_0^*) = T_0^*(P) \qquad (2)$$

where the point P is inside the body and T_0^* is a given function.

In the case of a sandwich plate with orthotropic layers, Eq.(1) can be written for each layer separately. Boundary conditions of two bodies in perfect thermal contact then hold true for the contact surfaces $x_3=\pm s$. Over the surfaces $x_3=\pm h$ either the prescribed heat flux, the prescribed surface temperature or the convection boundary conditions are to be considered. Along the boundary Γ of the plate, the surface temperature is usually prescribed.

Denote the temperature distribution T^* in the upper layer by the symbol T_1^*, in the core by T_2^*, and in the lower layer by T_3^*.

Deriving the approximate formulation of the heat conduction problem, we follow the same procedure as is used in the theory of homogeneous plates:

Have functions T_i^{**} (i = 1,2,3) continuous and smooth in the corresponding domains of definition of the respective functions T_i^* (i = 1,2,3). These functions-similarly to the functions T_i^* - define obviously certain piecewise smooth function T^{**} in the region $(-h \le x_3 \le h, R)$. Substituting functions T_i^{**} instead of T_i^* into equations (1) written for each layer, we find these equations generally not to be satisfied, i. e. the right - hand sides of these equations will no more be zeros but certain functions, say Φ_i (i = 1,2,3). We can now speak of a piecewise continuous function Φ defined by the following relations

$$\Phi(T^{**}) = \begin{cases} \Phi_1 & \text{in the region } (s \le x_3 \le h, R) \\ \Phi_2 & \text{in the region } (-s \le x_3 \le s, R) \\ \Phi_3 & \text{in the region } (-h \le x_3 \le -s, R) \end{cases}$$

According to this definition, we obviously have

$$\Phi(T^*) = 0 \quad \text{everywhere in } (-h \le x_3 \le h, R)$$

Now, the function T^{**} satisfying the conditions

$$\int_{-h}^{h} x_3^i \, \Phi(T^{**}) \, dx_3 = 0, \quad (i = 0,1) \quad \text{everywhere in } R \tag{3}$$

may be expected to be a good approximation to the function T^*.

In the next step in the derivation the functions T_i^{**} (i = 1,2,3) will be considered to be of the following form

$$T^{**} = \begin{cases} T_1^{**} = a + \left[\dfrac{k_3}{k_{3c}} s + (x_3 - s)\right] b, & h \le x_3 \le s \\ T_2^{**} = a + \dfrac{k_3}{k_{3c}} x_3 b, & -s \le x_3 \le s \\ T_3^{**} = a + \left[-\dfrac{k_3}{k_{3c}} s + (x_3 + s)\right] b, & -h \le x_3 \le -s \end{cases} \tag{4}$$

where k_3, k_{3c} denote the thermal conductivities in the x_3 - direction of the faces and the core, respect, a, b are certain functions of the variables x_1, x_2 having in R all necessary conditions. With these expressions, the conditions of perfect thermal contacts in $x_3 = \pm s$ planes are satisfied. The differential equations for the functions a, b are obtained when applying the conditions (3), i.e. the equations

$$\int_{-h}^{h} x_3^i \left(\sum_{j=1}^{3} k_j \frac{\partial^2}{\partial x_j^2} - c\rho^* \frac{\partial}{\partial t^*} \right) T^{**} dx_3 = 0, \quad (i = 0,1) \tag{5}$$

Assume a step heat input Q_u^* (Q_L^*) constant in R, to be applied over the surface $x_3 = h$ ($x_3 = -h$). Then, substituting for T^{**} from (4) into (5), integrating with respect to x_3 in corresponding limits, and employing the conditions of prescribed heat flux in $x_3 = \pm h$ and of perfect thermal contact in $x_3 = \pm s$, we find that the governing equations read

$$[\lambda \nabla_1^2 + (1-\lambda) \nabla_{1c}^2] a - [\lambda d + (1-\lambda) d_c] \dot{a} = -\frac{1}{2h}(Q_u^* + Q_L^*) \tag{6}$$

$$[r \nabla_1^2 + 2 \frac{k_3}{k_{3c}} (1-\lambda)^3 \nabla_{1c}^2] b - Dd\dot{b} - \frac{6k_3}{h^2} b = -\frac{3}{h^2}(Q_u^* - Q_L^*)$$

where to simplify the writing the following symbols have been introduced

$$\nabla_1^2 = k_1 \frac{\partial^2}{\partial x_1^2} + k_2 \frac{\partial^2}{\partial x_2^2}, \quad \nabla_{1c}^2 = k_{1c} \frac{\partial^2}{\partial x_1^2} + k_{2c} \frac{\partial^2}{\partial x_2^2} \tag{7}$$

$$\lambda = \frac{t}{h}, \quad d = c\rho^*, \quad d_c = c_c \rho_c^* \tag{8}$$

$$r = \lambda [\lambda(3-\lambda) + 3 \frac{k_3}{k_{3c}} (1-\lambda)(2-\lambda)], \quad D = r + 2 \frac{k_3}{k_{3c}} \frac{d_c}{d} (1-\lambda)^3$$

The dots in (6) mean the time derivatives. Analogical transformations of boundary conditions along Γ and of initial condition (2) result in

$$a = f^*, \quad b = f^{**} \text{ along } \Gamma \tag{9}$$

and

$$a(t_0^*) = A^*, \quad b(t_0^*) = B^* \tag{10}$$

respectively, where

$$f^* = \frac{1}{2h} \int_{-h}^{h} f \, dx_3, \quad f^{**} = \frac{3}{h^3 [r + 2 \frac{k_3}{k_{3c}} (1-\lambda)^3]} \int_{-h}^{h} x_3 f \, dx_3 \tag{11}$$

$$A^* = \frac{1}{2h} \int_{-h}^{h} T_0^* dx_3, \quad B^* = \frac{3}{h^3 [r + 2 \frac{k_3}{k_{3c}} (1-\lambda)^3]} \int_{-h}^{h} x_3 T_0^* dx_3$$

and f denotes the prescribed temperature along Γ.

Eqs.(6) are partial differential equations of parabolic types. The solution of these equations will now be shown for the case of an infinite plate in x_1, x_2 -plane. In such a case a, b are obviously functions of one single independent variable - the time t^*, and are easily found to be

$$a = A^* + \frac{Q_u^* + Q_L^*}{2h[\lambda d + (1-\lambda) d_c]} t^* \tag{12}$$

$$b = B^* e^{-[(6k_3)/(h^2 Dd)] t^*} + \frac{Q_u^* - Q_L^*}{2k_3} \left(1 - e^{-[(6k_3)/(h^2 Dd)] t^*}\right)$$

To illustrate the problem consider an infinite sandwich plate with isotropic layers under a uniform step heat input $\mathcal{Q}^*$ over the face $x_3=h$, while the second face $x_3=-h$ is insulated. The initial temperature T_0^* for $t_0^*=0$ let be zero.

Substitution of the appropriate parameters

$$A^*=B^*=0,\quad \mathcal{Q}_U^*=\mathcal{Q}^*,\quad \mathcal{Q}_L^*=0,\quad k_j=k,\quad k_{jc}=k_c \tag{13}$$

of the example considered into Eqs.(12) results in

$$a=\frac{\mathcal{Q}^*\tau}{\lambda d+(1-\lambda)d_c}\,\frac{2hd}{k},\qquad b=\frac{\mathcal{Q}^*}{2k}\left(1-e^{-\frac{24\tau}{D}}\right) \tag{14}$$

where

$$\tau=\frac{kt^*}{4h^2d} \tag{15}$$

is a nondimensional time parameter.

Let us further calculate a certain quantity $M^T(\tau)$ according to the general formula

$$M_{ii}^T=\int_{-h}^{h}x_3\,\delta_i\,T^{**}dx_3 \tag{16}$$

with

$$\delta_i=E_i^*(\alpha_i+\mu_{ki}\alpha_k),\qquad E_i^*=\frac{E_i}{1-\mu_{12}\mu_{21}} \tag{17}$$

where E_i are Young's moduli of elasticity, μ_{ij} the Poisson's ratios, and α_i the coefficients of thermal expansion. This quantity will be later called the thermal bending moment.

Since

$$\alpha_i=\alpha,\qquad \delta_i=\delta=\frac{E\alpha}{1-\mu} \tag{18}$$

for isotropic material, it follows from Eqs. (16), (4), (14) that

$$M^T(\tau)=\frac{\mathcal{Q}^*\alpha Eh^3}{6k(1-\mu)}\left[r+2\frac{\delta_c}{\delta}\frac{k}{k_c}(1-\lambda)^3\right]\left(1-e^{-\frac{24\tau}{D}}\right) \tag{19}$$

Limiting process to a homogeneous plate

$$k=k_c,\quad d=d_c,\quad \delta=\delta_c,\quad \alpha=\alpha_c,\quad E=E_c,\quad \mu=\mu_c,\quad \lambda=0 \tag{20}$$

i.e.

$$r=0,\quad D=2 \tag{21}$$

leads to the relations

$$a=\frac{2h\mathcal{Q}^*}{k}\tau,\qquad b=\frac{\mathcal{Q}^*}{2k}\left(1-e^{-12\tau}\right) \tag{22}$$

With these quantities Eqs.(4) reduce to one single equation

$$T^{**}(x_3,\tau)=\frac{2h\mathcal{Q}^*}{k}\left[\tau+\frac{x_3}{4h}\left(1-e^{-12\tau}\right)\right] \tag{23}$$

and Eq.(19) reads

$$M^T(\tau)=\frac{\mathcal{Q}^*\alpha Eh^3}{3k(1-\mu)}\left(1-e^{-12\tau}\right) \tag{24}$$

Nondimensional plots of the thermal moments M^T against τ according to the formula (24), and according to the formula given by Boley-Weiner (2), resp., are shown in Fig. 2.

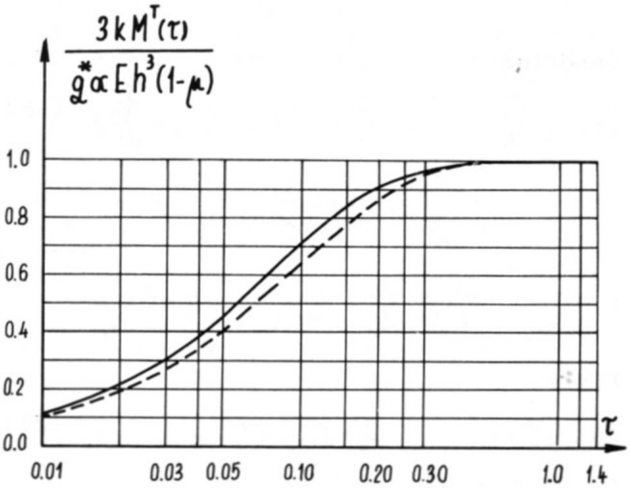

Figure 2. Thermal bending moment according to the present paper (heavy line) and according to Boley-Weiner (dashed line)

III. Thermoelastic problem

Formulation of the problem

In the first step in the development of the theory components of displacement u_j ($j=1,2,3$) are chosen in the following form†

$$u_i=u_{i0}+\left[\pm s+\varkappa_i(x_3\mp s)-\frac{\varkappa_i}{3t^2}(x_3\mp s)^3\right]u_{i1}+ \tag{25}$$

$$+\left[\varkappa_i-1-\frac{\varkappa_i}{3t^2}(x_3\mp s)^2\right](x_3\mp s)\frac{\partial u_{31}}{\partial x_i},\qquad (i=1,2)$$

$$u_3=u_{31}$$

in the faces, and

$$u_i=u_{i0}+x_3u_{i1},\quad (i=1,2)\qquad u_3=u_{31} \tag{26}$$

in the core. Upper (lower) sign in (25) holds true for the upper (lower) layer,

† Cf the papers by the author given in Ref.

and $\varkappa_i$ denotes the ratio of core shearing modulus G_{i3c} and face shearing modulus G_{i3}.

Making use of the strain - displacement relations, the components of strain (ε_i , γ_{ij}) are then found. Similarly, employing the stress - strain relations, the components of stress (σ_i , τ_{ij}) are arrived at. However, it is convenient in the plate theory to deal with forces and moments per unit length rather than with the stresses themselves; these new quantities are now defined by the formulae

$$S_{ii}=\int_{-h}^{h}\sigma_i\,dx_3\,,\quad S_{ik}=S_{ki}=\int_{-h}^{h}\tau_{12}\,dx_3\,,\quad T_i=\int_{-h}^{h}\tau_{i3}\,dx_3 \tag{27}$$

$$M_{ii}=\int_{-h}^{h}x_3\sigma_i\,dx_3\,,\quad M_{ik}=M_{ki}=\int_{-h}^{h}x_3\tau_{12}\,dx_3\,,\quad (i,k=1,2;\ i\neq k)$$

Denoting

$$S_{ii}^{o}=\sum_{j=1}^{2}A_{ij}\frac{\partial u_{jo}}{\partial x_j}\,,\quad S_{ik}^{o}=A_{33}\Big(\frac{\partial u_{10}}{\partial x_2}+\frac{\partial u_{20}}{\partial x_1}\Big),\quad T_i^{o}=B_i\Big(v_{i1}+\frac{\partial u_{31}}{\partial x_i}\Big) \tag{28}$$

$$M_{ii}^{o}=\sum_{j=1}^{2}\Big[D_{ij}\frac{\partial u_{j1}}{\partial x_j}+D_{i(j+2)}\frac{\partial^2 u_{31}}{\partial x_j^2}\Big]$$

$$M_{ik}^{o}=D_{31}\frac{\partial u_{11}}{\partial x_2}+D_{32}\frac{\partial u_{21}}{\partial x_1}+2D_{33}\frac{\partial^2 u_{31}}{\partial x_1\partial x_2}$$

where

$$A_{ii}=2(E_i^{*}t+E_{ic}^{*}s),\qquad A_{ik}=2(\mu_{ki}E_i^{*}t+\mu_{kic}E_{ic}^{*}s) \tag{29}$$

$$A_{33}=2(G_{12}t+G_{12c}s),\qquad B_i=\tfrac{2}{3}G_{i3c}(2h+s)$$

$$D_{ii}=\tfrac{1}{3}E_i^{*}t[3s(h+s)+\varkappa_i t(1.6h+0.9s)]+\tfrac{2}{3}E_{ic}^{*}s^3$$

$$D_{i(i+2)}=-\tfrac{1}{3}E_i^{*}t^2[2h+s-\varkappa_i(1.6h+0.9s)]$$

$$D_{i(k+2)}=-\tfrac{1}{3}E_i^{*}\mu_{ki}t^2[2h+s-\varkappa_k(1.6h+0.9s)]$$

$$D_{3i}=\tfrac{1}{3}G_{12}t[3s(h+s)+\varkappa_i t(1.6h+0.9s)]+\tfrac{2}{3}G_{12}s^3$$

$$D_{ik}=\tfrac{1}{3}E_i^{*}\mu_{ki}t\,[3s(h+s)+\varkappa_k t(1.6h+0.9s)+\tfrac{2}{3}E_{ic}^{*}\mu_{kic}s^3$$

$$D_{33}=-\tfrac{1}{3}G_{12}t[2h+s-0.5(\varkappa_1+\varkappa_2)(1.6h+0.9s)],\quad (i,k=1,2;\ i\neq k)$$

and further

$$S_{ii}^{T}=\int_{-h}^{h}\delta_i T^{**}dx_3=2\partial(\delta_i t+\delta_{ic}s) \tag{30}$$

$$M_{ii}^{T}=\int_{-h}^{h}x_3\delta_i T^{**}dx_3=\frac{h^3}{3}b\Big[\delta_i r+2\frac{k_3}{k_{3c}}(1-\lambda)^3\delta_{ic}\Big]$$

we obtain simple relations for the quantities (27)

$$S_{ii}=S_{ii}^{o}-S_{ii}^{T},\quad S_{ik}=S_{ik}^{o},\quad T_i=T_i^{o}-\frac{\partial M_{ii}^{T}}{\partial x_i} \tag{31}$$

$$M_{ii}=M_{ii}^{o}-M_{ii}^{T},\quad M_{ik}=M_{ik}^{o},\quad (i,k=1,2;\ i\neq k) \tag{31}$$

The conditions of equilibrium of the linear theory read

$$\sum_{j=1}^{2}\frac{\partial S_{ij}}{\partial x_j}=0,\quad (i=1,2) \tag{32}$$

$$T_i-\sum_{j=1}^{2}\frac{\partial M_{ij}}{\partial x_j}=0,\quad \sum_{i=1}^{2}\frac{\partial T_i}{\partial x_i}=-p,\quad (i=1,2) \tag{33}$$

where p denotes the eventual transverse loading.

The equations (32) imply the existence of a stress function $F(x_1,x_2)$ defined by the relations

$$S_{ii}=\frac{\partial^2 F}{\partial x_k^2}\,,\qquad S_{ik}=-\frac{\partial^2 F}{\partial x_i\partial x_k}\,,\quad (i,k=1,2;\ i\neq k) \tag{34}$$

the differential equation

$$A_{11}^{*}\frac{\partial^4 F}{\partial x_1^4}+(2A_{12}^{*}+A_{33}^{*})\frac{\partial^4 F}{\partial x_1^2\partial x_2^2}+A_{22}^{*}\frac{\partial^4 F}{\partial x_2^4}= \tag{35}$$

$$=-\Big(A_{21}^{*}\frac{\partial^2}{\partial x_1^2}+A_{22}^{*}\frac{\partial^2}{\partial x_2^2}\Big)S_{11}^{T}-\Big(A_{11}^{*}\frac{\partial^2}{\partial x_1^2}+A_{12}^{*}\frac{\partial^2}{\partial x_2^2}\Big)S_{22}^{T}$$

in R , and the proper boundary conditions along Γ . The constants A_{ij}^{*} in (35) are expressed in terms A_{ij} as follows

$$A_{ij}^{*}=(-1)^{(i+j)}\frac{A_{ij}}{A_{11}A_{22}-A_{12}A_{21}}\,,\quad A_{33}^{*}=\frac{1}{A_{33}}\quad (i,j=1,2) \tag{36}$$

For plates with isotropic layers (35) reduces to

$$\nabla^4 F=-\frac{A_{11}-A_{12}}{A_{11}}\nabla^2 S^{T} \tag{37}$$

where ∇^2 denotes the Laplacian operator.

Similarly, Eqs. (33) imply the existence of a displacement function $\omega(x_1,x_2)$ defined by the relations

$$u_{j1}=L_j\,\omega\,,\qquad (j=1,2,3) \tag{38}$$

the differential equation

$$L\omega=-p+\sum_{i=1}^{2}\frac{\partial^2 M_{ii}^{T}}{\partial x_i^2} \tag{39}$$

in R , an by proper boundary conditions† along Γ . Symbols L_j , L in Eqs. (38), (39) stand for certain partial differential operators which can be found in (3). In the case of isotropic layers, the expressions for these operators read simply

$$L=B(D_{11}-D_{13})\nabla^4(D_{31}\nabla^2-B),\quad L_i=-\frac{\partial}{\partial x_i}L_o\,,\ (i=1,2) \tag{40}$$

† All the boundary conditions mentioned can be found in (3).

$$L_0 = (D_{13}\nabla^2 - B)(D_{31}\nabla^2 - B)\,, \quad L_3 = (D_{11}\nabla^2 - B)(D_{31}\nabla^2 - B) \tag{40}$$

In the present problem the applied loads consist entirely of the negative of inertia forces, according to d'Alembert's principle; then

$$p = -m^* \ddot{u}_3 = -m^* \ddot{u}_{31} = -m^* L_3 \ddot{\omega} \tag{41}$$

provided that only deflections in the x_3 - direction need be considered. Symbol m^* represents the mass of the plate element of height $2h$

$$m^* = \frac{2(\gamma t + \gamma_c s)}{g} = 2(\rho^* t + \rho_c^* s) \tag{42}$$

γ, γ_c denote the specific gravities of the face and core materials, resp., ρ^*, ρ_c^* are the corresponding densities, and g the gravity constant. The dots mean again the time derivatives.

Inserting the inertia forces (41) into Eq.(39), we obtain

$$L\omega - m^* L_3 \ddot{\omega} = \sum_{i=1}^{2} \frac{\partial^2 M_{ii}^T}{\partial x_i^2} \tag{43}$$

In addition to the boundary conditions, the appropriate initial conditions for t^* 0 are to be given. Assume the following form of these conditions

$$u_3 = L_3\omega = u_{31}^o\,, \quad v_3 = \dot{u}_3^o = L_3\dot{\omega} = v_{31}^o \quad \text{for} \quad t^* = 0 \tag{44}$$

where u_{31}^o, v_{31}^o represent given functions of x_1, x_2.

Solution of the problem

The solution of the problem defined in the preceding Section consists of the solution of plane stress problem for the stress function F, and of the solution of the plate problem for the displacement function ω. These two solutions may be derived independently of each other. Since the first problem is formally identical to that of the theory of plane stress of homogeneous orthotropic bodies and has been discussed in many papers, we shall confine our attention to the solution of the second problem.

The problem for the displacement function ω consists generally of a nonhomogeneous differential equation (43), and of nonhomogeneous boundary and initial conditions, resp. Because of the linearity of the problem the solution can be expressed in the form

$$\omega = \omega_1 + \omega_2 \tag{45}$$

where ω_1 denotes the usual solution, namely one in which inertia effects are disregarded, and ω_2 is the solution which takes inertia into account. Following Boley-Weiner (2), the function ω_1 will be called here the static solution, similarly, the function ω_2 will be referred to as the dynamic solution.

It may thus be seen that the static solution ω_1 satisfies the equation

$$L\omega_1 = \sum_{i=1}^{2} \frac{\partial^2 M_{ii}^T}{\partial x_i^2} \tag{46}$$

in R, and the given boundary conditions along Γ. The dynamic solution ω_2 satisfies the differential equation

$$L\omega_2 - m^* L_3 \ddot{\omega}_2 = m^* L_3 \ddot{\omega}_1 \tag{47}$$

in the region R, and the corresponding homogeneous boundary conditions at the boundary Γ. For $t^* = 0$ the function ω_2 fulfills the initial conditions according to Eqs. (44), (45)

$$L_3\omega_2 = u_{31}^o - L_3\omega_1\,, \quad L_3\dot{\omega}_2 = v_{31}^o - L_3\dot{\omega}_1 \quad \text{for} \quad t^* = 0. \tag{48}$$

Our problem thus reduces to two particular problems - the static and dynamic one, respect. The static problem is identical to that of thermal bending and is solved in (3); the dynamic problem representing free forced vibrations of sandwich plates under the pulsating force (41) has been solved in (6).

IV. Example

To illustrate the solution of problems considered assume a rectangular simply-supported sandwich plate with isotropic layers occupying the space

$$R\{0 < x_i < l_i\,, \; i = 1,2\}\,, \quad -h < x_3 < h \tag{49}$$

with the boundary Γ ($x_i = 0$, $x_i = l_i$, $i = 1,2$) and $x_3 = \pm h$, respect. A step heat input q^*, constant in R, is assumed to be applied over the face $x_3 = h$, while the face $x_3 = -h$ is insulated. Initial deflection u_{31}^o, velocity v_{31}^o, and temperature T_0^* of the plate let altogether equal zero.

In the preceding Article, we have found that the thermal bending moment $M^T(\tau)$ due to the temperature considered is given by Eq.(19).Once the quantity $M^T(\tau)$ has been found the particular solutions of the static and dynamic problems, resp., can be sought.

The static problem is defined by the differential equation (46), the relations (40) and by the boundary conditions of simple supporting. In (3) we have shown that this formulation can be transformed into a more convenient form, namely,

$$B(D_{11} - D_{13})\nabla^2\omega_1 = M^T \tag{50}$$

in $\bar{R} = R + \Gamma$, and

$$\omega_1 = \frac{D_{11}M^T}{B^2(D_{11}-D_{13})} \qquad (51)$$

along Γ, the solution of which is found to be

$$\omega_1 = \frac{D_{11}M^T}{B^2(D_{11}-D_{13})} - \frac{M^T}{B}\sum_{mn}\alpha_{mn}\psi_{mn}, \quad (m,n \text{ odd}) \qquad (52)$$

where

$$\alpha_{mn} = \frac{16\ell_1^2}{(D_{11}-D_{13})\pi^4 mn(m^2+\chi^2 n^2)}, \quad \chi = \frac{\ell_1}{\ell_2} \qquad (53)$$

$$\psi_{mn} = \sin\frac{m\pi x_1}{\ell_1}\sin\frac{n\pi x_2}{\ell_2}$$

According to (3) the reduced form of operators L, L_j in the case of a plate with simply supported edges and isotropic layers is

$$L = B(D_{11}-D_{13})\nabla^4, \quad L_i = -\frac{\partial}{\partial x_i}(D_{13}\nabla^2 - B), \quad L_3 = D_{11}\nabla^2 - B \qquad (54)$$
$$(i=1,2)$$

The static part of the deflection u_3^{st} is obtained in the form

$$u_3^{st} = L_3\omega_1 = M^T\sum_{mn}\alpha_{mn}\psi_{mn}, \quad (m,n \text{ odd}) \qquad (55)$$

The dynamic problem is now defined by the differential equation (47) which with respect to Eqs.(52), (54) yields

$$L\omega_2 - m^* L_3\ddot{\omega}_2 = m^*(M^T)^{\cdot\cdot}\sum\alpha_{mn}\psi_{mn} \quad (m,n \text{ odd}) \qquad (56)$$

in the region R, and by boundary conditions

$$\omega_2 = \nabla^2\omega_2 = 0 \qquad (57)$$

at Γ. For $t^*=0$ the initial conditions

$$L_3\omega_2 = -(M^T)_0\sum_{mn}\alpha_{mn}\psi_{mn} \qquad (58)$$

$$L_3\dot{\omega}_2 = -(M^T)^{\cdot}_0\sum_{mn}\alpha_{mn}\psi_{mn}, \quad (m,n \text{ odd})$$

take place. Denoting

$$\tau = \frac{\eta D}{24}t^*, \qquad \eta = \frac{6k}{Dh^2 d} \qquad (59)$$

and further by

$$(M^T)^{\cdot}_0 = \frac{1}{1-\mu}\frac{\varrho^*\alpha E h^3}{6k}\left[r + \frac{\delta_c}{\delta}\frac{k}{k_c}(1-\lambda)^3\right]\eta \qquad (60)$$

the time derivative of M^T at $t^*=0$, Eqs. (58) can be rewritten to the form

$$L_3\omega_2 = 0, \quad L_3\dot{\omega}_2 = -(M^T)^{\cdot}_0\sum_{mn}\alpha_{mn}\psi_{mn}, \quad (m,n \text{ odd}) \qquad (61)$$

The present problem is solved in the very same manner as used in isothermal problems of free forced vibrations. In (5) we have found the natural frequencies η_{mn} of the plate under consideration to be as follows

$$\eta_{mn} = \frac{\pi^2}{\ell_1^2}(m^2+\chi^2 n^2)\sqrt{\frac{D_{11}-D_{13}}{1+\frac{D_{11}\pi^2(m^2+\chi^2n^2)}{B\ell_1^2}}\frac{1}{m^*}} \qquad (62)$$

The corresponding eigen-functions (natural modes of vibrations) represent the functions ψ_{mn} according to Eqs.(53). The eigen-values $p^{(mn)}$ of the problem are given by the simple formula

$$p^{(mn)} = m^*\eta_{mn}^2 \qquad (63)$$

Expanding the function ω_2 into an infinite series in eigen-functions ψ_{mn}

$$\omega_2 = \sum_{mn}\tilde{c}_{mn}(t^*)\psi_{mn} \qquad (64)$$

substituting (64) into (56), we arrive at the equations

$$\ddot{\tilde{c}}_{mn} + \eta_{mn}^2\tilde{c}_{mn} = -\frac{\alpha_{mn}}{\beta_{mn}}\eta e^{-\eta t^*}(M^T)^{\cdot}_0, \quad (m,n \text{ odd}) \qquad (65)$$

because of the relations

$$(M^T)^{\cdot\cdot} = -(M^T)^{\cdot}_0\eta e^{-\eta t^*}, \quad L_3\psi_{mn} = -\beta_{mn}\psi_{mn} \qquad (66)$$

$$\beta_{mn} = D_{11}\frac{\pi^2}{\ell_1^2}(m^2+\chi^2n^2) + B$$

$$L\psi_{mn} = -p^{(mn)}L_3\psi_{mn} = p^{(mn)}\beta_{mn}\psi_{mn} = m^*\eta_{mn}^2\beta_{mn}\psi_{mn}$$

the last of which implies the fact $p^{(mn)}$ is the eigen-value of the problem considered.

The general solution of (65) can be expressed as follows

$$\tilde{c}_{mn} = \frac{\alpha_{mn}}{\beta_{mn}}\frac{\eta}{\eta^2+\eta_{mn}^2}(M^T)^{\cdot}_0(e^{-\eta t^*} + a_{mn}\cos\eta_{mn}t^* + b_{mn}\sin\eta_{mn}t^*), \qquad (67)$$
$$(m,n \text{ odd})$$

where a_{mn}, b_{mn} are certain constants to be calculated from the initial conditions (61). Making use of these conditions, we obtain

$$a_{mn} = -1, \quad b_{mn} = -\frac{\eta_{mn}}{\eta}, \quad (m,n \text{ odd}) \qquad (68)$$

With these results Eqs.(54), (64), (67) yield the dynamic part of the deflection

$$u_3^D = L_3\omega_2 = (M^T)^{\cdot}_0\sum_{mn}\alpha_{mn}\psi_{mn}\frac{\eta}{\eta^2+\eta_{mn}^2}\left(e^{-\eta t^*} - \cos\eta_{mn}t^* - \frac{\eta_{mn}}{\eta}\sin\eta_{mn}t^*\right) \qquad (69)$$
$$(m,n \text{ odd})$$

Summarizing u_3^{st} and u_3^D the total deflection u_3 is obtained.

V. Conclusion

At conclusion let us analyze the results obtained in the preceding Article. Denoting

$$\vartheta_0 = \frac{\max u_3}{\max u_3^{st}} \tag{70}$$

we can write

$$\max u_3 = \max u_3^{st}(1+\vartheta_0) \tag{71}$$

In order to gain a better insight into the significance of the parameter ϑ_0 let us now consider, without any loss of generality, an infinite strip in the x_2 -direction. By performing the appropriate limiting process, we obtain first

$$\alpha_m = \frac{4\ell_1^2}{D_{11}-D_{13}}\,\frac{1}{\pi^3 m^3}\,,\ (m\ odd) \qquad \eta_1 = \frac{\pi^2}{\ell_1^2}\sqrt{\frac{D_{11}-D_{13}}{m^3}} \tag{72}$$

and denoting

$$B_0 = \frac{2h}{\ell\sqrt{\frac{k}{d}}}\left(\frac{D_{11}-D_{13}}{m^*}\right)^{1/4}$$

further

$$\vartheta_0 = -\frac{32}{\pi^2}\sum_m (-1)^{\frac{m-1}{2}}\frac{1}{m^3}\,\frac{1}{1+B_0^4\pi^4 m^4\left(\frac{D}{24}\right)^2}\left(\cos m^2\tau_0 + \frac{B_0^2\pi^2 D}{24}m^2\sin m^2\tau_0\right), \qquad (m\ odd) \tag{73}$$

provided

$$\frac{D_{11}\pi^2}{B\ell_1^2} \ll 1 \tag{74}$$

The time parameter τ_0 corresponds to a practically stationary value of u_3^{st} ; the following value can be adopted for this parameter

$$\tau_0 = 1.5 - \mathrm{arctg}\,\frac{24}{B_0^2 D\pi^2} \tag{75}$$

The function $\vartheta_0(B_0)$ according to (73), (75) can often be approximated by a piecewise smooth curve

$$\vartheta_0^* = 1 \qquad \text{for } 0 \le B_0 \le 0.5 \tag{76}$$

$$\vartheta_0^* = 2\left[1-0.36788\sqrt{\frac{2}{D}} - B_0\left(1-0.73575\sqrt{\frac{2}{D}}\right)\right] \qquad \text{for } 0.5 \le B_0 \le 1$$

$$\vartheta_0^* = 2\sqrt{\frac{2}{D}}\,e^{-B_0} \qquad \text{for } B_0 \ge 1$$

If another limiting process, namely the transition to a homogeneous beam is performed, a comparison between the results of the present paper and of Boley-Weiner (2) can be made. Nondimensional plots of reduced values of deflection

$$\bar{u}_3 = \frac{\pi^4 k\, u_3}{192\, g^* \alpha\, \ell_1^2} \quad at \quad x_1 = \frac{\ell_1}{2}$$

according to the present paper (heavy line) and to Boley-Weiner (2) (dashed line) are shown in Fig.3. It is seen that the dynamic solution oscillates about the static one in a very similar manner

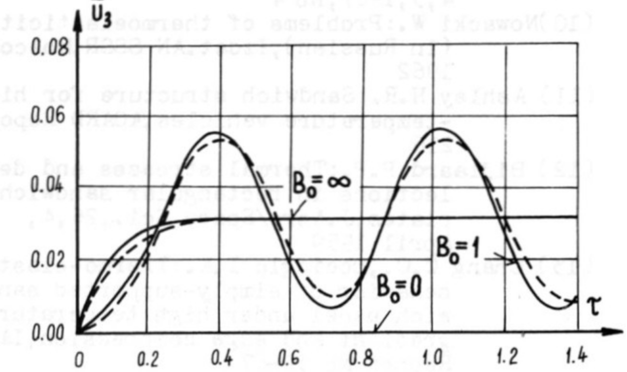

Figure 3. Deflection of heated plate according to present paper (heavy line) and according to Boley-Weiner (dashed line)

Collecting our results we conclude that the effect of inertia is to be taken into account whenever the characteristic parameter

$$B_0 = \sqrt{\frac{\eta_{11}}{\eta}\,\frac{24}{\pi^2(1+\chi^2)D}} \tag{77}$$

of the sandwich plate is lower or equal to three. Even in the cases $3 < B_0 \le 4$ this effect seems to be worth noting. The significance of this effect can be evaluated by means of the formula (71) with ϑ_0^* instead of ϑ_0 . With certain amount of inaccuracy we can say that the effect of inertia will be considerable in the case of relatively (small $\rho = h/\ell_1$) or absolutely (small h) thin plates, or in the case of plates with very thin facings.

References

(1) Boley B.A.: Thermally induced vibrations of beams, Journ.of the Aero. Sci., 23, 1956, No 2

(2) Boley B.A., Weiner J.H.: Theory of thermal stresses, John Wiley and Sons, New York-London 1960

(3) Kovařík V.: On thermal stresses in sandwich plates, Acta Technica ČSAV, 12, 1967, No 4

(4) Kovařík V., Šlapák P.: Stabilitätsuntersuchung der Sandwich-Platten, Acta Technica ČSAV, 11, 1966, No 6

(5) Kovařík V., Šlapák P.: Natural transverse vibrations of sandwich plates, Acta Technica ČSAV, 12, 1967, No 2

(6) Kovařík V., Šlapák P.: Forced transverse vibrations of sandwich plates, Acta Technica ČSAV, 12, 1967, No 3

(7) Melan E.,Parcus H.:Wärmespannungen infolge stationärer Temperaturfelder,Springer Verlag,Wien 1953
(8) Mindlin R.D.,Goodman L.E.:Beam vibrations with time-dependent boundary conditions,Journ.of Appl.Mech.,17, 1950
(9) Nowacki W.:Boundary problem of heat conduction,Bull.Acad.Polon.Sci., 4,5,1967,No 4
(10)Nowacki W.:Problems of thermoelasticity (in Russian),Izdat.AN SSSR,Moscow 1962
(11) Ashley H.R.:Sandwich structure for high -temperature vehicles,AGARD Report 216
(12) Bijlaard P.P.:Thermal stresses and deflections in rectangular sandwich plates,J.Aero/Space Sci.,26,4, April 1959
(13) Chang C.C.,Ebcioglu I.K.:Thermo-elastic behavior of simply-supported sandwich panel under high temperature gradient and edge compression,IAS Report No 59-67
(14) Chang C.C.,Ebcioglu I.K.:Thermoelastic behavior of a simply-supported sandwich panel under large temperature gradients and edge compression,J.Aerospace Sci.,28,6,June 1961
(15) Ebcioglu I.K.:Thermo-elastic equations for a sandwich panel under arbitrary temperature distribution, transverse load, and edge compression,Proc.4th U.S.Natl.Congr.Appl. Mech.,I
(16) Kuenzi E.W.,Jahnke W.E.:Mechanical properties of some heat-resistant metal honeycomb cores,FPL Report 1872
(17) Kuranishi M.:The behavior of sandwich structures involving stress, temperature and time dependent factors,IUTAM Symp.on Non-Homogeneity in Elasticity and Plasticity, Warsaw,Sept.1958; Bull.Acad.Pol. des Sciences,VII,2-3
(18) Mathauser E.E.,Pride R.A.:Compressive strength of stainless-steel sandwiches at elevated temperature, NASA Meno 6-2-59 L
(19) Schoeller W.C.:Calculating thermal stresses in sandwich panels,Aviation Age Res. and Dev.Tech.Handbook 1957-1958
(20) Semonian J.W.,Crawford R.F.:Some methods for the structural design of wings for application either at ambient or elevated temperatures,Trans.ASME,80,2,Feb.1958
(21) Swann R.T.:Heat transfer and thermal stresses in sandwich panels,NACA Tech.Note 4349
(22) Swann R.T.:Calculated effective thermal conductivities of honeycomb sandwich panels, NASA TN D-171