ICAS Paper No. 68-15

THE ACCURATE NUMERICAL CALCULATION OF THE FLOW AROUND BLUNT BODIES (IDEAL GAS AND SPHERICAL SHAPE) MOVING AT SUPERSONIC SPEEDS

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The Sixth Congress of the International Council of the Aeronautical Sciences

DEUTSCHES MUSEUM, MÜNCHEN, GERMANY/SEPTEMBER 9-13, 1968

Preis: DM 2.00

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Abstract

An accurate inverse method for the numerical calculation of the supersonic flow around an axisymmetric blunt body is presented. By a suitable transformation the field under consideration is mapped on a rectangular strip. Integration of the equations of motion in the transformed domain is performed by simultaneous application of a third order predictor-corrector process in the subsonic part, and the method of characteristics in the supersonic part of the field.

An iterative technique, based on the minimization of a certain function along the body contour, is used in order to obtain the correct shock wave shape and flow field belonging to a given body contour at a given Mach number.

Computational results for spherical bodies, prepared with the assumption of ideal gas properties, have been obtained, but the method is not restricted to spherical shaped bodies nor to the assumption of ideal gas properties.

Contents

List of symbols

B, C parameters in shock or body equation
c_v specific heat at constant volume
F p/ρ^γ (sections 2 and 3); function defined along body contour (section 5)
g weight factor
G weight factor
K parameter in shock or body equation
M Mach number
N number of steps $\Delta\tau$ in inverse computation
p dimensionless pressure, measured relative to $\rho_\infty q_\infty^2$
q total velocity $\sqrt{(u^2+v^2)}$; $q_\infty = 1$
r polar radius (fig.1)
S entropy
u, v components of velocity in direction of increasing r and ϑ respectively
$\bar{u}$, $\bar{v}$ components of velocity in direction of increasing x and y respectively
x co-ordinate along axis of symmetry (fig.1)
y cylindrical co-ordinate (fig.1)
β $\sqrt{(M^2-1)}$
γ ratio of specific heats
ε shock distance, measured along axis of symmetry
ϑ polar angle (fig.1)
ψ stream function
τ variable, used as co-ordinate and defined by eq (7)
ρ density; $\rho_\infty = 1$
σ shock angle

Subscripts:

b refers to "body"
s refers to "shock"
0 refers to "$\vartheta = 0$"
∞ refers to "free stream"

1 Introduction

Though the amount of methods and existing programs for solution of the blunt body supersonic flow problem is considerable, the final word in the matter has not yet been written. It appears namely that the results of calculations, performed by using different methods for the same test case, can show a considerable scatter. When the need for the availability of accurate information about blunt nose flow fields in supersonic free stream became apparent at the NLR, it was decided to develope a reliable and accurate method. After a comparative literature study the choice was fixed upon an inverse method, the basic idea of which was developed in [1]. It appeared to be possible to construct a computer program with such properties, that the first requirement, imposed by the need of information about blunt nose fields, could be met, i.e. the flow fields around spheres could be computed with unique results for free stream Mach numbers in the range from infinity down to about 1.8, with the assumption of ideal gas properties. Unique in the sense: independent from the stepsizes used (within certain limits) in the calculation.

Further requirements are the computation of flow fields around spheres in the range of Mach numbers below 1.8, and the computation of flow fields around bodies other than spheres. The successful operation of the program up to now and the good hope of successful operation in more difficult cases are consequences of two important factors:

(i) the accuracy of the inverse computation and the full control of eventually occurring instabilities
(ii) application of a powerful technique for the selection of a new shock wave shape from previous results, a technique which is based on the definition of a norm on the body contour.

2 Basic equations.

In the polar co-ordinate system r, ϑ (figure 1) the equations of motion for an axi-symmetric ideal gas flow can be written as follows:

$$u\frac{du}{dr} + \frac{v}{r}\frac{du}{d\vartheta} + \frac{1}{\rho}\frac{dp}{dr} = \frac{v^2}{r} \tag{1}$$

$$u\frac{dv}{dr} + \frac{v}{r}\frac{dv}{d\vartheta} + \frac{1}{\rho r}\frac{dp}{d\vartheta} = -\frac{uv}{r} \tag{2}$$

$$\frac{d}{dr}(\rho u r^2 \sin\vartheta) + \frac{d}{d\vartheta}(\rho v r \sin\vartheta) = 0 \tag{3}$$

$$u^2 + v^2 + \frac{2\gamma}{\gamma-1}\frac{p}{\rho} = 1 + \frac{2}{(\gamma-1)M_\infty^2} \tag{4}$$

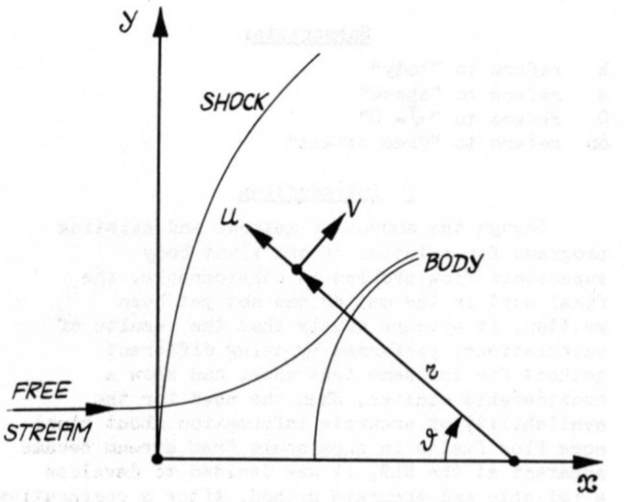

Figure 1 Configuration.

A stream function ψ can be defined by the equations

$$\frac{d\psi}{d\vartheta} = -\rho u r^2 \sin\vartheta, \quad \frac{d\psi}{dr} = \rho v r \sin\vartheta \quad , \tag{5}$$

satisfying the continuity equation (3). From (5) follows

$$d\psi = -\rho r \sin\vartheta\ (ur\, d\vartheta - v dr)$$

so that $d\psi = 0$ for $\frac{1}{r}\frac{dr}{d\vartheta} = \frac{u}{v}$, hence ψ = constant along streamlines. Upstream of the shock wave ψ can be found by integration of

$$d\psi = d\left(\tfrac{1}{2} r^2 \sin^2\vartheta\right).$$

This formula can be derived on account of the relations $u = -\cos\vartheta$
$v = \sin\vartheta$
$\rho = 1,$
valid in the free stream.
With $\psi = 0$ for $\vartheta = 0$, the integration delivers simply

$$\psi = \tfrac{1}{2} r^2 \sin^2\vartheta .$$

Just behind the shock wave ψ is then also known, ψ being continuous through the shock wave:

$$\psi_s = \tfrac{1}{2} r_s^2 \sin^2\vartheta \quad , \tag{6}$$

the subscript s referring to "shock".
Along the body contour $\psi_b = 0$.
At this point a variable τ is introduced, defined by

$$\tau = \frac{2\psi}{r^2 \sin^2\vartheta} \tag{7}$$

It follows that $\tau_s = 1$
and $\tau_b = 0$.

Along streamlines holds $\tau r^2 \sin^2\vartheta$ = constant. Introducing the co-ordinate system τ, ϑ and transforming the equations of motion accordingly, the following result is obtained:

$$(\rho u + \tau\cos\vartheta)\frac{du}{d\tau} + (\rho v - \tau\sin\vartheta)\frac{dv}{d\tau} = \frac{pr^2\sin^2\vartheta}{2(1-\gamma)F}\frac{dF}{d\psi} - \frac{1}{2}\sin\vartheta\left(v - \frac{du}{d\vartheta}\right) \tag{8}$$

$$\left[\tau\sin\vartheta - \rho v - \frac{\tau\rho u}{\gamma p}(u\sin\vartheta + v\cos\vartheta)\right]\frac{du}{d\tau} + \left[\tau\cos\vartheta + \rho u - \frac{\tau\rho v}{\gamma p}(u\sin\vartheta + v\cos\vartheta)\right]\frac{dv}{d\tau} = \frac{1}{2}\left[v\cos\vartheta + \sin\vartheta\left\{2u + \frac{dv}{d\vartheta} - \frac{\rho v}{\gamma p}\left(u\frac{du}{d\vartheta} + v\frac{dv}{d\vartheta}\right)\right\}\right] \tag{9}$$

In these equations only u and v occur as integration variables. The derivatives of p and ρ have been eliminated by means of the relation (4) and the relation

$$p/\rho^\gamma = F(\psi). \tag{10}$$

The equation (10) expresses the constancy of entropy along streamlines, p/ρ^γ being related to entropy as follows:

$$\frac{p}{\rho^\gamma} = \frac{1}{\gamma M_\infty^2}\exp\left\{c_v\,(S - S_\infty)\right\}.$$

For the purpose of the elimination of the derivatives of p and ρ the following relations have been used:

$$\left(\frac{d\psi}{d\tau}\right)_\vartheta = \frac{\rho v\, r^2 \sin^2\vartheta}{2(\rho v - \tau\sin\vartheta)}$$

$$\left(\frac{d\psi}{d\vartheta}\right)_\tau = \tau r^2\, \rho\sin\vartheta\,\frac{v\cos\vartheta + u\sin\vartheta}{\rho v - \tau\sin\vartheta}$$

One relation still must be added in order to complete the set of equations needed for the description of the flow field, i.e. the relation,

$$\left(\frac{dr}{d\tau}\right)_\vartheta = \frac{r\sin\vartheta}{2(\rho v - \tau\sin\vartheta)} \tag{11}$$

The complete set is now available and it consists of the differential equations (8), (9) and (11), together with the algebraic relations (4), (7) and (10). The quantity $dF/d\psi$, occurring in (8), is a known quantity as a function of τ, r and ϑ, because along the given shock wave F is a known function of ψ.

In the supersonic part of the flow field real characteristics exist. Because the computation of this part of the field is performed by using the method of characteristics, the characteristic equations and directions will be given here. These characteristic relations can most easily be derived by starting from the equations (1) through (4), obtaining the relations in the co-ordinate system r, ϑ. The next step is to transform them to the co-ordinate system τ, ϑ. Finally the derivatives of p and ρ are eliminated, using (4) and (10). Three families, of characteristics are obtained in this way, characterized by the relations given below:

(i) $\frac{d\vartheta}{d\tau} = \frac{1}{2}\,\frac{(\beta v-u)\sin\vartheta}{\rho q^2-\tau(\bar{u}+\beta\bar{v})}$ (local direction),

$$(\beta u-v)\frac{du}{d\tau} + (\beta v+u)\frac{dv}{d\tau} =$$

$$= \frac{q^2}{2}\left\{\frac{\bar{v}-(\beta v-u)\sin\vartheta-\frac{\beta\rho^{\gamma}\,r^2\,\sin^2\vartheta}{\gamma-1}\,\frac{dF}{d\Psi}}{\rho\,q^2-\tau(\bar{u}+\beta\bar{v})}\right\}$$

(characteristic equation),

(ii) $\frac{d\vartheta}{d\tau} = -\frac{1}{2}\,\frac{(\beta v+u)\,\sin\vartheta}{\rho q^2-\tau(\bar{u}-\beta\bar{v})}$ (local direction),

$$(\beta u+v)\frac{du}{d\tau} + (\beta v-u)\frac{dv}{d\tau} =$$

$$= -\frac{q^2}{2}\left\{\frac{\bar{v}+(\beta v+u)\,\sin\vartheta+\frac{\beta\rho^{\gamma}\,r^2\,\sin^2\vartheta}{\gamma-1}\,\frac{dF}{d\Psi}}{\rho q^2-\tau(\bar{u}-\beta\bar{v})}\right\}$$

(characteristic equation),

(iii) $\tau r^2\,\sin^2\vartheta$ = constant (streamlines),
p/ρ^{γ} = constant along each streamline.

The relations (iii) have been used already to derive (8) and (9) and to eliminate $dp/d\tau$ and $d\rho/d\tau$ from the characteristic differential relations.
To complete the set of equations one equation still must be added:

$$\left(\frac{dr}{d\tau}\right)_{\substack{\text{char.of}\\ \text{fam.(i)}}} = \frac{(\beta u+v)r\,\sin\vartheta}{2\left\{\rho q^2-\tau(\bar{u}+\beta\bar{v})\right\}} \qquad (12)$$

In figure 2 the transformed domain has been sketched.

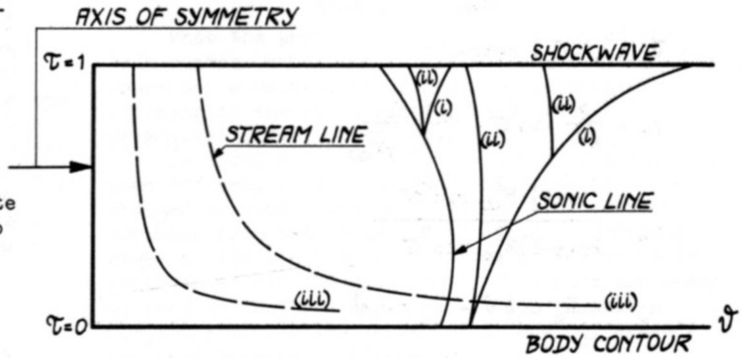

Figure 2 Characteristics of first (i) and second (ii) family, streamlines (iii) and sonic line in the τ, ϑ-domain.

3 Boundaries.

In the τ, ϑ-co-ordinate system three boundaries, namely the shock wave, the body contour and the axis of symmetry coincide with co-ordinate planes. In the supersonic field a characteristic of the first family has been choosen as a boundary. The treatment of the boundaries will be described now subsequently.

3.1 The shock wave.

In the inverse problem the shape of the shock wave is assumed to be known in advance:

$r_s = f(\vartheta)$.

In connection to this the following choice has been made:

$$y_s^2 = 2\,K_s x - B_s x^2 + C_s x^3, \qquad (13)$$

where

$y_s = r_s\,\sin\vartheta$,

$x = r_{s,0} - r_s\cos\vartheta$,

$r_{s,0} = f(0)$,

and

K_s, B_s and C_s are constants (shock parameters).

The subscript "s" refers to "shock".
The slope of the shock wave is determined by the angle σ, to be obtained from $tg\sigma = dy_s/dx$.

The field quantities just behind the shock wave can be expressed as follows:

$$\bar{u}_s = 1 + \frac{2}{\gamma+1}\left(\frac{1}{M_\infty^2} - \sin^2\sigma\right)$$

$$\bar{v}_s = (1-\bar{u}_s)\ \mathrm{cotg}\,\sigma$$

$$p_s = \frac{2}{\gamma+1}\sin^2\sigma - \frac{\gamma-1}{\gamma+1}\,\frac{1}{\gamma M_\infty^2}$$

$$\rho_s = \frac{(\gamma+1)\,M_\infty^2\,\sin^2\sigma}{2+(\gamma-1)M_\infty^2\,\sin^2\sigma}$$

$$u_s = -\,\bar{u}_s\cos\vartheta + \bar{v}_s\sin\vartheta$$

$$v_s = \bar{u}_s\sin\vartheta + \bar{v}_s\cos\vartheta$$

Thus for $\tau = 1$ all field quantities are known as a function of ϑ.
Care must be taken that $\vartheta < \vartheta_{max}$ where ϑ_{max} is the angle for which the strength of the shock is zero, i.e. the angle for which applies: $\sin\sigma = 1/M_\infty$.

The quantity $dF/d\psi$, occurring in (8) and in the characteristic equations, can be expressed algebraically in the quantities p, ρ, r, ϑ, τ, K_s, B_s and C_s.

3.2 The axis of symmetry.

For $\vartheta = 0$ the equations of motion (8) and (9) degenerate. From (8) can be derived

$$\rho u + \tau = 0 \qquad (14)$$

It is possible to compute the exact values of ρ, u and p as functions of τ. For any τ, ρ can be solved numerically from the following equation, derived from (4) by substituting (14) and noting that v = 0 on the axis of symmetry:

$$\frac{\tau^2}{\rho^2} + \frac{2\gamma}{\gamma-1}\left(\frac{p_s}{\rho_s^\gamma}\right)_{\vartheta=0}\rho^{\gamma-1} = 1 + \frac{2}{(\gamma-1)\,M_\infty^2} \qquad (15)$$

Next u follows from (14) and p from (4) or from the entropy relation.
The integration of r along the axis of symmetry can be accomplished by using the following formula, derived from (11):

$$\left(\frac{dr}{d\tau}\right)_{\vartheta=0} = \frac{r}{2\left[\rho\left(\frac{dv}{d\vartheta}\right)_{\vartheta=0} - \tau\right]} \qquad (16)$$

The value of $(dv/d\vartheta)_{\vartheta=0}$ is determined by numerical differentiation.

3.3 The boundary characteristic and the body contour.

In the supersonic part of the flow field the hyperbolic character of the differential equations allows the generation of a stable solution of the initial value problem using the method of characteristics. A natural boundary for the region of computation is a characteristic of the first family, as indicated in figure 2. This boundary is not known in advance, but it is generated while the computation proceeds. The solution of the hyperbolic problem then furnishes a known boundary for computation of the subsonic flow field.

The integration process ultimately delivers the field quantities along the line $\tau = 0$. As this line is identified with the body contour, no interpolation or extrapolation is required in order to obtain the body quantities.

4 Method of computation.

4.1 The subsonic field.

In order to be able to perform the computation numerically, a finite difference lattice network is fixed in the region of computation and steps of magnitude $\Delta\tau$ and $\Delta\vartheta$ are assumed.

Treating the problem as an initial value problem, the quantities at $\tau = 1$ being the initial values, the integration of the eqs (8) and (9) has been achieved using a predictor-corrector scheme.

It is supposed that for a certain $\tau = \tau_i$ (where $0 \leq \tau_i = 1 - i.\Delta\tau \leq 1$; i=0,1,2,---,N) all field quantities are known in the points $\vartheta = \vartheta_k = k.\Delta\vartheta$ (k = 0,1,----,n).

Derivatives in ϑ-direction along the line $\tau = \tau_i$ can be found by numerical differentiation. The ϑ-derivatives once being known, τ-derivatives can be calculated from the eqs (8) and (9).

The application of the predictor-corrector process is the next step in the computation.
Predictor:

$$u_{i+1,k} = u_{i,k} + \frac{1}{2}\Delta\tau\left\{3\left(\frac{du}{d\tau}\right)_{i,k} - \left(\frac{du}{d\tau}\right)_{i-1,k}\right\} \qquad (17)$$

Corrector:

$$u_{i+1,k} = u_{i,k} + \frac{1}{12}\Delta\tau\left\{5\left(\frac{du}{d\tau}\right)_{i+1,k} + 8\left(\frac{du}{d\tau}\right)_{i,k} - \left(\frac{du}{d\tau}\right)_{i-1,k}\right\} \qquad (18)$$

This integration process, the third order Adams-Moulton process, is a stable one, the extraneous roots of the characteristic equation being located within the unit circle in the complex plane, provided the stepsize $|\Delta\tau|$ not being too large.

In the above $\Delta\tau = \tau_{i+1} - \tau_i = -1/N$, N being the total number of steps from $\tau = 1$ to $\tau = 0$. Because the integration process (17), (18) is not self-starting, a special arrangement has to be made in order to achieve the first step $\Delta\tau$. For this purpose the second order Euler-Cauchy process appears to be very suited and this process is therefore used, covering the first step $\Delta\tau$ by 4 second order steps.

In experimental computations the application of a smoothing technique appeared to have a favourable influence on the preservation of stability near the axis of symmetry, especially at the lower free stream Mach numbers. The smoothing is realized in the following way. Taking together a group of 7 or 9 given function values in the interval to be smoothed, a third degree polynomial is fitted to these values in the sense of least squares, and the polynomial value of the central point of the group is stored. After having dealt with the entire interval in this way, the stored function values are assigned to the corresponding points. Near the boundaries of the interval considered, some minor modifications of the process are necessary.

4.2 The supersonic field.

In the supersonic field the method of characteristics yields a solution of the flow problem. The same finite difference network as used in the subsonic field is applied, except near the boundary characteristic.

The supersonic and subsonic calculations are performed simultaneously in the computer program, but it is also possible in principle to separate both calculations and to solve first the supersonic part in a separate program.

In section 2 the characteristic equations and directions are given. After transformation of these equations to finite difference form, an iteration process will yield the numerical solution.

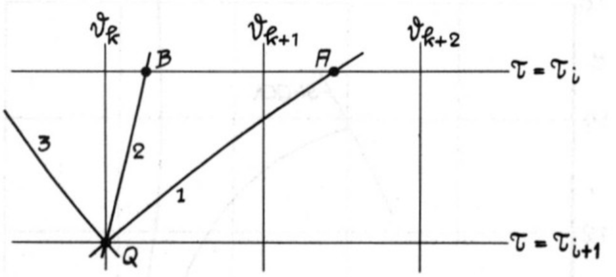

Figure 3 Characteristics of first (1) and second (2) family and streamline (3) passing through an arbitrary lattice point (Q) in the supersonic field.

In this iteration process the velocity components u and v in the lattice point Q (fig.3) are solved using the finite difference representations of the characteristic equations along AQ and BQ, and the radius vector r using the finite difference equivalent of (12) along AQ, the positions of A and B being included in the iteration cycle using the finite difference representations of the equations for the characteristic directions. The quantities p and ρ can be found at any τ and ϑ from the eqs (4) and (10) when u, v and r are known. Quadratic interpolation along the line $\tau=\tau_i$ is used in order to obtain u, v and r in the points A and B.

5 Generation of the desired body contour.

Once the problem of the calculation of the body contour belonging to a given shock wave shape being solved, there remains the problem of changing the shock contour such, that a prescribed body contour results.

In order to solve this problem a norm has been defined, on the basis of which can be checked to what extent an obtained body contour deviates from the desired body contour and which gives a rule for changing the shock wave shape in order to adjust the body shape. This norm has been defined to be the minimum value of a function $F(B_s, C_s)$ of the two variables B_s and C_s, where F has been defined as follows:

$$F(B_s,C_s)=(B_b-B_b^*)^2+G\sum_i g_i\left\{\left(\frac{dy}{dx}\right)_i-\frac{\bar{v}_i}{\bar{u}_i}\right\}_b^2 \qquad (19)$$

In this formula B_s and C_s are shock parameters (see (13)), while B_b is a body parameter, the body contour being represented by the equation

$$y_b^2 = 2K_b(x-\varepsilon)-B_b(x-\varepsilon)^2 \qquad (20)$$

Here ε is the shock distance, measured along the axis of symmetry.

The G and g_i are weight factors.

B_b^* is the desired body parameter value, while B_b is the value of this parameter obtained by fitting the equation (20) to the body points belonging to the shock described by B_s and C_s. The derivative $(dy/dx)_i$ is the slope of the desired body shape at the given location y_i; while $\bar{v}_i/\bar{u}_i$ is the slope of the obtained body contour at the same location y_i.

A gradient method has been applied to find the minimum value of the function $F(B_s,C_s)$, starting with an approximating set B_s,C_s.

By means of the factor G the influence of the second expression on the right side of (19) can be increased. A value of about 20 appears to be favourable. The factors g_i have been chosen in such a way that the weight gradually increases when moving along the body surface in streamdirection.

The choice (19) for the function F and the choice of the above mentioned values for the weight factors furnish quite satisfactory results. The accent which is laid on the correct representation of the flow expansion rate in the supersonic region is an important reason for the successful operation of the shock selection subroutine, and such an accent even seems absolutely necessary for the purpose of obtaining representative flow field results.

During the iterative shock selection process the parameter K_s is held constant. Once having found the final shock wave, the scale of the computational result can be changed, if desired, by changing K_s, C_s and some additional parameters.

6 A numerical example.

The field around a spherically shaped nose cap has been computed using the program described before. The free stream Mach number has been chosen to be 1.8.

Starting with a trial shock wave obtained, by extrapolation, from results of computations performed at higher Mach numbers and specified by

$$K_{s,0} = 2.2$$
$$B_s = -0.46$$
$$C_s = 0.1,$$

seven iteration steps were needed before the resulting field was satisfactory. The shock parameters, obtained then, were

$$K_{s,0} = 2.2$$
$$B_s = -0.51424$$
$$C_s = 0.04426,$$

and the radius of the spherical nose cap appeared to be 0.9536 ± .0008. In order to normalize the mean body radius to 1 the scale was changed, an operation which resulted in the following shock parameters:

$$K_{s,0} = 2.30694$$
$$B_s = -0.51424$$
$$C_s = 0.04221.$$

At last the final computation was performed. Some results are presented here in graphical form.

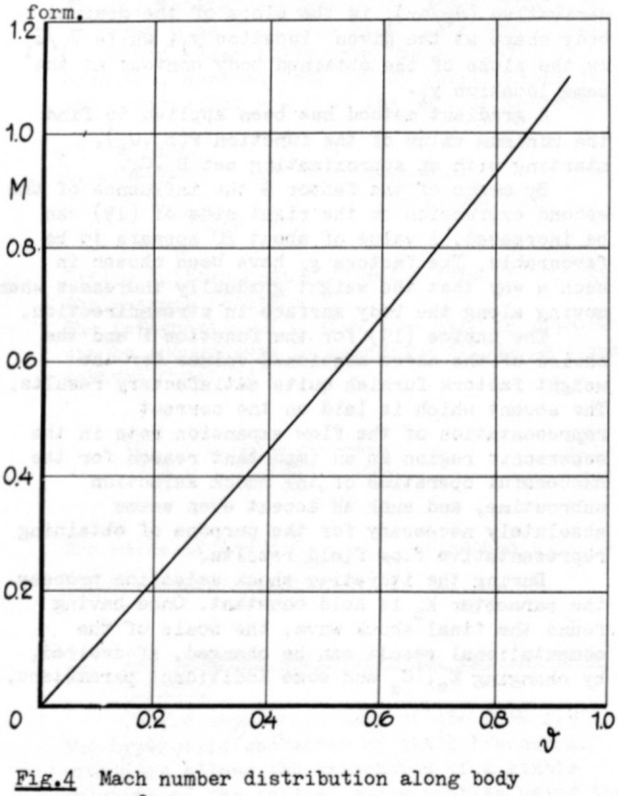

Fig.4 Mach number distribution along body surface.

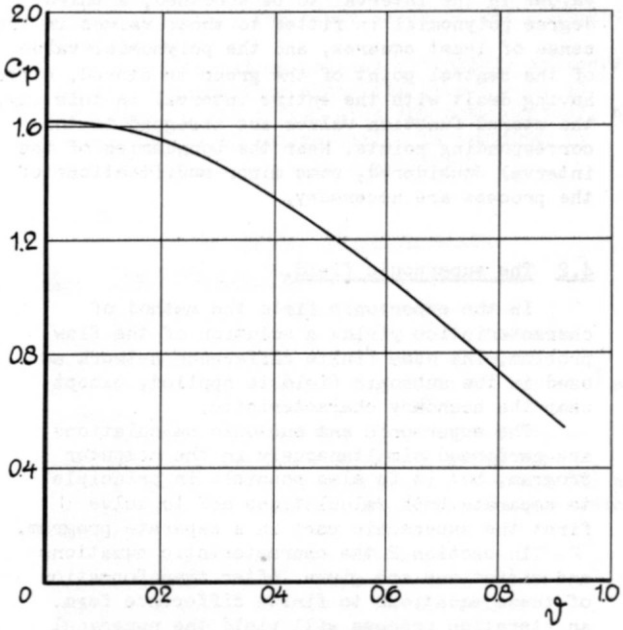

Fig.5 c_p-distribution along body surface

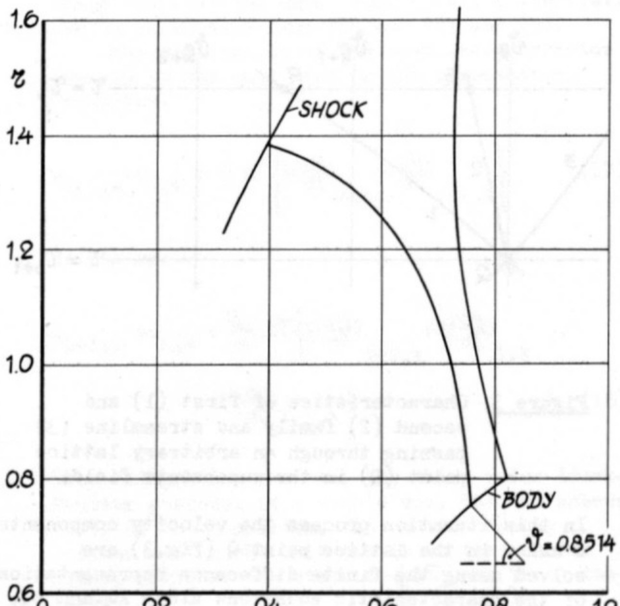

Fig.6 Sonic line and part of boundary characteristic.

7 Concluding remarks.

The primary purpose of this paper has been to describe a method for the accurate inverse computation of blunt nose flow fields. It appears from literature that, especially at lower free stream Mach numbers, a considerable scatter is present in the results of different methods when applied to one test case. The above mentioned method gives very satisfactory results in the case of spherical bodies, down to a Mach number of 1.8. It may be expected that extension to lower Mach numbers and to non-spherical bodies will not be a serious problem.

Reference.

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