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ALLOWABLE FATIGUE STRESSES FOR A GIVEN LIFETIME

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The objective of this method is to correlate the dynamic response stress with a function of fatigue damage as caused by sine or random loadings. The approach is concerned with structures rather than machines and may be applied at any stage of the design process, from initial design of a new structure, when the resonant frequencies and transmissibilities can only be estimated, on through to the later stages when vibration test data from models provides values of these parameters. In redesigns and extensions of present designs, the natural frequencies and transmissibilities can usually be extrapolated from prior data with reasonable accuracy and, in general, a knowledge of these numbers is essential for a practical solution.

There are at least a dozen readily identifiable methods for the prediction of fatigue life or the calculation of an allowable stress and essentially all of them involve the concept of the gradual accumulation of damage during the loading period. The differences among the methods appear from the emphasis placed on a particular aspect or equation for representing the load spectra or the S-N data. The methods fall naturally into three categories according to these differences:

a) Linear Cumulative Damage
b) Non-linear Cumulative Damage
c) Damage Boundaries

and evaluation on the basis of available data, minimal assumption and acceptable accuracy indicate a preference for the linear function in the form of the Palmgren-Miner Hypothesis. This concept is fundamental to the method described here. A second aspect of this method is a practical treatment of the non-linear S-N function, which has not yet been expressed in acceptable form for the whole range of life. Use of the power laws has been fairly successful but they suffer considerable inaccuracies in both the upper and lower ranges of life. The equations in the present treatment are of the form of a power law, and so arranged that they may be written for any incremental length of the log S-log N curve, thus determining the slope, or exponent, with an accuracy dependent only on that of the data itself. As a third element of the method, the dynamic transfer function is introduced to accommodate properly the relation between the external forcing function and the response stress at any chosen location.

Since fatigue damage is due predominantly to the number of response stress peaks, the Rayleigh distribution is taken on the basis for the derivation. Figure 1 shows this correlation schematically and Equation 1 is applied under the assumption that the response stress is a reasonably linear function of the input acceleration:

$$S_A = K(G_r)_B \qquad (1)$$

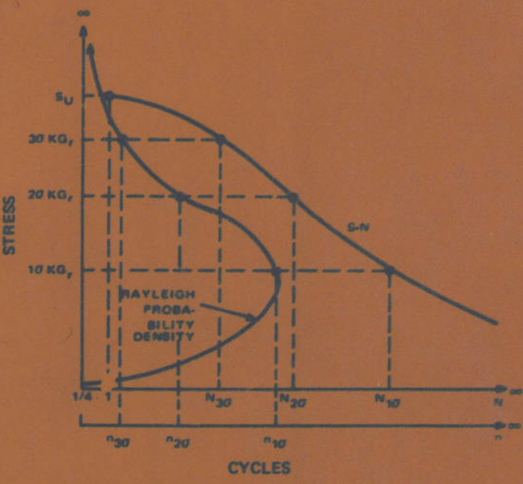

Figure 1. Correlation of Fatigue and Peak-Stress Probability

where S_A is the response stress at any point A in the structure and G_r is the acceleration at any other point B, normally taken as the input. The factor K may be regarded as a scale adjustment for the ordinate, and carries the dimensions of stress per unit acceleration, psi/g. It is a spatial function of the geometry and consequently varies throughout the structure, but is constant at any given point, regardless of the acceleration input level. The assumption is also made that the peak response and stress occur at resonance of the assembly. Should a local spring-mass be sufficiently detuned from this resonance to have a significant amplitude at its local resonance, the assumption may require modification. The working area of Figure 1 is bounded by the log S-log N curve. The Rayleigh response stress is:

$$S_{rr} = \sigma K \qquad (1a)$$

the same as Equation (1) with an acceptable standard deviation, σ, applied. In order to determine the number of cycles, n, for which this stress will occur, the product is formed of the Rayleigh probability density, the natural frequency and the time:

$$P_s f_n t = n_s \qquad (2)$$

the subscript s denoting any chosen stress level. The apparent double-value nature of the density curve when placed on these axes is negated by the fact that, although for any n a pair of stress values exist, the damage fractions for the two stresses will be different because their n's are ratioed with different N's on the S-N curve. The density is drawn with zero probability of failure at 1/4 cycle, and will fall within the allowable bounds only if the design is correct; that is, if the summation of damage fractions is one or less, $\Sigma\, n/N \leq 1$. If this curve intersects the S-log N curve at $n \geq 1$, then $\Sigma\, n/N \geq 1$ automatically, resulting in failure. The summation of damage fractions may be performed for a discrete set of stresses, or for a continuously varying set; in the latter case the summation sign is replaced by an integral sign. The S-N curve is interpreted as a plot of damage and its equation written as:

$$N = f(S) = (S_o/S)^{\alpha} \qquad (3a)$$

or

$$S_o = SN^{1/\alpha} \qquad (3b)$$

where α is a constant derived from a log S-log N plot as

$$\alpha = \frac{\log N_2 - \log N_1}{\log S_2 - \log S_1} \qquad (3c)$$

S is the cyclic stress to cause failure at N, and S_o is the failure stress at one-quarter cycle, usually taken as the ultimate tensile strength.

A basic ground-rule of the method is that the fatigue curve, log S vs log N, may be expressed analytically, in increments, as

$$\left.\begin{aligned} N_{ab} &= (S_{ab}/S)^{\alpha_{ab}} \\ N_{bc} &= (S_{bc}/S)^{\alpha_{bc}} \\ &\cdot \\ &\cdot \\ &\cdot \\ N_{ij} &= (S_{ij}/S)^{\alpha_{ij}} \end{aligned}\right\} \qquad (4)$$

shown schematically in Figure 2. Since few materials and conditions have been found for which the plot may be reasonably made as a single increment for an appreciable range of cycles, it is seen that α is not a true constant for any material. For the light metals it varies from about 4 to 50, Figure 2; the common alloy steels show a somewhat narrower range. The method is sensitive to the value of α; to fair in the plot line with only one or two slopes (increments) over 10 or more decades on the abscissan is to negate the possible precision. For some magnesium alloys it has been found necessary to calculate α for each 1 or 2 decades, especially in the low-cycle region.

Consider now the case of a structure subjected to a band-limited spectrum of random accelerations, the band containing the structure's lowest natural frequency. The input has a Gaussian distribution at a constant acceleration spectral density. (ASD is often loosely referred to as power spectral density, PSD). For a linear system, the instantaneous response acceleration distribution

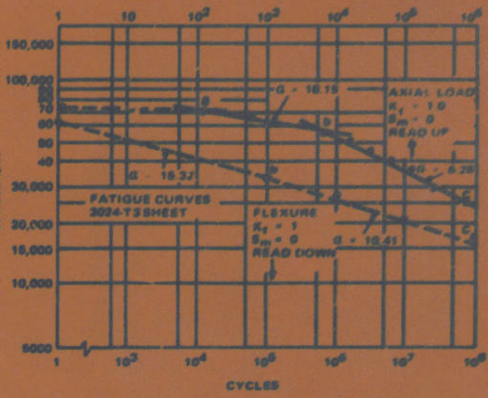

Figure 2. Fatigue Curves for Aluminum Sheet, 2024-T3

is Gaussian also, but the peaks follow the Rayleigh distribution. Since fatigue damage is dependent on the number of acceleration peaks, the probability density of response stress peaks is that characteristic of the Rayleigh distribution:

$$\lambda = \frac{G_r}{\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{G_r}{\sigma}\right)^2\right] \qquad (5)$$

where σ is the rms value of the acceleration response amplitudes, G_r. The expression for the rms response, for $-\infty \leq G_r \leq +\infty$:

$$\sigma = \sqrt{\frac{\pi}{2} G_o f_n (Q + 1/Q)} \qquad (5a)$$

is reduced for simplicity to

$$\sigma = \sqrt{\frac{\pi}{2} G_o f_n Q} \qquad (5b)$$

where G_o is the input acceleration spectral density level, f_n is the lowest natural frequency and Q is the transmissibility. For systems of medium to high transmissibility, $Q > 4$, the error introduced by this simplification is negligible.

Since a fatigue cycle is defined as a complete stress reversal, the probability density for the acceleration per cycle may be defined as λ, for the range $0 \leq G_r \leq +\infty$. More specifically, the probability of occurrence of any G_r is $\lambda_i dG_r$. Hence, multiplying $\lambda_i dG_r$ by the total number of occurrences or cycles gives the number of cycles, n_i, corresponding to $(G_r)_i$. To obtain the total number of acceleration reversals, it is noted that the number of zero crossings per period is 2. Thus, the total number of stress cycles is $f_n t$, where t is the total time of random vibration, in seconds. Then the number of cycles, n_i, at any response stress level $(G_r)_i$ is

$$n_i = f_n t\, \lambda_i dG_r = f_n t \frac{G_r}{\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{G_r}{\sigma}\right)^2\right] dG_r \qquad (6)$$

Correlation of the damage criterion, $\Sigma n/N = C_1$, with an allowable stress level is obtained by substitution from the previous equations, resulting in, finally:

$$C_1 = \Sigma n/N = \int_0^\infty \frac{(f_n t/\sigma)(G_r/\sigma)\exp\left[-\frac{1}{2}(G_r/\sigma)^2\right]}{(S_o/KG_r)^\alpha} dG_r \quad (7)$$

which is simplified by letting $X = G_r/\sigma$ and $dX = dGr/\sigma$, to

$$C_1 = \Sigma n/N = f_n t \left(\frac{K\sigma}{S_o}\right)^\alpha \int_0^\infty X^{1+\alpha} \exp\left(\frac{X^2}{2}\right) dX \quad (7a)$$

where K is the allowable stress in psi/g. Letting I_r equal the integral term in Eq. (7a) results in further simplification:

$$C_1 = \Sigma n/N = f_n t\, I_r \left(\frac{K\sigma}{S_o}\right)^\alpha \quad (8)$$

and solving for the unknown, K, yields

$$K = \frac{S_o}{\sigma}\left(\frac{C_1}{f_n t\, I_r}\right)^{1/\alpha} \quad (9)$$

or

$$\log K = \log(S_o/\sigma) + \frac{1}{\alpha}\log\left(\frac{C_1}{f_n t\, I_r}\right) \quad (10)$$

The integral, I_r, has been evaluated by computer for a convenient range of α values; I_r vs α is plotted in Figure 3.

In a normal problem, Q and S_o are known; and f_n is known or estimated; σ is calculated from Equation (5b); α is calculated from fatigue data, as Figure 2 and I_r is determinable from Figure 3. If f_n should be totally unknown and the vibration modes so complex that no estimate is possible, this method becomes inapplicable. Upon evaluation of K, the allowable fatigue stress becomes $3\sigma K$ and the design load due to acceleration is 3σ, ignoring responses greater than 3σ. The response is trimmed off at this level because the probability of responses (acceleration peaks) occurring above 3σ in the Rayleigh distribution is extremely small, about 1%, and to design for $G_r > 3\sigma$ would seem to penalize most structures rather severely. For those cases wherein the reliability requirements are very high, this latter statement may not be applicable, and more standard deviations may be indicated.

As an example, consider a spacecraft structure, made of riveted aluminum alloy 2024-T3, and which approximates a single-degree-of-freedom system. Other structures similar to this in size, material, and mode of fabrication have indicated a transmissibility of 10 at a first mode resonance of 40 cps. Find the allowable fatigue stress for the condition that $\Sigma n/N = 1$ with the random vibration input:

Input level $G_o = 0.10 g^2/\text{cps}$, constant
Time $t = 15$ minutes
Bandwidth, $B = 20$ to 200 cps, or 6.62 octaves

Then, for yielding as a failure criterion, $S_o = 48000$ psi $= F_{ty}$ (from MIL-HDBK-5a), or for fracture, $S_o = 65000$ psi $= F_{tu}$.

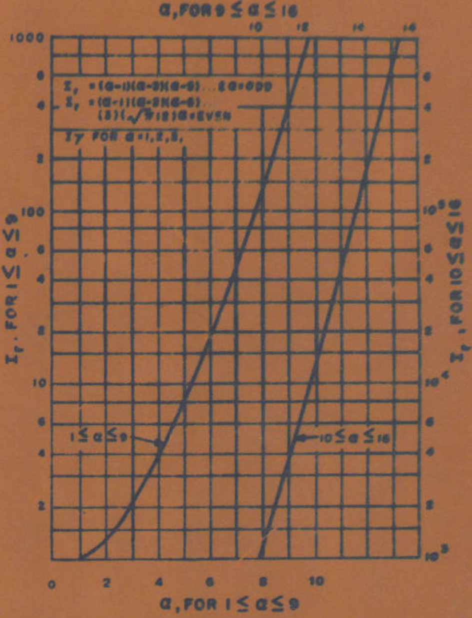

Figure 3. Plot of the "Random-Integral" vs Slope of the S-N Curve, I_r vs α

From Equation (3c) and a plot, Figure 2, of log S vs log N for 2024-T3 in the region of 10^4 to 10^5 cycles, ($f_n t = 3.6 \times 10^4$ cycles), α is determined as 6.28, and from Figure 3, I_r is found as equal to 30. σ is obtained from Equation (5b) as equal to 7.92, then by Equation (10):

$$\log K = \log\left(\frac{48000}{7.92}\right) + \frac{1}{6.28}\log\left(\frac{1\,(2.5)}{40(15)\,(60)\,(30)}\right)$$

$$\log K = 3.78176 + 0.159\,(-5.63545) = 2.88650$$

$$K = 770 \text{ psi/g, for } S_o = F_{ty}$$

similarly:

$$K = 1045 \text{ psi/g, for } S_o = F_{tu}$$

The allowable fatigue stress is $3\sigma K$ or 3(7.92)(770) = 18,400 psi for yielding as the failure criterion, or 24,800 psi for fracture.

Similar investigation, by this method, of the response to sinusoidal sweeps may be done under the assumptions:

1) The rate of sweep frequency is proportional to the frequency: $df/dt = K_2 f$, or $f = f_o(f_f/f_o)\,p$, where $p = t/t_s$.
2) The input acceleration level, g_o, is constant over the bandwidth of the sweep.
3) Steady-state response is attained at each frequency; a conservative condition which has been observed at normal sweep rates.

The instantaneous number of cycles, n_j, for any frequency, f_j, is then:

$$n_j = f_j dt = \frac{t_s\, df_j}{\ln (f_f/f_o)} \qquad (11)$$

Substitution of Eqs. (11) and (4) into (2) yields:

$$C_2 = \sum_{f_a}^{f_b} \frac{n_j}{N_j} = \int_{f_a}^{f_b} \frac{df/K_2}{(S_{ab}/S)^{\alpha}} + \int_{f_b}^{f_c} \frac{df/K_2}{(S_{bc}/S)^{\alpha}} \qquad (12)$$

For simplification, the consideration of the S-N curve is limited to one increment, (a-b), and by use of Eq. (1) there follows:

$$C_2 = \frac{1}{K_2 (S_i)^{\alpha}} \int_{f_a}^{f_b} (K\, G_r)^{\alpha}\, df$$

$$C_2 = \frac{t_s}{\ln (f_f/f_o)\, (S_i)^{\alpha}} \int_{f_a}^{f_b} (K\, G_r)^{\alpha}\, df \qquad (13)$$

Now substituting into Eq. (13): $Z = f/f_n$; $dZ = df/f_n$; $Z_a = f_a/f_n$; $Z_b = f_b/f_n$, and taking the standard expression for amplification or magnification, M, as:

$$M = \frac{1}{\left\{ \left[1 - (f/f_n)^2\right]^2 + (f/Qf_n)^2 \right\}^{1/2}} \qquad (14)$$

Equation (13) becomes:

$$C_2 = \frac{f_n\, t_s}{\ln (f_f/f_o)} \left(\frac{K\, g_o}{S_i}\right)^{\alpha} X$$

$$\int_{Z_a}^{Z_b} \frac{dZ}{\left[(1 - Z^2)^2 + (Z/Q)^2\right]^{\alpha/2}} \qquad (15)$$

Defining the "sine integral," I_s, as the integral term in Eq. (15), taken from $0 < Z < \infty$, which, with substitution from Eq. (14), leads to:

$$I_s = \int_0^{\infty} M^{\alpha}\, dZ \qquad (16)$$

and the final solution takes the form:

$$C_2 = \frac{I_s\, t_s\, f_n}{\ln (f_f/f_o)} \left(\frac{K\, g_o}{S_i}\right)^{\alpha} \qquad (17)$$

Values of I_s vs. α are plotted in Figs. 4a thru d. After evaluation of Eq. (17) for K, the allowable stress is Kg_oQ and the design load is g_oQ. Eq. (17) may be simplified to the same form as Eq. (10):

$$\log K = \log\left(\frac{S_o}{g_o}\right) + \frac{1}{\alpha} \log \frac{C_2}{I_s t_s f_n} \qquad (18)$$

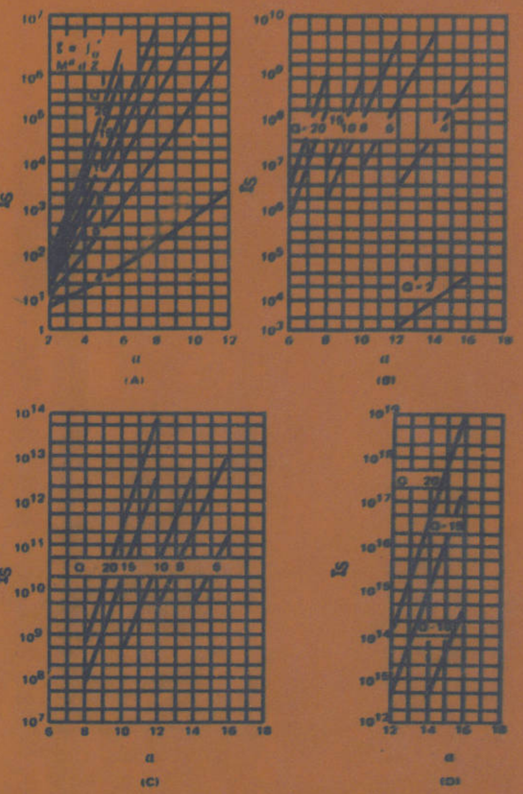

Figure 4. Plot of the "Sine-Integral" vs Slope of the S-N Curve, I_s vs α

As an example, take the structure described above as subjected to a 5g zero-to-peak acceleration in the form of a sine-wave swept from 20 to 2000 cps at a rate proportional to frequency. As before, $\alpha = 6.28$; $Q = 10$, and, from Fig. 4a, $I_s = 2.0 \times 10^5$. Substituting these values into Eq. (18) yields:

$$\log K = \log \frac{48000}{5.0} + \frac{1}{6.28} \log \frac{1 \times 2.5}{(40)(10)(60)(2.0 \times 10^5)}$$

$$\log K = 3.98227 + 0.159\,(-9.284)$$

$$\log K = 2.49683$$

K = 314 psi/g, for $S_o = F_{ty}$, and for a 10 min. sweep.

And for varying sweep periods:

K = 300 psi/g for $S_o = F_{ty}$, and a 15 min. sweep
k = 281 psi/g for $S_o = F_{ty}$, and a 20 min. sweep

for $S_o = F_{tu}$:

K = 425 psi/g $S_o = F_{tu}$; for 10 min. sweep
K = 406 $S_o = F_{tu}$; for 15 min. sweep
K = 381 $S_o = F_{tu}$; for 20 min. sweep

The decreasing value of the allowable stress with increasing time is explained by the fact that, although the number of cycles applied during the resonances ($n = f_n t$), is rising, so is the number of cycles, N, allowed to failure; thus the stress "slides down" the curve with increasing time.

For a comparison of the effects of changing the input, take $g_o = 7.92$, the equivalent to 0.1 g^2/cps from Eq. (5b). Then by Eq. (10):

K = 198 psi/g for $S_o = F_{ty}$ and a 10 min. sweep
K = 189 psi/g for $S_o = F_{ty}$ and a 15 min. sweep
K = 177 psi/g for $S_o = F_{ty}$ and a 20 min. sweep

K = 269 psi/g for $S_o = F_{tu}$ and a 10 min. sweep
K = 256 psi/g for $S_o = F_{tu}$ and a 15 min. sweep
K = 240 psi/g for $S_o = F_{tu}$ and a 20 min. sweep

These are the maximum stresses which result from the input loadings as described in the examples, and which would in the time periods given, be applied for the correct number of cycles to bring the Miner summation to unity, that is, n occurs with a 3σ probability. The types of stress such as tension or bending would be derived from the deflection mode as dictated by the structural and loading geometries.

Thus, it appears that the allowable fatigue stress, as given by this method of solution:

a) Varies directly with the input load level.
b) Varies directly with the failure limit stress, as F_{tu} or F_{ty}.
c) Varies slowly with the time of cyclic stressing.
d) Varies sharply with the fatigue resistance in the form of the material parameter α.

The use of the Rayleigh distribution is correct for fatigue work because of the dependence of fatigue damage on the numbers of peaks or, equivalently, on numbers of stress cycles. The mathematical operations, can, of course, be performed on whatever distribution may be assumed but a comparison may be made only at a single frequency, the fundamental of the system. Treatment under the Gaussian assumption yields the time at any amplitude; treatment under the Rayleigh assumption yields the probability that peak amplitudes will be reached. From either assumption then, a number of stress cycles may be derived, but since fatigue damage is dependent predominately on the higher stresses (in the vicinity of the peak amplitudes), the use of the Rayleigh distribution is correct. In a few trial cases, stress values so determined were about 10% more conservative than those from the Gaussian.

These examples were simplified by the assumption that the response was in the same direction as the input with no cross-coupling. The degree to which a modally-coupled response is excited depends upon the structural and loading geometries, and it is not uncommon for the amplitude of such response to exceed that of the direct response. Also, the excitation was generally assumed to be on a single axis and for a single time period - so, for spacecraft being qualified to conventional specifications, the times per axis per test must be summed. Conservatism also dictates that a re-test factor, somewhat in excess of 1.0, be applied to the total time of the specification, the value depending on local conditions. For aircraft, the number and duration of the sorties in the design life would be summed for the total time, and adjustment factors, depending on the type of service, applied as deemed advisable.